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A Spectral Characterization of KMS States

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Abstract. Let ω be a state on a C^*-dynamical system $(\mathfrak{A}, \mathbb{R}, \alpha)$. For each of the following properties of ω: (1) ω is β-KMS with respect to α for some given β, $0 \leqq \beta < +\infty$, (2) ω is either a KMS state or a ground state, necessary and sufficient conditions are given involving only the spectral subspaces of $\mathfrak{A}$ associated with α. The results provide a new insight in the concept of passivity, introduced by W. Pusz and S. L. Woronowicz.

1. Introduction, Main Results, Preliminaries

Let $(\mathfrak{A}, \mathbb{R}, \alpha)$ be a C^*-dynamical system [8, Chapter 7], β a nonzero positive real number. A state ω on $\mathfrak{A}$ is said to be β-KMS *with respect to* α if for every pair $x, y \in \mathfrak{A}$ there exists a bounded continuous complex function F on the closure of the strip $D = \{z | 0 < \operatorname{Im} z < \beta\}$ that is holomorphic on D and has boundary values

$$\begin{aligned} F(t) &= \omega(y\alpha_t(x)) \\ F(t+i\beta) &= \omega(\alpha_t(x)y) \quad (t \in \mathbb{R}). \end{aligned} \tag{1.1}$$

This condition was introduced in the algebraic statistical theory of infinite quantum systems by R. Haag, N. M. Hugenholtz and M. Winnink [4] to provide a substitute for Gibbs states. The point of view that the KMS condition is to be satisfied by equilibrium states at inverse temperature β is supported by a fair amount of physically relevant mathematical evidence, a lot of which can be found in [2; 6]. Moreover the KMS condition plays a central role in the Tomita–Takesaki theory [10] and in non-commutative integration.

On the other hand, one of the main tools to study C^*-dynamical systems in general is the notion of spectral subspaces, the introduction of which in the theory of operator algebras is due to W. B. Arveson [1; 8, Chapter 8]. It is known, for instance, that α is completely determined if the spectral subspaces $R(-\infty, \lambda) \subset \mathfrak{A}$

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(defined below) are given for all $\lambda \in \mathbb{R}$. Hence it is natural (and, as it turns out, useful) to ask for a "harmonic analysis" version of the KMS condition (1.1), in which the action α is to enter only via the associated "spectral resolution" of $\mathfrak{A}$ in subspaces $R(-\infty, \lambda)$.

The main results of this paper, which we now proceed to summarize, provide an answer to that question. In what follows ($\mathfrak{A}$, $\mathbb{R}$, α) is a C^*-dynamical system.

Theorem 1.1. *Let ω be a state on $\mathfrak{A}$, and $0 < \beta < +\infty$. Then ω is β-KMS with respect to α if and only if it is α-invariant and satisfies the following spectral condition*:

$$\textit{for all } \lambda \in \mathbb{R} \textit{ and } x \in R(-\infty, \lambda), \textit{ one has } \omega(x^*x) \leqq e^{\beta\lambda}\omega(xx^*). \tag{1.2}$$

Definition 1.2. A state ω on $\mathfrak{A}$ will be called *n-spectrally passive* with respect to α (where $n \in \mathbb{N}_0$) if it is α-invariant and if the following property holds: for every n-tuple of real numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that $\sum_{j=1}^{n} \lambda_j \leqq 0$, and every n-tuple $x_1, x_2, \ldots, x_n$ of elements of $\mathfrak{A}$ such that

$$x_j \in R(-\infty, \lambda_j) \text{ for all } j = 1, 2, \ldots, n,$$

the inequality

$$\omega(x_1^* x_1)\omega(x_2^* x_2)\ldots\omega(x_n^* x_n) \leqq \omega(x_1 x_1^*)\omega(x_2 x_2^*)\ldots\omega(x_n x_n^*) \tag{1.3}$$

is valid. If ω is n-spectrally passive for all $n \in \mathbb{N}_0$, it is said to be *completely spectrally passive* with respect to α. An invariant state ω satisfying

$$x \in R(-\infty, 0) \Rightarrow \omega(x^*x) \leqq \omega(xx^*) \tag{1.4}$$

is simply said to be *spectrally passive* (rather than 1-spectrally passive).

Theorem 1.3. *For a state ω on $\mathfrak{A}$ the following are equivalent*:

(i) *ω is completely spectrally passive with respect to α.*

(ii) *ω is either a* KMS *state or a ground state with respect to α.*

In (ii) above, "ω is KMS" means either that ω is β-KMS for some strictly positive β, or that it is an α-invariant tracial state (in which case we say that ω is 0-KMS); and ω is called a *ground state* if the "hamiltonian" H (defined below) in the GNS representation generated by ω is a positive operator. As is well known [2; 8] the ground state case corresponds to the limit $\beta = +\infty$.

The reason for the terminology introduced in Definition 1.2 lies in the fact that condition (1.4) contains the mathematical essence of the notion of *passivity*, introduced on physical grounds by W. Pusz and S. L. Woronowicz in [9]. This will be shown in Sect. 3. Actually, although of a purely mathematical nature, the results in the present paper are closely related to those in [9]. The relationship is discussed in some detail in our final Sect. 5.

The proofs of the Theorems 1.1 and 1.3 are to be found in Sect. 2 and 4, respectively. Section 4 moreover contains both an example of an n-spectrally passive state that is not $(n+1)$-spectrally passive, and a theorem to the effect that spectral passivity does imply complete spectral passivity in the special case of a weakly clustering state ω.

We conclude this section with some preliminary material. Throughout it is assumed that ω is an α-invariant state on $\mathfrak{A}$. Let $(\mathscr{H}, \pi, \Omega)$ be the GNS representation generated by ω[2, 2.3.3; 8, 3.3]: specifically, we have $\omega(x) = (\Omega, \pi(x)\Omega)$ for $x \in \mathfrak{A}$, where $(\cdot, \cdot)$ denotes the scalar product in $\mathscr{H}$. There exists a unique strongly continuous group of unitaries $\{U_t\}_{t \in \mathbb{R}}$ on $\mathscr{H}$ with the properties

$$\pi(\alpha_t(x)) = U_t \pi(x) U_t^* \qquad (x \in \mathfrak{A}, t \in \mathbb{R})$$

and

$$U_t \Omega = \Omega \qquad (t \in \mathbb{R}).$$

Invoking Stone's theorem we write $U_t = e^{itH}$, where H, the "hamiltonian," is a self-adjoint operator on $\mathscr{H}$. On the von Neumann algebra $\mathscr{A} = \pi(\mathfrak{A})''$ generated by $\pi(\mathfrak{A})$ we define a continuous one-parameter group of automorphisms $\{\bar{\alpha}_t\}_{t \in \mathbb{R}}$ by

$$\bar{\alpha}_t(A) = U_t A U_t^* \qquad (A \in \mathscr{A}, t \in \mathbb{R}).$$

We want to recall the relationship between the spectral subspaces associated with α, $\bar{\alpha}$ and U. Let $\mathcal{O}$ be an open subset of $\mathbb{R}$. With the definition[1]

$$\hat{f}(\lambda) = \int e^{i\lambda t} f(t) dt \qquad (\lambda \in \mathbb{R})$$

for the Fourier (co)transform of $f \in L^1(\mathbb{R})$, let us define the spectral subspace $R(\mathcal{O})$ (respectively $\bar{R}(\mathcal{O})$) to be the norm (respectively the σ-weak) closure of the space spanned by all elements of $\mathfrak{A}$ (respectively $\mathscr{A}$) of the form $\int f(t)\alpha_t(x)dt$ with $x \in \mathfrak{A}$ (respectively $\int f(t)\bar{\alpha}_t(A)dt$ with $A \in \mathscr{A}$), where $f \in L^1(\mathbb{R})$ and $\hat{f}$ has compact support contained in $\mathcal{O}$[8, p. 300][2]. Also P will denote the projection-valued measure associated to H, so that $H = \int \lambda dP_\lambda$ and $U_t = \int e^{i\lambda t} dP_\lambda$. The subspace $P(\mathcal{O})\mathscr{H}$ of $\mathscr{H}$ is precisely the spectral subspace of $\mathscr{H}$ associated with U and corresponding to the open subset $\mathcal{O}$ of $\mathbb{R}$ [8, Proposition 8.3.2].

Lemma 1.4 *Let $\mathcal{O}$ be an open subset of $\mathbb{R}$.*
(i) *$\bar{R}(\mathcal{O})$ is the σ-weak closure of $\pi(R(\mathcal{O}))$ in $\mathscr{A}$.*
(ii) *$P(\mathcal{O})\mathscr{H}$ is the closure in $\mathscr{H}$ of either of the spaces $\pi(R(\mathcal{O}))\Omega$ or $\bar{R}(\mathcal{O})\Omega$.*

Proof. (i) As $\pi(\int f(t)\alpha_t(x)dt) = \int f(t)\bar{\alpha}_t(\pi(x))dt$ for all $x \in \mathfrak{A}$ and $f \in L^1(\mathbb{R})$, it is obvious that $\pi(R(\mathcal{O}))^- \subset R(\mathcal{O})$. The converse inclusion follows from the fact that the linear map $A \mapsto \int f(t)\bar{\alpha}_t(A)dt\,(A \in \mathscr{A})$ is σ-weakly continuous [1, Proposition 1.6].

