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Asymptotic Behaviour of the Classical Scalar Fields and Topological Charges

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Abstract. The existence and the properties of the limit at spatial infinity are studied for the finite-energy scalar fields with respect to the topological charge introduction. The limit is shown to be constant in time and in almost all spatial directions. The proof of the existence of the limit given by Parenti, Strocchi and Velo is extended to two-dimensional space. A generalized definition of the topological charge is suggested for a σ-model as an example.

1. Introduction and Conclusions

The possibility of introducing conserved topological charges [1] widely used in the soliton and instanton physics depends substantially on the asymptotic behaviour of the fields at spatial infinity. In the present paper, the existence and the properties of the limit at spatial infinity for a system of classical scalar fields with a finite energy are discussed from this point of view.

We consider a system of real scalar fields $\varphi : \mathbb{R}^s \times \mathbb{R} \to \mathbb{R}^n$,

$$\varphi(x,t) = \begin{pmatrix} \varphi_1(x,t) \\ \vdots \\ \varphi_n(x,t) \end{pmatrix}, \qquad x = (x^1, \ldots, x^s)$$

with continuous first derivatives[1]. The Lagrangian of the system is assumed to be of the form

$$\mathscr{L} = \tfrac{1}{2}(\partial_\mu \varphi)(\partial^\mu \varphi) - U(\varphi)$$

leading to the field equations

$$\Box \varphi_j(x,t) + \frac{\partial}{\partial \varphi_j} U(\varphi(x,t)) = 0 \qquad (j = 1, \ldots, n)$$

1 The second derivatives of the fields and the first derivatives of the potential U appear in the field equations but most of the following statements are valid for the field configurations of finite energy regardless of field equations

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and the energy

$$E=\int_{\mathbb{R}^s}\left[\frac{1}{2}\left(\frac{\partial\varphi}{\partial t}\right)^2+\frac{1}{2}(\nabla\varphi)^2+U(\varphi)\right]d^sx\,.$$

The map $U:\mathbb{R}^n\to\mathbb{R}$ is assumed to be continuous[1]. We shall call it the potential. Let us denote M the set of zero-points of the potential.

The topological charges can be defined as numbers characterising the homotopy class of the function $\psi_t:\mathbb{S}^{s-1}\to M$ if there exists the limit

$$\psi_t(\xi)=\lim_{r\to+\infty}\varphi(r\xi,t)\in M \tag{1.1}$$

and if it is a continuous function of a unit vector $\xi\in\mathbb{S}^{s-1}$ and of time $t\in\mathbb{R}$.

Under very general assumptions, Parenti et al. [2] showed that for arbitrary $t\in\mathbb{R}$

$$\psi_t(\xi)=\psi_{t_0}(\xi)$$

at almost all $\xi\in\mathbb{S}^{s-1}$, if the limit (1.1) exists at an initial time t_0. Therefore, not only the homotopy class of ψ_t but even the limit (1.1) itself is conserved [if necessary, limit (1.1) is redefined on a set of zero measure at each time]. In [2], the limit ψ_t was defined as a dynamical charge of the field. The limit (1.1) was shown to exist in two cases: for the fields with

$$\nabla\varphi(x,t)\in L^2(\mathbb{R}^s)\,, \tag{1.2}$$

$$U(\varphi(x,t))\in L^1(\mathbb{R}^s) \tag{1.3}$$

if $s=n=1$, and for $\nabla\varphi(x,t)\in L^2(\mathbb{R}^s)$ if $s\geqq 3$. For $s=1$ and $n>1$, the weaker statement that the $\lim_{r\to+\infty}|\varphi(r\xi,t)|$ exists for almost all $\xi\in\mathbb{S}^{s-1}$ can be proved. We extend these results from the one-dimensional space ($s=1$) to the two-dimensional one ($s=2$) in Propositions 1 and 2 of Sect. 2. The original method of proofs for $s=1$ can be used for $s=2$ without substantial modifications.

The integrability conditions (1.2) and (1.3) are evidently valid for the fields with finite conserved energy if the potential $U(z)\geqq 0$ for $z\in\mathbb{R}^n$. In Sect. 2, we give a simple proof that the dynamical charge is conserved for such fields (Proposition 3). Under certain conditions on the potential, the energy conservation was proved for the solutions of the field equations in the completed space of smooth functions with compact support [2, 3]. It is a nontrivial assumption in general.