(ii) Clearly $(\int f(t)\bar{\alpha}_t(A)dt)\Omega = \int f(t)U_t A\Omega dt\,(A \in \mathscr{A})$, hence $(\bar{R}(\mathcal{O})\Omega)^- \subset P(\mathcal{O})\mathscr{H}$. Using the cyclicity of Ω one shows immediately that $P(\mathcal{O})\mathscr{H} \subset (\pi(R(\mathcal{O}))\Omega)^-$. Since (i) implies $(\bar{R}(\mathcal{O})\Omega)^- = (\pi(R(\mathcal{O}))\Omega)^-$, the lemma follows. □

Remark 1.5. The spectra $\mathrm{sp}(\alpha)$ and $\mathrm{sp}(\bar{\alpha})$ of α and $\bar{\alpha}$ in the sense of Arveson are by definition the smallest of all closed subsets $\mathscr{F}$ of $\mathbb{R}$ such that $R(\mathbb{R}\backslash\mathscr{F}) = \{0\}$, respectively $\bar{R}(\mathbb{R}\backslash\mathscr{F}) = \{0\}$ [8, 8.1.6]. The lemma entails the inclusions $\sigma(H) \subset \mathrm{sp}(\bar{\alpha}) \subset \mathrm{sp}(\alpha)$ [8, Proposition 8.1.9], where $\sigma(H)$ is the operator spectrum of H (or, equivalently, the spectrum of U in the sense of Arveson). If π is faithful, we have

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1 Unless the limits of integration are explicitly stated, all integrations extend from $-\infty$ to $+\infty$

2 We use $R(-\infty, \lambda)$, $P[\lambda, \mu]$, etc. instead of $R((-\infty, \lambda))$, $P([\lambda, \mu])$, etc.

$\mathrm{sp}(\bar{\alpha}) = \mathrm{sp}(\alpha)$. If Ω is separating for $\pi(\mathfrak{A})$, then $\sigma(H) = \mathrm{sp}(\bar{\alpha})$ (and, in particular, H has a symmetric spectrum [8, Corollary 8.3.4]). Hence if ω is faithful, the three sets coincide [6, Remark 4.1].

Let x be an element of $\mathfrak{A}$. As the complex valued functions $t \mapsto \omega(x^*\alpha_t(x))$ and $t \mapsto \omega(\alpha_t(x)x^*)$ are positive definite (by the α-invariance of ω) they are the Fourier transforms of bounded positive Radon measures. It follows by polarization that there exist (unique) bounded Radon measures $\mu_{x,y}$ and $\nu_{x,y}$ for every pair $x, y \in \mathfrak{A}$ such that

$$\left.\begin{aligned} \omega(x^*\alpha_t(y)) &= \int e^{i\lambda t}\, d\mu_{x,y}(\lambda), \\ \text{and} \qquad \omega(\alpha_t(y)x^*) &= \int e^{i\lambda t}\, d\nu_{x,y}(\lambda). \end{aligned}\right\} \tag{1.5}$$

These measures are obviously related to the projection-valued measure P and to each other. It is not difficult to check that

$$d\mu_{x,y}(\lambda) = d(\pi(x)\Omega, P_\lambda \pi(y)\Omega), \tag{1.6}$$

$$d\mu_{x,y}(-\lambda) = d\nu_{y^*,x^*}(\lambda). \tag{1.7}$$

The spectrum $\mathrm{sp}(x)$ of an element $x \in \mathfrak{A}$ (with respect to α) is defined by [1, Definition 2.1]:

$$\lambda \in \mathrm{sp}(x) \Leftrightarrow \hat{f}(\lambda) = 0 \text{ whenever } \int f(t)\alpha_t(x)dt = 0. \tag{1.8}$$

Lemma 1.6. *Let x, y be elements of $\mathfrak{A}$. Then* $\mathrm{supp}\,\mu_{x,y} \subset \mathrm{sp}(y)$ *and* $\mathrm{supp}\,\nu_{x,y} \subset \mathrm{sp}(y)$.

Proof. Let $f \in L^1(\mathbb{R})$ be such that $\hat{f}$ has compact support contained in the complement of $\mathrm{sp}(y)$. Then

$$\begin{aligned} \int \hat{f}(\lambda)d\mu_{x,y}(\lambda) &= \int\int f(t)e^{i\lambda t}dt\, d\mu_{x,y}(\lambda) \\ &= \int f(t)\omega(x^*\alpha_t(y))dt \qquad \text{by (1.5)}, \\ &= 0 \qquad [1, \text{p. } 225, \text{Remarks}]. \end{aligned}$$

Now any $g \in C_c(\mathbb{R})$ (the space of complex continuous functions on $\mathbb{R}$ with compact support) whose support is contained in $\mathbb{R}\backslash\mathrm{sp}(y)$ can be approximated uniformly by functions $\hat{f}$ as above. Thus the first inclusion follows, whereas the second one is proved similarly. □

Let us introduce the abbreviations μ_x and ν_x for the positive measures $\mu_{x,x}$ and $\nu_{x,x}$, respectively. The following characterization of KMS states is well known [4, 3.32; 2, Proposition 5.3.14]. It will be our working definition of the KMS condition (1.1).

Lemma 1.7. *Let ω be an α-invariant state, $0 < \beta < +\infty$. Then the following are equivalent:*

(i) *ω is β-KMS with respect to α.*

(ii) *For all $x \in \mathfrak{A}$, the measures μ_x and ν_x are equivalent, with Radon–Nikodým derivative given by*

$$\frac{d\mu_x}{d\nu_x}(\lambda) = e^{\beta\lambda}$$

for ν_x-almost all λ.

The notation introduced above will be used throughout the paper, mostly without explicit reference.

2. KMS States at Given Inverse Temperature

This short section is entirely devoted to the proof of Theorem 1.1. Therefore we let $(\mathfrak{A}, \mathbb{R}, \alpha)$ be a C^*-dynamical system, $0 < \beta < +\infty$, and we first assume that the state ω on $\mathfrak{A}$ is β-KMS with respect to α. Then ω is α-invariant [8, Prop. 8.12.4]. Let λ be a real number and $x \in R(-\infty, \lambda)$. We have

$$\omega(x^*x) = \int d\mu_x(\xi) = \int e^{\beta\xi} d\nu_x(\xi)$$

by (1.5) and Lemma 1.7. Since $\operatorname{sp}(x) \subset (-\infty, \lambda]$, as one easily checks using (1.8), the support of ν_x is contained in $(-\infty, \lambda]$ as a consequence of Lemma 1.6. As $\beta > 0$ this implies

$$\int e^{\beta\xi} d\nu_x(\xi) \leqq e^{\beta\lambda} \int d\nu_x(\xi) = e^{\beta\lambda}\omega(xx^*)$$

(again by (1.5)), which proves the "only if" part of Theorem 1.1.

Conversely, suppose that the α-invariant state ω satisfies the spectral condition (1.2). Let $x \in \mathfrak{A}$ and $\lambda_1 < \lambda_2$; we claim that, for all positive $g \in C_c(\mathbb{R})$ with support in (λ_1, λ_2), the inequalities

$$e^{\beta\lambda_1} \int g(\xi) d\nu_x(\xi) \leqq \int g(\xi) d\mu_x(\xi) \leqq e^{\beta\lambda_2} \int g(\xi) d\nu_x(\xi) \tag{2.1}$$

hold. As every such g can be uniformly approximated by functions $|\hat{f}|^2$, where $f \in L^1(\mathbb{R})$ and $\operatorname{supp}\hat{f} \subset (\lambda_1, \lambda_2)$, it suffices to check the claim for $g = |\hat{f}|^2$. If we put $y = \int f(t)\alpha_t(x)dt$, then clearly $y \in R(-\infty, \lambda_2) \cap R(-\infty, -\lambda_1)^*$. Hence by assumption

$$e^{\beta\lambda_1}\omega(yy^*) \leqq \omega(y^*y) \leqq e^{\beta\lambda_2}\omega(yy^*). \tag{2.2}$$

On the other hand,

$$\begin{aligned}\omega(y^*y) &= \int\int \overline{f(s)} f(t) \omega(\alpha_s(x^*)\alpha_t(x)) ds\, dt \\ &= \int\int \overline{f(s)} f(t+s) \omega(x^*\alpha_t(x)) ds\, dt \\ &= \int \left(\int \overline{f(s)} e^{-i\lambda s} ds\right)\left(\int f(t+s) e^{i\lambda(t+s)} dt\right) d\mu_x(\lambda),\end{aligned}$$

by (1.5) and Fubini's theorem

$$= \int |\hat{f}(\lambda)|^2 d\mu_x(\lambda).$$

Similarly $\omega(yy^*) = \int |\hat{f}(\lambda)|^2 d\nu_x(\lambda)$. Consequently (2.2) becomes

$$e^{\beta\lambda_1} \int |\hat{f}(\lambda)|^2 d\nu_x(\lambda) \leqq \int |\hat{f}(\lambda)|^2 d\mu_x(\lambda) \leqq e^{\beta\lambda_2} \int |\hat{f}(\lambda)|^2 d\nu_x(\lambda).$$

This establishes our claim (2.1). By the Lebesgue–Radon–Nikodým theorem [5, Theorem 12.17], μ_x is absolutely continuous with respect to ν_x (and vice versa), and moreover the Radon–Nikodým derivative $h = d\mu_x/d\nu_x$ verifies $e^{\beta\lambda_1} \leqq h \leqq e^{\beta\lambda_2}$ ν_x-almost everywhere on (λ_1, λ_2).

It is now an easy exercise in measure theory to show that $h(\lambda) = e^{\beta\lambda}$ for ν_x-almost all λ. For suppose there exist $\varepsilon > 0$ and a ν_x-measurable subset K of $\mathbb{R}$ such that

$\nu_x(K) > 0$ and $|h(\lambda) - e^{\beta\lambda}| \geqq \varepsilon$ for all $\lambda \in K$. By the inner regularity of ν_x we may assume that K is compact. Then we can cover K with a finite number (n, say) of open intervals (ξ_j, ξ_j') such that $e^{\beta\xi_j'} - e^{\beta\xi_j} < \varepsilon$ for all $j = 1, 2, \ldots, n$. Since $e^{\beta\xi_j} \leqq h(\lambda) \leqq e^{\beta\xi_j'}$ for ν_x-almost all $\lambda \in (\xi_j, \xi_j')$, we also have $|e^{\beta\lambda} - h(\lambda)| < \varepsilon$ for these values of λ. But at least one of the n intervals (ξ_j, ξ_j') has a non-negligible intersection with K (with respect to ν_x), which leads to a contradiction.

Hence we have shown that $\dfrac{d\mu_x}{d\nu_x}(\lambda) = e^{\beta\lambda}$ for ν_x-almost all λ. By Lemma 1.7, ω is β-KMS with respect to α. $\square$

Remarks 2.1 (i) It is useful to observe that Theorem 1.1 remains true in the limiting cases $\beta = 0$ and (at least formally) $\beta = +\infty$. For if an α-invariant state ω satisfies

$$\omega(x^*x) \leqq \omega(xx^*) \quad \text{whenever } x \in R(-\infty, \lambda)$$

for all $\lambda \in \mathbb{R}$, then clearly $x \in R(-\lambda, \lambda)$ (with $\lambda > 0$) will imply $\omega(x^*x) = \omega(xx^*)$. As $\bigcup_{\lambda > 0} R(-\lambda, \lambda)$ is dense in $\mathfrak{A}$ [8, Theorem 8.1.4], we conclude that ω is an α-invariant tracial state, i.e. a 0-KMS state.

On the other hand, by Lemma 1.4, ω is a ground state if and only if it satisfies

$$x \in R(-\infty, 0) \Rightarrow \omega(x^*x) = 0 \tag{2.3}$$

(the α-invariance follows from the inequality $|\omega(x)|^2 \leqq \omega(x^*x)$, cf. next remark). This is exactly the condition one obtains by taking the limit $\beta = +\infty$ in (1.2) for all $\lambda < 0$. This characterization of ground states is well known, cf. e.g. [2, Proposition 5.3.19].

(ii) The α-invariance of ω is of course a spectral property itself. In terms of the spectral subspaces corresponding to open half lines it can be formulated as

$$x \in R(-\infty, 0) \Rightarrow \omega(x) = 0.$$

We do not know whether the spectral condition (1.2) already implies that ω is α-invariant, as does the ground state condition (2.3).

(iii) Recall that to each closed subset $\mathscr{F}$ of the real line there corresponds a spectral subspace

$$M(\mathscr{F}) = \{x \in \mathfrak{A} \mid \mathrm{sp}(x) \subset \mathscr{F}\}$$

[1, Definition 2.1]. Theorem 1.1 still holds when $R(-\infty, \lambda)$ is replaced with $M(-\infty, \lambda]$ in (1.2).

3. Spectral Passivity

In view of Theorem 1.1, spectral passivity (Definition 1.2) appears as the most elementary property of spectral type shared by all KMS states: it simply follows by virtue of the observation that $e^{\beta 0} = 1$, whatever the value of β. It is remarkable that this property turns out to be equivalent with a "differential inequality" ((3.1) below) derived by W. Pusz and S. L. Woronowicz on physical grounds [9, 3.4] (for a somewhat different derivation of this inequality, cf. [3]). This is the content of the main theorem of this section.

Lemma 3.1. *Let ω be an invariant state of the C^*-dynamical system $(\mathfrak{A}, \mathbb{R}, \alpha)$. Then the following are equivalent:*

(i) *ω is spectrally passive.*

(ii) *$\mu_x \leqq \nu_x$ on $(-\infty, 0)$ for all $x \in \mathfrak{A}$.*

(iii) *$\mu_x \geqq \nu_x$ on $(0, +\infty)$ for all $x \in \mathfrak{A}$.*

Proof. The equivalence of (ii) and (iii) is an immediate consequence of (1.7). Assume ω is spectrally passive, $x \in \mathfrak{A}$, and the Fourier transform of $f \in L^1(R)$ has compact support contained in $(-\infty, 0)$. Then $y = \int f(t)\alpha_t(x)dt \in R(-\infty, 0)$ and one computes as in the previous section that

$$\int |\hat{f}(\lambda)|^2 d\mu_x(\lambda) = \omega(y^*y) \leqq \omega(yy^*) = |\hat{f}(\lambda)|^2 d\nu_x(\lambda),$$

using the passivity assumption. This implies (ii).

Conversely, assume (ii) holds. Let ε be positive, and $x \in M(-\infty, -\varepsilon]$. Then the support of μ_x and ν_x is contained in $(-\infty, -\varepsilon]$ by Lemma 1.6, and hence $\mu_x \leqq \nu_x$. This implies

$$\omega(x^*x) = \int d\mu_x(\lambda) \leqq \int d\nu_x(\lambda) = \omega(xx^*).$$

Since $\bigcup_{\varepsilon > 0} M(-\infty, -\varepsilon]$ is dense in $R(-\infty, 0)$, we conclude that ω is spectrally passive. □

Remark 3.2. The condition

$$\omega(x^*x) \leqq \omega(xx^*) \quad \text{whenever } x \in M(-\infty, 0],$$

where ω is an α-invariant state, is strictly stronger than spectral passivity (this is particularly obvious if α is trivial, but non-trivial examples are readily available in low dimensions). In terms of the measures introduced in Sect. 1 it can be stated as

$$\mathrm{sp}(x) \subset (-\infty, 0] \Rightarrow \mu_x \leqq \nu_x.$$

Theorem 3.3. *Let $(\mathfrak{A}, \mathbb{R}, \alpha)$ be a C^*-dynamical system, and let δ (with domain $D(\delta)$) be the infinitesimal generator of the group $\{\alpha_t\}_{t \in \mathbb{R}}$. Then ω is spectrally passive with respect to α if and only if*

$$-i\omega(x\delta(x)) \geqq 0 \quad \textit{for all } x = x^* \in D(\delta). \tag{3.1}$$

Proof. First we observe as in [9, p. 283] that the condition (3.1) is equivalent with

$$-i\omega(x^*\delta(x)) - i\omega(x\delta(x^*)) \geqq 0 \quad \text{for all } x \in D(\delta). \tag{3.2}$$

Suppose now ω is spectrally passive and $x \in D(\delta)$. Then $\pi(x)\Omega$ belongs to the domain of H and

$$\begin{aligned} -i\omega(x^*\delta(x)) &= -i(\pi(x)\Omega, \pi(\delta(x))\Omega) \\ &= (\pi(x)\Omega, H\pi(x)\Omega) \\ &= \int \lambda d(\pi(x)\Omega, P_\lambda \pi(x)\Omega) \\ &= \int \lambda d\mu_x(\lambda) \end{aligned}$$

by (1.6); and, using (1.7),

$$-i\omega(x\delta(x^*)) = -\int \lambda \, d\nu_x(\lambda).$$

Summing these equations we obtain

$$\begin{aligned} -i\omega(x^*\delta(x)) - i\omega(x\delta(x^*)) &= \int_{(-\infty,0)} (-\lambda)(d\nu_x(\lambda) - d\mu_x(\lambda)) \\ &\quad + \int_{(0,+\infty)} \lambda(d\mu_x(\lambda) - d\nu_x(\lambda)) \geqq 0 \end{aligned}$$

by Lemma 3.1. This proves (3.2) and hence the "only if" part of the Theorem.