The definition of the topological charge as a homotopy class of the limit (1.1) is applicable to the finite-energy scalar fields in one-dimensional space because the limit (1.1) exists for both directions $\xi=\pm 1$ there. In the spaces of higher dimensions $s\geqq 2$, the finiteness of energy if insufficient for the existence of a continuous limit (1.1) for all the directions $\xi\in\mathbb{S}^{s-1}$. If the limit (1.1) differs from a continuous function on a set of zero measure in $\mathbb{S}^{s-1}$, we could still speak about the topological charge. We shall show (Proposition 4 of Sect. 2) that for $\nabla\varphi\in L^2(\mathbb{R}^s)$ the limit $\psi_t(\xi)$ (1.1) has the same value for almost all $\xi\in\mathbb{S}^{s-1}$ if $s\geqq 3$, and for all ξ at which it exists if $s=2$. The topological charge is then zero. The proof of this fact, formerly accepted by heuristic arguments [1] only, is our main result.

The fact that a large class of finite-energy (finite-action in the Euclidean case) fields has the limit at infinity which is constant in time and in all directions allows for another definition of the topological charges based on the compactification of the space by adding a "point at infinity" [4]. The Euclidean space $\mathbb{R}^s$ becomes topologically equivalent to the sphere $\mathbb{S}^s$ after the compactification. The field φ can be continuously extended on the compactified space if the limit $\psi_t(\xi)$ (1.1) exists uniformly with respect to the time t in every bounded interval and all $\xi \in \mathbb{S}^{s-1}$, and if it is constant in ξ. Let us assume now that the values of the field are constrained to some subspace $T \subset \mathbb{R}^n$ with a nontrivial homotopy group $\pi_s(T)$. Then the extended field defines a map $\mathbb{S}^s \to T$ and the topological charge can be defined as a number characterising the homotopy class of this map.

We propose a generalized definition of the topological charge applicable to a class of fields which cannot be continuously extended on the compactified space. The fields of this class can be approximated by a sequence of auxiliary fields with well defined topological charges. The (generalized) topological charge of the original field can be defined as a limit of the sequence of the topological charges for auxiliary fields, if this limit exists and if it is independent of the choice of auxiliary fields. We shall discuss the generalized definition of the topological charges for a σ-model in Sect. 3, but the general approach can be used for other models as well.

2. Limits at Spatial Infinity for Fields of Finite Energy

We prove several propositions on the existence and properties of the limits of the fields at spatial infinity in this section. The following two propositions are the generalizations of Theorems C.1 and C.2 of [2] to the case $s>1$. Stronger results are known for $s \geqq 3$ (Lemma 6 of [2]) and for all s if $U(z)=z^2$ (Lemma 5 of [2]).

Proposition 1. *Let M be the set of zero-points of continuous function $U:\mathbb{R}\to\mathbb{R}$, and*

$$\int_{-\infty}^{z_0} |U(z)|^{1/2}\,dz = \int_{z_0}^{+\infty} |U(z)|^{1/2}\,dz = +\infty \tag{2.1}$$

for some $z_0 \in \mathbb{R}$. If M is a discrete nonempty set and $\varphi:\mathbb{R}^s\to\mathbb{R}$ is a function with continuous first derivatives such that

$$\int_{\mathbb{R}^s} \left[\tfrac{1}{2}(\nabla\varphi(x))^2 + |U(\varphi(x))|\right] d^s x < +\infty\,, \tag{2.2}$$

then there exists

$$\lim_{r\to+\infty} \varphi(r\xi) \in M$$

for almost all unit vectors $\xi \in \mathbb{S}^{s-1}$ (for $\xi = \pm 1$ if $s=1$). If $M=\emptyset$, then no function $\varphi:\mathbb{R}^s\to\mathbb{R}$ with continuous first derivatives satisfying (2.2) *exists.*

Proof. Let M be at most a discrete set and φ be a function satisfying (2.2). Then

$$\int_1^{+\infty} \left[\frac{1}{2}\left(\frac{\partial\varphi(r\xi)}{\partial r}\right)^2 + |U(\varphi(r\xi))|\right] dr < +\infty$$

for almost all $\xi \in \mathbb{S}^{s-1}$, and we can repeat the proof of Theorem C.1 from [2] keeping ξ fixed.