Conversely, let us assume that (3.1) holds. Then ω is α-invariant [2, Lemma 5.3.16]. Reasoning as in the second part of the proof of Lemma 3.1 we see that it is enough to show that $\mu_x \leqq \nu_x$ whenever $x \in M\left[-n, -\frac{1}{n}\right]$, where n is an arbitrary natural number. Under these assumptions, let g be a continuously differentiable complex valued function on $\mathbb{R}$ with compact support (i.e. $g \in C_c^1(\mathbb{R})$), and put $y = \int g(t)\alpha_t(x)dt$. Clearly $y \in D(\delta)$ and

$$\delta(y) = -\int g'(t)\alpha_t(x)dt,$$

where g' denotes the derivative of g. One easily computes

$$\begin{aligned} -i\omega(y^*\delta(y)) &= i\int\int g'(s)\overline{g(t)}\omega(\alpha_t(x^*)\alpha_s(x))\,ds\,dt \\ &= i\int\int g'(s)\overline{g(t)}e^{i(s-t)\lambda}d\mu_x(\lambda)\,ds\,dt \\ &= i\int (g')^\wedge(\lambda)\overline{\hat{g}(\lambda)}\,d\mu_x(\lambda) \\ &= \int \lambda|\hat{g}(\lambda)|^2 d\mu_x(\lambda) \end{aligned}$$

(notice that μ_x has compact support!), and similarly

$$-i\omega(y\delta(y^*)) = -\int \lambda|\hat{g}(\lambda)|^2 d\nu_x(\lambda).$$

Therefore (3.2) implies

$$\int \lambda|\hat{g}(\lambda)|^2(d\mu_x(\lambda) - d\nu_x(\lambda)) \geqq 0 \quad \text{for all } g \in C_c^1(\mathbb{R}). \tag{3.3}$$

Using the facts that μ_x and ν_x have compact support and that the image of $C_c^1(\mathbb{R})$ under Fourier transformation is uniformly dense in $C_0(\mathbb{R})$, we can strengthen (3.3):

$$\int \lambda h(\lambda)(d\mu_x(\lambda) - d\nu_x(\lambda)) \geqq 0 \quad \text{for all } h \in C_0(\mathbb{R})_+. \tag{3.4}$$

Finally, suppose f is a continuous positive function with compact support contained in $(-\infty, 0)$. As the function h defined by

$$\begin{aligned} h(\lambda) &= -\lambda^{-1}f(\lambda), && \text{if } \lambda \in (-\infty, 0) \\ &= 0, && \text{if } \lambda \in [0, +\infty) \end{aligned}$$

belongs to $C_0(\mathbb{R})_+$, substitution in (3.4) leads to the conclusion that $\mu_x \leqq \nu_x$

on $(-\infty, 0)$. But then $\mu_x \leqq \nu_x$, since both μ_x and ν_x are supported in $\left[-n, -\frac{1}{n}\right]$, by Lemma 1.6. □

4. KMS States at Unspecified Inverse Temperatures

It follows immediately from Theorem 1.1 and Remark 2.1 (i) that all KMS states (in the broad sense: ground states and invariant traces are included) are completely spectrally passive (Definition 1.2). To prove Theorem 1.3 we have to show that the converse holds as well. This will be achieved through a series of lemmas concerning a state ω of $(\mathfrak{A}, \mathbb{R}, \alpha)$ on which, for the sake of clarity, the requirement of complete spectral passivity with respect to α will be imposed only gradually.

Definition 4.1. For every $\lambda \in \mathbb{R}$ we define two closed sets of nonnegative real numbers

$$S_\lambda = \{a \geqq 0 | \omega(x^*x) \leqq a\omega(xx^*), \quad \forall x \in R(-\infty, \lambda)\}$$

and

$$T_\lambda = \{b \geqq 0 | \omega(x^*x) \geqq b\omega(xx^*), \quad \forall x \in R(\lambda, +\infty)\}.$$

They determine two functions s and t of a real variable, with values in $[0, +\infty]$:

$$\begin{aligned} s(\lambda) &= +\infty, && \text{if } S_\lambda = \emptyset, \\ &= \inf S_\lambda, && \text{otherwise}, \\ t(\lambda) &= \sup T_\lambda && (\text{notice that } 0 \in T_\lambda \text{ for all } \lambda). \end{aligned}$$

Since $\lambda \leqq \mu$ implies $S_\mu \subset S_\lambda$ and $T_\lambda \subset T_\mu$, it is clear that both s and t are monotonically increasing. Other elementary properties of s and t are described next.

Lemma 4.2. (i) $s(\lambda) = 0$ *if and only if* $t(-\lambda) = +\infty$.

$$s(\lambda) = +\infty \quad \textit{if and only if} \quad t(-\lambda) = 0.$$

If $0 < s(\lambda) < +\infty$, *then* $t(-\lambda) = s(\lambda)^{-1}$.

(ii) *The function* s *(respectively* t*) is everywhere continuous from the left (respectively right).*

(iii) *Let* ω *be* α*-invariant. If* $\lambda < \mu$, $[\lambda, \mu) \cap \operatorname{sp}(\bar{\alpha}) = \emptyset$, *then* $s(\lambda) = s(\mu)$ *(a similar property holds for* t*, of course).*

Proof. (i) Notice that $s(\lambda) = 0$ if and only if $\omega(x^*x) = 0$ for all $x \in R(-\infty, \lambda)$. Now $x \in R(-\infty, \lambda)$ is equivalent with $x^* \in R(-\lambda, +\infty)$ [8, Lemma 8.3.3]. Hence $s(\lambda) = 0$ if and only if $T_{-\lambda} = [0, +\infty)$, or $t(-\lambda) = +\infty$. On the other hand $s(\lambda) = +\infty$ if and only if for every $\varepsilon > 0$ there exists $x \in R(-\infty, \lambda)$ such that $\omega(x^*x) > \varepsilon^{-1}\omega(xx^*)$. Equivalently (putting $y = x^*$), for every $\varepsilon > 0$ there exists $y \in R(-\lambda, +\infty)$ such that $\omega(y^*y) < \varepsilon\omega(yy^*)$, which means precisely that $T_{-\lambda} = \{0\}$. Finally, if $0 < s(\lambda) < +\infty$, one has $0 \notin S_\lambda \neq \emptyset$, and $a \in S_\lambda$ if and only if $a^{-1} \in T_{-\lambda}$.

(ii) By (i) it is sufficient to show that t is continuous from the right at every $\lambda \in \mathbb{R}$. If $t(\lambda) = +\infty$ this is obviously true, so assume $t(\lambda) < +\infty$. Then

$$t(\lambda) = \inf\{\omega(x^*x)\omega(xx^*)^{-1} | x \in R(\lambda, +\infty), \quad \omega(xx^*) \neq 0\}, \tag{4.1}$$

this set being nonempty by assumption. Hence for all $\varepsilon > 0$ there exists $x \in R(\lambda, +\infty)$ with $\omega(xx^*) \neq 0$ and $\omega(x^*x)\omega(xx^*)^{-1} < t(\lambda) + \varepsilon/2$. Since $R(\lambda, +\infty)$ is the closure of $\bigcup_{\mu > \lambda} R(\mu, +\infty)$ [8, Theorem 8.1.4], there exist $\mu > \lambda$ and $y \in R(\mu, +\infty)$ such that $\omega(yy^*) \neq 0$ and $|\omega(y^*y)\omega(yy^*)^{-1} - \omega(x^*x)\omega(xx^*)^{-1}| < \varepsilon/2$. Consequently $t(\mu) \leqq \omega(y^*y)\omega(yy^*)^{-1} < t(\lambda) + \varepsilon$, which implies the conclusion as t is increasing.