Proposition 2. *Let $U : \mathbb{R}^n \to \mathbb{R}$ be a continuous function, let*

$$F(\varrho) = \left[\min_{|z|=\varrho} |U(z)|\right]^{1/2},$$

for $\varrho \geqq 0$, where $|z| = \left(\sum_{j=1}^{n} z_j^2\right)^{1/2}$, let M be the set of zero-points of the function F, and let

$$\int_0^{+\infty} F(\varrho)\, d\varrho = +\infty . \tag{2.3}$$

If M is a discrete nonempty set and $\varphi : \mathbb{R}^s \to \mathbb{R}^n$ is a map with continuous first derivatives satisfying the relation (2.2), *then there exists*

$$\lim_{r\to+\infty} |\varphi(r\xi)| \in M$$

for almost all $\xi \in \mathbb{S}^{s-1}$ (for $\xi = \pm 1$ if $s=1$). If $M = \emptyset$, then no map $\varphi : \mathbb{R}^s \to \mathbb{R}^n$ with continuous first derivatives satisfying (2.2) *exists.*

The proof is similar to the proof of Proposition 1.

The next proposition is fundamental for the introduction of dynamical and topological charges of scalar fields since it gives sufficient conditions for their conservation. The proposition states that the field with finite conserved energy [if $U(z) \geqq 0$] remains in one Hilbert space sector in the sense of [2]. We offer a simple proof of this fact for those who take the energy conservation as a physical requirement.

Proposition 3. *Let $\varphi : \mathbb{R}^s \times \mathbb{R} \to \mathbb{R}^n$ be a map with continuous first derivatives and*

$$\int_{\mathbb{R}^s} \left[\left(\frac{\partial \varphi(x,t)}{\partial t}\right)^2 + (\nabla\varphi(x,t))^2\right] d^s x < C \tag{2.4}$$

for a finite constant C and all $t \in \mathbb{R}$. Then for all $t, t_0 \in \mathbb{R}$

$$\int_{\mathbb{R}^s} [\varphi(x,t) - \varphi(x,t_0)]^2\, d^s x < +\infty \tag{2.5}$$

and

$$\lim_{r\to+\infty} [\varphi(r\xi,t) - \varphi(r\xi,t_0)] = 0 \tag{2.6}$$

for almost all $\xi \in \mathbb{S}^{s-1}$ (for $\xi = \pm 1$ if $s=1$). Especially if the limit $\lim_{r\to+\infty} \varphi(r\xi,t_0)$ or $\lim_{r\to+\infty} |\varphi(r\xi,t_0)|$ exists at some $t_0 \in \mathbb{R}$ for almost all $\xi \in \mathbb{S}^{s-1}$, then at all $t \in \mathbb{R}$

$$\lim_{r\to+\infty} \varphi(r\xi,t) = \lim_{r\to+\infty} \varphi(r\xi,t_0)$$

or

$$\lim_{r\to+\infty} |\varphi(r\xi, t)| = \lim_{r\to+\infty} |\varphi(r\xi, t_0)|$$

for almost all $\xi \in \mathbb{S}^{s-1}$.

Proof. By Schwarz inequality

$$|\varphi(x,t)-\varphi(x,t_0)| = \left|\int_{t_0}^{t} \frac{\partial\varphi(x,\tau)}{\partial\tau} d\tau\right| \leqq \left|\int_{t_0}^{t} \left(\frac{\partial\varphi}{\partial\tau}\right)^2 d\tau\right|^{1/2} |t-t_0|^{1/2}.$$

Now, according to (2.4),

$$\int_{\mathbb{R}^s} [\varphi(x,t)-\varphi(x,t_0)]^2 \, d^s x \leqq C(t-t_0)^2$$

and (2.5) is proved. Equation (2.6) then follows, e.g., from Proposition 2 applied to the map $\varphi(x,t)-\varphi(x,t_0)$ and the function $U(z)=z^2$.

Remark. In Proposition 3, we gave the condition (2.4) evidently valid for the fields of finite conserved energy if the potential is non-negative. Weaker assumptions

$$\left|\int_{t_0}^{t} \int_{\mathbb{R}^s} \left(\frac{\partial\varphi(x,\tau)}{\partial\tau}\right)^2 d^s x \, d\tau\right| < +\infty$$

and

$$\int_{\mathbb{R}^s} [\nabla\varphi(x,t)-\nabla\varphi(x,t_0)]^2 \, d^s x < +\infty$$

were used in the proof.

We gave the sufficient conditions for the existence of the limit of the field for almost all directions at spatial infinity. Now we show that this limit has the same value at almost all directions for $s \geqq 2$.