(iii) The property will be established if we prove the following facts:

$$S_\lambda = \{a \geqq 0 \mid \|A\Omega\|^2 \leqq a\|A^*\Omega\|^2, \quad \forall A \in \bar{R}(-\infty, \lambda)\} \tag{4.2}$$

$$\text{if } \lambda < \mu, [\lambda, \mu) \cap \mathrm{sp}(\bar{\alpha}) = \emptyset, \quad \text{then } \bar{R}(-\infty, \lambda) = \bar{R}(-\infty, \mu). \tag{4.3}$$

To show (4.2), one observes (using Lemma 1.4 and [2, Theorem 2.4.7]) that every $A \in \bar{R}(-\infty, \lambda)$ is the σ-strong* limit of a net $\{\pi(x_i)\}$ with $\{x_i\} \subset R(-\infty, \lambda)$, so that $\|A\Omega\|^2 = \lim \omega(x_i^* x_i)$ and $\|A^*\Omega\|^2 = \lim \omega(x_i x_i^*)$. To prove (4.3), let $\bar{M}(-\infty, v]$ denote the spectral subspace of $\mathscr{A}$ associated to $\bar{\alpha}$ and corresponding to the closed subset $(-\infty, v]$ of $\mathbb{R}$. If $[\lambda, \mu) \cap \mathrm{sp}(\bar{\alpha}) = \emptyset$, there exists $\varepsilon > 0$ such that $(\lambda - \varepsilon, \mu) \cap \mathrm{sp}(\bar{\alpha}) = \emptyset$, since $\mathrm{sp}(\bar{\alpha})$ is closed. Suppose $\lambda \leqq v < \mu$, $A \in \mathscr{A}$:

$$\begin{aligned} A \in \bar{M}(-\infty, v] &\Leftrightarrow \mathrm{sp}(A) \subset (-\infty, v], \quad \text{by definition} \\ &\Leftrightarrow \mathrm{sp}(A) \subset (-\infty, \lambda - \varepsilon], \quad \text{since } \mathrm{sp}(A) \subset \mathrm{sp}(\bar{\alpha}), \\ &\Leftrightarrow A \in \bar{M}(-\infty, \lambda - \varepsilon]. \end{aligned}$$

But $\bar{R}(-\infty, \mu) = (\bigcup_{v < \mu} \bar{M}(-\infty, v])^-$ (σ-weak closure) by [8, Theorem 8.1.4], hence

$$\bar{R}(-\infty, \mu) = \bar{M}(-\infty, \lambda - \varepsilon] \subset \bar{R}(-\infty, \lambda).$$

The converse inclusion is obvious. □

Definition 4.3. Putting $f(\lambda) = \inf_{\mu > \lambda} s(\mu)$, $g(\lambda) = \sup_{\mu < \lambda} t(\mu)$, we define two functions f and g on $\mathbb{R}$ with positive (possibly infinite) values; both are increasing, and moreover f is right continuous, whereas g is left continuous. Obviously $s \leqq f$, $g \leqq t$. The next lemma gives an important property of the pair (f, g) not shared by the pair (s, t).

Lemma 4.4. *If ω is an α-invariant state on $\mathfrak{A}$, then $g(\lambda) \leqq f(\lambda)$ for all $\lambda \in \sigma(H) \cup (-\sigma(H))$.*

Proof. First suppose $-\lambda \in \sigma(H)$. If $\mu < \lambda < v$ one has $P(-v, -\mu)\mathscr{H} \neq \{0\}$. By Lemma 1.4 it follows that there exists $x \in R(\mu, v)$ such that $\omega(xx^*) = \|\pi(x^*)\Omega\|^2 \neq 0$. Since $R(\mu, v) \subset R(\mu, +\infty) \cap R(-\infty, v)$, we conclude that

$$t(\mu) \leqq \omega(x^*x)\omega(xx^*)^{-1} \leqq s(v).$$

Hence

$$g(\lambda) = \sup_{\mu < \lambda} t(\mu) \leqq \inf_{v > \lambda} s(v) = f(\lambda).$$

If $\lambda \in \sigma(H)$, the first part of the proof implies $g(-\lambda) \leqq f(-\lambda)$. But it clearly follows from Lemma 4.2 (i) that $g(-\lambda) = f(\lambda)^{-1}$ and $f(-\lambda) = g(\lambda)^{-1}$ (if we agree that $1/0 = +\infty$ and $1/+\infty = 0$). Hence $g(\lambda) \leqq f(\lambda)$ in this case as well. □

Next we dispose of the ground state case.

Lemma 4.5. *Let ω be a state on $\mathfrak{A}$. Then ω is a ground state if and only if $s(0)=0$, or equivalently $g(\lambda)=+\infty$ for all $\lambda>0$.*

Proof. The state ω is a ground state if and only if $\omega(x^*x)=\|\pi(x)\Omega\|^2=0$ for all $x\in R(-\infty,0)$ (Remark 2.1. (i)), i.e. if and only if $s(0)=0$. On the other hand $s(0)=0$ is equivalent with $t(0)=+\infty$ (Lemma 4.2. (i)); since $g\leqq t$ and t is right continuous, this in turn is equivalent with $g(\lambda)=+\infty$ for all $\lambda>0$. □

Now we want to show that, if ω is completely spectrally passive but not a ground state, there exists a nonnegative β such that $s(\lambda)\leqq e^{\beta\lambda}$ for all λ. This will imply that ω is KMS by Theorem 1.1 (and Remark 2.1 (i)). The exclusion of the ground state case, combined with complete spectral passivity, has some very important consequences (as shown in the next lemma), most surprisingly a strengthening of the complete spectral passivity condition (1.3) itself.

Lemma 4.6. *Let ω be completely spectrally passive but not a ground state:*

(i) *$s(\lambda)<+\infty$ and (equivalently) $g(\lambda)>0$ for all $\lambda\in\mathbb{R}$.*

(ii) *Ω is separating for $\mathscr{A}$, and hence $\sigma(H)=\mathrm{sp}(\bar{\alpha})$.*

(iii) *$g(\lambda)<+\infty$ whenever $\lambda\in\mathrm{sp}(\bar{\alpha})$.*

(iv) *For all $n\in\mathbb{N}_0$ and $\varepsilon>0$ there exists $\delta=\delta(n,\varepsilon)>0$ with the following property: if $\lambda_1,\lambda_2,\dots,\lambda_n$ and $x_1,x_2,\dots,x_n$ are n-tuples of real numbers, respectively of elements of $\mathfrak{A}$ such that $\sum_{j=1}^{n}\lambda_j\leqq\delta$ and $x_j\in R(-\infty,\lambda_j)$ for all j, $1\leqq j\leqq n$, then*

$$\omega(x_1{}^*x_1)\omega(x_2{}^*x_2)\dots\omega(x_n{}^*x_n)\leqq(1+\varepsilon)\omega(x_1x_1{}^*)\omega(x_2x_2{}^*)\dots\omega(x_nx_n{}^*).$$

Proof. (i) Since ω is not a ground state, Lemma 4.5 implies the existence of some $\mu>0$ such that $s(-\mu)>0$. Hence there exists $y\in R(\mu,+\infty)$ such that $\omega(yy^*)>0$. Let n be any natural number and $x\in R(-\infty,n\mu)$. Since ω is $(n+1)$-spectrally passive,

$$\omega(x^*x)\omega(yy^*)^n\leqq\omega(xx^*)\omega(y^*y)^n,$$

and consequently

$$s(n\mu)\leqq\omega(y^*y)^n\omega(yy^*)^{-n}<+\infty.$$

As s is increasing, this suffices to conclude that s is everywhere finite. The equivalence with the nonvanishing of g follows from Lemma 4.2 (i).

(ii) Let Q denote the orthogonal projection onto the closure of $\mathscr{A}'\Omega$ in $\mathscr{H}$. We have to show that $Q=1$. Choose $\lambda\in\mathbb{R}$, $A\in\bar{R}(-\infty,\lambda)$ and $a\in S_\lambda$ (that S_λ is nonempty follows from the first part of the Lemma). Since $Q\Omega=\Omega$, one has $A^*(1-Q)\Omega=0$.

On the other hand Q is α-invariant, so that $(1-Q)A\in\bar{R}(-\infty,\lambda)$. Hence by (4.2)

$$\|(1-Q)A\Omega\|^2\leqq a\|A^*(1-Q)\Omega\|^2=0,$$

or $(1-Q)A\Omega=0$. But $\bigcup_{\lambda\in\mathbb{R}}\bar{R}(-\infty,\lambda)$ is σ-strongly dense in $\mathscr{A}$. By the cyclicity of Ω, it follows that $1-Q=0$, as desired, and also $\sigma(H)=\mathrm{sp}(\bar{\alpha})$, as was pointed out in Remark 1.5.

(iii) If $\lambda\in\mathrm{sp}(\bar{\alpha})=\sigma(H)$ and $g(\lambda)=+\infty$, we would have $f(\lambda)=+\infty$ by Lemma 4.4, contradicting the finiteness of s ((i) above).

(iv) Suppose there are $n \in \mathbb{N}_0$ and $\varepsilon > 0$ such that for all $\delta > 0$ there exist n-tuples $\lambda_1, \lambda_2, \ldots, \lambda_n$ and $x_1, x_2, \ldots, x_n$ satisfying $\sum_{j=1}^{n} \lambda_j \leqq \delta$, $x_j \in R(-\infty, \lambda_j)$ for $j = 1, 2, \ldots, n$ and

$$\omega(x_1{}^*x_1)\omega(x_2{}^*x_2)\ldots\omega(x_n{}^*x_n) > (1+\varepsilon)\omega(x_1x_1{}^*)\omega(x_2x_2{}^*)\ldots\omega(x_nx_n{}^*). \quad (4.4)$$

Fix $\lambda > 0$. For an arbitrary nonzero positive integer m, put $\delta = \frac{\lambda}{m}$ and find $\lambda_1, \lambda_2, \ldots, \lambda_n$, $x_1, x_2, \ldots, x_n$ as above. For all $y \in R(\lambda, +\infty)$, we have

$$\omega(x_1{}^*x_1)^m\omega(x_2{}^*x_2)^m\ldots\omega(x_n{}^*x_n)^m\omega(yy^*) \leqq \omega(x_1x_1{}^*)^m \cdot\omega(x_2x_2{}^*)^m\ldots\omega(x_nx_n{}^*)^m\omega(y^*y), \quad (4.5)$$

since ω is $(nm+1)$-spectrally passive. Now either $\omega(x_jx_j{}^*) = 0$ for some j, $1 \leqq j \leqq n$, or $\omega(x_jx_j{}^*) \neq 0$ for all j. In the first case (4.5) implies that $\omega(yy^*) = 0$, since $\omega(x_j{}^*x_j) \neq 0$ for all j by (4.4); hence $t(\lambda) = +\infty$. In the second case, (4.4) and (4.5) lead to

$$\omega(y^*y) \geqq \omega(x_1{}^*x_1)^m\omega(x_1x_1{}^*)^{-m}\omega(x_2{}^*x_2)^m\omega(x_2x_2{}^*)^{-m}\ldots \ldots\omega(x_n{}^*x_n)^m\omega(x_nx_n{}^*)^{-m}\omega(yy^*) > (1+\varepsilon)^m\omega(yy^*).$$

Hence $t(\lambda) \geqq (1+\varepsilon)^m$ for all m, so $t(\lambda) = +\infty$ in this case as well. Anyway we find $g(\lambda) = +\infty$ for all $\lambda > 0$, contradicting the fact that ω is not a ground state, by Lemma 4.5. □

The proof of (i) above gives the first indication to the effect that the growth of s is at most exponential. Also the property (iv) for $n = 2$ implies $f \leqq g$, whereas 2-spectral passivity only entails the weaker inequality $s \leqq t$. More generally we have the following:

Lemma 4.7. *Let ω be a completely spectrally passive state, but not a ground state.*

(i) *Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be elements of* $\mathrm{sp}(\bar{\alpha})$. *Then*

$$f(\lambda_1 + \lambda_2 + \ldots + \lambda_n) \leqq g(\lambda_1)g(\lambda_2)\ldots g(\lambda_n).$$

If $\mu_1, \mu_2, \ldots, \mu_m$ are real numbers with $\sum_{j=1}^{m} \mu_j \leqq \sum_{k=1}^{n} \lambda_k$, one has

$$f(\mu_1)f(\mu_2)\ldots f(\mu_m) \leqq g(\lambda_1)g(\lambda_2)\ldots g(\lambda_n).$$

(ii) *If moreover $\mu_1, \mu_2, \ldots, \mu_m \in \mathrm{sp}(\bar{\alpha})$, then*

$$g(\mu_1)g(\mu_2)\ldots g(\mu_m) \leqq g(\lambda_1)g(\lambda_2)\ldots g(\lambda_n).$$

Proof. (i) Let $\lambda_1, \lambda_2, \ldots, \lambda_n, \mu_1, \mu_2, \ldots, \mu_m$ be as in the statement, and fix $\varepsilon > 0$. Choose $\delta' = \delta(\varepsilon, n+1)$ and $\delta'' = \delta(\varepsilon, m+n)$ according to Lemma 4.6 (iv), and $\zeta > 0$. With

$$x \in R(-\infty, \lambda_1 + \lambda_2 + \ldots + \lambda_n + \delta' - \zeta),$$

$$x_j \in R\left(-\infty, \mu_j + \frac{1}{m}(\delta'' - \zeta)\right), \quad j = 1, 2, \ldots, m,$$

$$y_k \in R\left(\lambda_k - \frac{1}{n}\zeta, +\infty\right), \qquad k = 1, 2, \ldots, n,$$

Lemma 4.6 (iv) implies that

$$\omega(x^*x)\omega(y_1y_1^*)\omega(y_2y_2^*)\ldots\omega(y_ny_n^*)$$
$$\leqq (1+\varepsilon)\omega(xx^*)\omega(y_1^*y_1)\omega(y_2^*y_2)\ldots\omega(y_n^*y_n)$$

and

$$\omega(x_1^*x_1)\omega(x_2^*x_2)\ldots\omega(x_m^*x_m)\omega(y_1y_1^*)\omega(y_2y_2^*)\ldots\omega(y_ny_n^*)$$
$$\leqq (1+\varepsilon)\omega(x_1x_1^*)\omega(x_2x_2^*)\ldots\omega(x_mx_m^*)\omega(y_1^*y_1)\omega(y_2^*y_2)\ldots\omega(y_n^*y_n).$$

Requiring moreover that $\omega(y_ky_k^*) \neq 0$ for $k = 1, 2, \ldots, n$ (which is possible since $t\left(\lambda_k - \frac{1}{n}\zeta\right) \leqq g(\lambda_k) < +\infty$), by Lemma 4.6 (iii) we obtain

$$s(\lambda_1 + \lambda_2 + \ldots + \lambda_n + \delta' - \zeta)$$
$$\leqq (1+\varepsilon)\omega(y_1^*y_1)\omega(y_1y_1^*)^{-1}\omega(y_2^*y_2)\omega(y_2y_2^*)^{-1}\ldots\omega(y_n^*y_n)\omega(y_ny_n^*)^{-1}$$

and

$$s\left(\mu_1 + \frac{1}{m}(\delta'' - \zeta)\right)s\left(\mu_2 + \frac{1}{m}(\delta'' - \zeta)\right)\ldots s\left(\mu_m + \frac{1}{m}(\delta'' - \zeta)\right)$$
$$\leqq (1+\varepsilon)\omega(y_1^*y_1)\omega(y_1y_1^*)^{-1}\omega(y_2^*y_2)\omega(y_2y_2^*)^{-1}\ldots\omega(y_n^*y_n)\omega(y_ny_n^*)^{-1}.$$

Using (4.1) (cf. the proof of Lemma 4.2), this yields

$$s(\lambda_1 + \lambda_2 + \ldots + \lambda_n + \delta' - \zeta)$$
$$\leqq (1+\varepsilon)t\left(\lambda_1 - \frac{1}{n}\zeta\right)t\left(\lambda_2 - \frac{1}{n}\zeta\right)\ldots t\left(\lambda_n - \frac{1}{n}\zeta\right),$$

and

$$s\left(\mu_1 + \frac{1}{m}(\delta'' - \zeta)\right)s\left(\mu_2 + \frac{1}{m}(\delta'' - \zeta)\right)\ldots s\left(\mu_m + \frac{1}{m}(\delta'' - \zeta)\right)$$
$$\leqq (1+\varepsilon)t\left(\lambda_1 - \frac{1}{n}\zeta\right)t\left(\lambda_2 - \frac{1}{n}\zeta\right)\ldots t\left(\lambda_n - \frac{1}{n}\zeta\right).$$

Passing to the limit $\zeta \gtrapprox 0$, we obtain

$$f(\lambda_1 + \lambda_2 + \ldots + \lambda_n) \leqq s(\lambda_1 + \lambda_2 + \ldots + \lambda_n + \delta')$$
$$\leqq (1+\varepsilon)g(\lambda_1)g(\lambda_2)\ldots g(\lambda_n)$$

and

$$f(\mu_1)f(\mu_2)\ldots f(\mu_m) \leqq s\left(\mu_1 + \frac{1}{m}\delta''\right)s\left(\mu_2 + \frac{1}{m}\delta''\right)\ldots s\left(\mu_m + \frac{1}{m}\delta''\right)$$
$$\leqq (1+\varepsilon)g(\lambda_1)g(\lambda_2)\ldots g(\lambda_n),$$

because s is increasing and continuous from the left (Lemma 4.2 (ii)). As ε was arbitrary, this proves (i).

(ii) If $\mu_1, \mu_2, \ldots, \mu_m \in \mathrm{sp}(\bar{\alpha}) = \sigma(H)$, then $g(\mu_j) \leqq f(\mu_j)$ for all $j = 1, 2, \ldots, m$ by

Lemma 4.4, hence (ii) follows immediately from (i). □

Now we are ready to prove Theorem 1.3. We suppose as before that ω is completely spectrally passive but not a ground state. Let Γ denote the subgroup of $\mathbb{R}$ generated by $\mathrm{sp}(\bar{\alpha})$. Define a function γ on Γ with positive real values by

$$\gamma(\lambda_1 + \lambda_2 + \ldots + \lambda_n) = g(\lambda_1)g(\lambda_2)\ldots g(\lambda_n),$$

where $\lambda_j \in \mathrm{sp}(\bar{\alpha})$ for $j = 1, 2, \ldots, n$. Then γ is well-defined and increasing by Lemma 4.7 (ii) and $f|_\Gamma \leqq \gamma$ by the first part of Lemma 4.7 (i). Moreover, as g never vanishes (Lemma 4.6 (i)), γ is by its very definition a homomorphism of Γ into the multiplicative group $\mathbb{R}_0^+$ of the nonzero positive real numbers.

Now we make a distinction between two possible cases:

(a) Γ has no smallest positive element, i.e. Γ is dense in $\mathbb{R}$. As γ is an increasing homomorphism of Γ into $\mathbb{R}_0^+$, it extends to a continuous increasing homomorphism on the whole of $\mathbb{R}$. Hence there exists $\beta \geqq 0$ such that $\gamma(\lambda) = e^{\beta\lambda}, \lambda \in \Gamma$. The inequalities $s(\lambda) \leqq f(\lambda) \leqq \gamma(\lambda) (\lambda \in \Gamma)$ imply $s(\lambda) \leqq e^{\beta\lambda}$ for all $\lambda \in \mathbb{R}$, by the left continuity of s. By the definition of s and Theorem 1.1 (and Remark 2.1 (i)), we conclude that ω is β-KMS with respect to α.

(b) Γ has a smallest positive element $\mu > 0$, i.e. $\Gamma = \mathbb{Z}\mu$ (notice that $\Gamma = \{0\}$ corresponds to the ground state case). Then again $\gamma(\lambda) = e^{\beta\lambda}$ for all $\lambda \in \Gamma$ (where now $\beta = \mu^{-1} \log \gamma(\mu)$. To show that $s(\lambda) \leqq e^{\beta\lambda}$ for all real λ, however, we need a last lemma (a slight modification of Lemma 4.2 (iii)):

Lemma 4.8. *Let ω be an α-invariant state on $\mathfrak{A}$. If $\lambda < \mu$, $(\lambda, \mu] \cap \mathrm{sp}(\bar{\alpha}) = \emptyset$, then $f(\lambda) = f(\mu)$.*

Proof. As $\mathrm{sp}(\bar{\alpha})$ is closed, there exists $\varepsilon > 0$ such that $(\lambda, \mu + \varepsilon) \cap \mathrm{sp}(\bar{\alpha}) = \emptyset$. For all real δ such that $0 < \delta \leqq \mu - \lambda$, Lemma 4.2 (iii) implies $s(\lambda + \delta) = s(\mu + \varepsilon)$. In the limit $\delta \underset{\rightarrow}{>} 0$ this yields $f(\lambda) = s(\mu + \varepsilon)$. The desired conclusion follows from the inequalities $f(\lambda) \leqq f(\mu) \leqq s(\mu + \varepsilon)$. □

Let us now finish the proof of Theorem 1.3 by showing that $s(\lambda) \leqq e^{\beta\lambda}$ for all $\lambda \in \mathbb{R}$ under assumption (b) above. This is clear for $\lambda \in \Gamma$. If $n\mu < \lambda < (n+1)\mu$ for some $n \in \mathbb{Z}$, we have

$$s(\lambda) \leqq f(\lambda) = f(n\mu) \leqq \gamma(n\mu) = e^{\beta n\mu} \leqq e^{\beta\lambda},$$

where the first equality follows from Lemma 4.8, since $\mathrm{sp}(\bar{\alpha}) \subset \mathbb{Z}\mu$. Hence we are done.

Example 4.9. In order to show that one really needs *complete* spectral passivity in Theorem 1.3, we give an example of a C^*-dynamical system $(\mathfrak{A}, \mathbb{R}, \alpha)$ and an α-invariant state ω on $\mathfrak{A}$ such that ω is m-spectrally passive for all $m \leqq n$, but not $(n+1)$-spectrally passive (where $n \in \mathbb{N}_0$ is arbitrary).

In fact, let $\mathfrak{A}$ be the algebra of complex $q \times q$ matrices ("finite quantum spin system"), and $e_{jk} (1 \leqq j, k \leqq q)$ the usual matrix units. We define α by $\alpha_t(x) = e^{ith}\, ix\, e^{-ith}$, where $h = \sum_{j=1}^{q} \mu_j e_{jj}$ and μ_j $(1 \leqq j \leqq q)$ are real numbers. It is immediate that $x=\sum_{j,k=1}^{q} x_{jk}e_{jk}$ $(x_{jk}\in\mathbb{C})$ belongs to $R(-\infty,\lambda)$ if and only if $x_{jk}=0$ whenever $\mu_j-\mu_k\geqq\lambda$.

Let τ be the trace on $\mathfrak{A}$ (i.e. $\tau(e_{jk})=\delta_{jk}$). Any state ω on $\mathfrak{A}$ is given by a density matrix ρ with the properties $\rho\geqq 0$, $\tau(\rho)=1$, $\omega(x)=\tau(\rho x)$ for all $x\in\mathfrak{A}$. If ω is α-invariant, ρ and h commute. Hence we may suppose

$$\rho=\sum_{j=1}^{q}\rho_j e_{jj},\quad \rho_j\geqq 0\quad (1\leqq j\leqq q),\quad \sum_{j=1}^{q}\rho_j=1$$

(performing a unitary transformation in $\mathbb{C}^q$ if necessary).
Since

$$\omega(x^*x)=\sum_{h,k=1}^{q}\rho_k|x_{jk}|^2,$$

$$\omega(xx^*)=\sum_{j,k=1}^{q}\rho_j|x_{jk}|^2,$$

if $x=\sum_{j,k=1}^{q}x_{jk}e_{jk}$, one finds that ω is spectrally passive with respect to α if and only if

$$\mu_j-\mu_k<0\Rightarrow\rho_k\leqq\rho_j \tag{4.6}$$

(cf. [7, p. 579]).

Similarly it is easy to show that m-spectral passivity of ω is equivalent with the following property: for all choices of integers $j_l, k_l\in\{1,2,\dots,q\}$, with $1\leqq l\leqq m$, such that $\sum_{l=1}^{m}(\mu_{j_l}-\mu_{k_l})<0$, one has

$$\rho_{k_1}\rho_{k_2}\cdots\rho_{k_m}\leqq\rho_{j_1}\rho_{j_2}\cdots\rho_{j_m} \tag{4.7}$$

(cf. [7, p. 583]).

To construct our example we fix a nonzero natural number n and put $q=3$. We choose

$$\mu_1=0,\quad \mu_2=1,\quad \mu_3=n+\frac{1}{2}, \tag{4.8}$$

and require

$$\rho_1=\rho_2>\rho_3. \tag{4.9}$$

Then ω is m-spectrally passive with respect to α for all $m\leqq n$. The reason is that if

$$\sum_{l=1}^{m}\mu_{j_l}<\sum_{l=1}^{m}\mu_{k_l},$$

then because of our choice (4.8) the term μ_3 has to appear at least as often in the right hand side as in the left hand side (as long as $m\leqq n$). By (4.9) it follows that

$$\rho_{k_1}\rho_{k_2}\cdots\rho_{k_m}\leqq\rho_{l_1}\rho_{l_2}\cdots\rho_{l_m},$$

because the factor ρ_3 appears at least as often in the left hand side as in the right hand side. On the other hand ω is *not* $(n+1)$-spectrally passive, since $n\mu_1 + \mu_3 < (n+1)\mu_2$ but $\rho_2^{n+1} > \rho_1^n \rho_3$, by (4.8) and (4.9).

There is an important case, however, where complete spectral passivity is implied by spectral passivity, viz. the case where ω is weakly clustering.

Definition 4.10. Let $(\mathfrak{A}, G, \gamma)$ be a C^*-dynamical system, and let ω be a γ-invariant state on $\mathfrak{A}$. Then ω is called *G-weakly clustering* if for all $x, y \in \mathfrak{A}$

$$\inf |\omega(xy') - \omega(x)\omega(y)| = 0$$

when y' runs over the convex hull of $\{\gamma_g(y) | g \in G\}$ in $\mathfrak{A}$.

Theorem 4.11. *Let $(\mathfrak{A}, \mathbb{R}, \alpha)$ be a C^*-dynamical system, and let G be a locally compact group acting continuously on $\mathfrak{A}$ by *-automorphisms $\gamma_g(g \in G)$, which all commute with the $\alpha_t(t \in \mathbb{R})$. Let the γ-invariant state ω on $\mathfrak{A}$ be spectrally passive with respect to α and G-weakly clustering. Then ω is completely spectrally passive with respect to α (and hence a* KMS *state or a ground state).*

Proof. We shall show that under the above assumptions n-spectral passivity implies $(n+1)$-spectral passivity. The hypothesis of spectral passivity therefore provides a basis for induction on n. First, however, we have to introduce some notation and establish a few preliminary facts.

As ω is γ-invariant, we can perform the constructions of Sect. 1 with respect to γ instead of α: we define the unitary operators V_g on $\mathscr{H}$ and the *-automorphisms $\bar{\gamma}_g$ on $\mathscr{A}$ by

$$V_g \pi(x)\Omega = \pi(\gamma_g(x))\Omega$$

and

$$\bar{\gamma}_g(A) = V_g A V_{g^{-1}} \quad (g \in G, x \in \mathfrak{A}, A \in \mathscr{A}).$$

Also (as in the proof of Lemma 4.6 (ii)) let Q denote the orthogonal projection on the closure of $\mathscr{A}'\Omega$. Finally, we use E for the orthogonal projection onto the space of V-invariant vectors in $\mathscr{H}$.

By the mean ergodic theorem [2, Prop. 4.3.4], E is the strong operator limit of a net $\left\{ \sum_{k=1}^{n_i} c_{ik} V_{g_{ik}} | i \in I \right\}$, where $g_{ik} \in G$, $0 < c_{ik} \leqq 1$ and $\sum_{k=1}^{n_i} c_{ik} = 1$ for all $i \in I$ and $1 \leqq k \leqq n_i$. As is well known, the assumption that ω is G-weakly clustering is equivalent with the fact that E is the projection onto the space of scalar multiples of Ω [2, Theorem 4.3.22]. It follows immediately that for all $A \in \mathscr{A}$,

$$\lim \sum_{k=1}^{n_i} c_{ik} \bar{\gamma}_{g_{ik}}(A) Q = \lim \sum_{k=1}^{n_i} c_{ik} \bar{\gamma}_{g_{ik}^{-1}}(A) Q = (\Omega, A\Omega) Q$$

in the weak operator sense, and hence that

$$\lim \sum_{k=1}^{n_i} c_{ik} \| \bar{\gamma}_{g_{ik}}(A) Q B \Omega \|^2 = \lim \sum_{k=1}^{n_i} c_{ik} \| \bar{\gamma}_{g_{ik}^{-1}}(A) Q B \Omega \|^2 = \| A\Omega \|^2 \| QB\Omega \|^2 \quad (4.10)$$

for all $A, B \in \mathscr{A}$.

To conclude these introductory remarks, suppose that ω is spectrally passive.

Then by Lemma 1.4 (i) we have $\|B\Omega\| \leqq \|B^*\Omega\|$ for all $B \in \bar{R}(-\infty, 0)$. In particular, since Q is $\bar{\alpha}$-invariant,

$$\|(1-Q)B\Omega\| \leqq \|B^*(1-Q)\Omega\| = 0,$$

or

$$QB\Omega = B\Omega \quad \text{for all } B \in \bar{R}(-\infty, 0). \tag{4.11}$$

Now we prove the induction step outlined in the beginning of the proof: we assume that ω is G-weakly clustering, spectrally passive and n-spectrally passive with respect to α. By Lemma 1.4 (i) the last assumption is equivalent with the following: if $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{R}$, $A_1, A_2, \ldots, A_n \in \mathscr{A}$ satisfy $\sum_{j=1}^{n} \lambda_j \leqq 0$ and $A_j \in \bar{R}(-\infty, \lambda_j)(j = 1, 2, \ldots, n)$, then

$$\|A_1\Omega\| \, \|A_2\Omega\| \ldots \|A_n\Omega\| \leqq \|A_1{}^*\Omega\| \, \|A_2{}^*\Omega\| \ldots \|A_n{}^*\Omega\|. \tag{4.12}$$

Suppose now $\mu_0, \mu_1, \ldots, \mu_n \in \mathbb{R}$, $\sum_{j=0}^{n} \mu_j \leqq 0$ and $x_j \in R(-\infty, \mu_j)$ $(j = 1, \ldots, n)$. Rearranging if necessary we may assume $\mu_0 \leqq 0$, Since $\bar{\gamma}$ and $\bar{\alpha}$ commute, and since Q is $\bar{\alpha}$-invariant, we have

$$\bar{\gamma}_g(\pi(x_1))Q\pi(x_0) \in \bar{R}(-\infty, \mu_0 + \mu_1)$$

for all $g \in G$ [8, Lemma 8.3.3 (iii)]. Hence by (4.12) (with $\lambda_1 = \mu_0 + \mu_1$, $\lambda_j = \mu_j$ for $j \geqq 2$) we obtain

$$\begin{aligned} &\|\bar{\gamma}_g(\pi(x_1))Q\pi(x_0)\Omega\|^2 \omega(x_2{}^*x_2)\omega(x_3{}^*x_3)\ldots\omega(x_n{}^*x_n) \\ &\quad \leqq \|\pi(x_0{}^*)Q\bar{\gamma}_g(\pi(x_1{}^*))\Omega\|^2 \omega(x_2x_2{}^*)\omega(x_3x_3{}^*)\ldots\omega(x_nx_n{}^*). \end{aligned} \tag{4.13}$$

On the other hand, with c_{ik} and g_{ik} as above,

$$\begin{aligned} \lim \sum_{k=1}^{n_i} c_{ik} \|\bar{\gamma}_{g_{ik}}(\pi(x_1))Q\pi(x_0)\Omega\|^2 &= \|\pi(x_1)\Omega\|^2 \, \|Q\pi(x_0)\Omega\|^2, && \text{by (4.10)} \\ &= \|\pi(x_1)\Omega\|^2 \, \|\pi(x_0)\Omega\|^2 \\ &= \omega(x_0{}^*x_0)\omega(x_1{}^*x_1), && \text{by (4.11)} \end{aligned}$$

as $\pi(x_0) \in \bar{R}(-\infty, 0)$. Moreover, using the fact that Q is also $\bar{\gamma}$-invariant, we have

$$\begin{aligned} \lim \sum_{k=1}^{n_i} c_{ik} \|\pi(x_0{}^*)Q\bar{\gamma}_{g_{ik}}(\pi(x_1{}^*))\Omega\|^2 &= \lim \sum_{k=1}^{n_i} c_{ik} \|\bar{\gamma}_{g_{ik}^{-1}}(\pi(x_0{}^*))Q\pi(x_1{}^*)\Omega\|^2 \\ &= \|\pi(x_0{}^*)\Omega\|^2 \, \|Q\pi(x_1{}^*)\Omega\|^2, && \text{by (4.10)} \\ &\leqq \|\pi(x_0{}^*)\Omega\|^2 \, \|\pi(x_1{}^*)\Omega\|^2 \\ &= \omega(x_0{}^*x_0)\omega(x_1{}^*x_1). \end{aligned}$$

Hence taking the limit of the convex combinations of the inequality (4.13) with $g = g_{ik}$ and coefficients c_{ik} as above, we obtain

$$\omega(x_0{}^*x_0)\omega(x_1{}^*x_1)\ldots\omega(x_n{}^*x_n) \leqq \omega(x_0x_0{}^*)\omega(x_1x_1{}^*)\ldots\omega(x_nx_n{}^*).$$

The conclusion is that ω is $(n+1)$-spectrally passive, thus completely passive by induction. □

5. Discussion

In [9], a state ω of a C^*-dynamical system $(\mathfrak{A}, \mathbb{R}, \alpha)$ is called *passive* if (roughly speaking) the system is unable to perform work in a cyclic process with initial state ω. As the precise definition of this concept [9, Definition 1.1] has been shown to entail the inequality (3.1) above, it follows from Theorem 3.3 that passivity implies spectral passivity. Hence our Theorem 4.11 provides a new proof of [9, Theorem 1.3]. Actually, an inspection of the proof of that theorem reveals that it still goes through if the assumption of passivity is replaced with (3.1) (see also [2, Theorem 5.3.22]).

A similar relationship exists between the notions of complete passivity [9, Definition 1.3] and complete spectral passivity on the one hand, and between [9, Theorem 1.4] and Theorem 1.3 above on the other hand. A state ω is said to be completely passive in [9] if all its tensor powers $\overset{n}{\otimes}\,\omega$ are passive states of the corresponding C^*-dynamical systems $\left(\overset{n}{\otimes}\,\mathfrak{A}, \mathbb{R}, \overset{n}{\otimes}\,\alpha\right)$ $(n \in \mathbb{N}_0)$. Likewise a state whose n^{th} tensor power is spectrally passive has to be n-spectrally passive; however the converse is not obvious *a priori*. One of the main benefits of our approach is indeed that we got rid of tensor products and of the cumbersome technicalities that inevitably go with them. It is also worthwhile to observe that our proofs are independent of the Tomita–Takesaki theory.

Finally, one should notice that our results generalize those obtained by A. Lenard for finite quantum spin systems [7]. In this case (i.e. when $\mathfrak{A}$ is a full matrix algebra) passivity and spectral passivity are equivalent. We do not know whether this holds in general.

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