Proposition 4. *Let* $\varphi: \mathbb{R}^s \to \mathbb{R}^n$ $(s \geqq 2)$ *be a map with continuous first derivatives,* $\nabla\varphi \in L^2(\mathbb{R}^s)$. *If* $s \geqq 3$, *then there exists a constant* $a \in \mathbb{R}^n$ *such that*

$$\lim_{r\to+\infty} \varphi(r\xi) = a \tag{2.7}$$

for almost all $\xi \in \mathbb{S}^{s-1}$. *If* $s=2$, *then the limit* (2.7) $\left(\text{or } \lim_{r\to+\infty} |\varphi(r\xi)|\right)$ *has the same (finite or infinite) value at all* $\xi \in \mathbb{S}^1$ *for which it exists.*

Proof. It is sufficient to give the proof for $n=1$ and then to apply the result to each component of φ separately $\left(\text{the proof for } \lim_{r\to+\infty} |\varphi(r\xi)| \text{ in the case } s=2 \text{ is also similar}\right)$. Let us start with the simpler case $s=2$. If there exist

$$\lim_{r\to+\infty} \varphi(r\xi_1) < \lim_{r\to+\infty} \varphi(r\xi_2)$$

for some $\xi_1, \xi_2 \in \mathbb{S}^1$ with polar angles $\omega_1 \neq \omega_2$, then there exist $r_0 \geqq 0$ and numbers $b, c \in \mathbb{R}$ such that

$$\varphi(r\xi_1) < b < c < \varphi(r\xi_2)$$

for all $r > r_0$. Integrating over polar angle ω, we have now for all $r > r_0$

$$\int_0^{2\pi} (\nabla\varphi)^2 \, d\omega \geqq \frac{1}{r^2} \left| \int_{\omega_1}^{\omega_2} \left(\frac{\partial\varphi}{\partial\omega} \right)^2 d\omega \right| > \frac{(c-b)^2}{r^2 |\omega_2 - \omega_1|}$$

because of Schwarz inequality

$$|\varphi(r\xi_2) - \varphi(r\xi_1)| = \left| \int_{\omega_1}^{\omega_2} \frac{\partial\varphi}{\partial\omega} d\omega \right| \leqq |\omega_2 - \omega_1|^{1/2} \left| \int_{\omega_1}^{\omega_2} \left(\frac{\partial\varphi}{\partial\omega} \right)^2 d\omega \right|^{1/2}.$$

The contradiction with the assumption $\int (\nabla\varphi)^2 r \, dr \, d\omega < +\infty$ is thus obtained.

The same arguments are used for $s \geqq 3$, but they are applicable only after some preliminary considerations. Let us first realize that the finite limit in the left-hand side of Eq. (2.7) exists for almost all $\xi \in \mathbb{S}^{s-1}$ according to Lemma 6 of [2] and that it is a measurable function of ξ as a limit of measurable functions. It can be seen now that either the statement (2.7) is valid or there exist numbers $b < c$ such that the sets

$$N = \left\{ \xi \in \mathbb{S}^{s-1} \,\middle|\, \lim_{r \to +\infty} \varphi(r\xi) < b \right\},$$
$$L = \left\{ \xi \in \mathbb{S}^{s-1} \,\middle|\, \lim_{r \to +\infty} \varphi(r\xi) > c \right\}$$

have positive measures in $\mathbb{S}^{s-1}$. Let us assume that the last possibility takes place. The set N can be rotated to the position $\mathcal{R}N$ where it has an intersection of nonzero measure with the set L:

$$\mu(L \cap \mathcal{R}N) > 0,$$

where μ is Lebesgue measure on $\mathbb{S}^{s-1}$ and $\mathcal{R} \in SO(s)$ is a suitable rotation. The existence of $\mathcal{R}$ is proved in Appendix. We can choose such an orthogonal system of coordinates that the matrix $\mathcal{R}$ has a canonical form [5]

$$\mathcal{R} = \begin{pmatrix} \cos\alpha_1 & -\sin\alpha_1 & & & & & \\ \sin\alpha_1 & \cos\alpha_1 & & & & & \\ & & \ddots & & & & \\ & & & \cos\alpha_k & -\sin\alpha_k & & \\ & & & \sin\alpha_k & \cos\alpha_k & & \\ & & & & & 1 & \\ & & & & & & \ddots \\ & & & & & & & 1 \end{pmatrix},$$

where $0 < |\alpha_l| \leqq \pi$ for $l = 1, \ldots, k$.

Let us introduce in $\mathbb{R}^s$ a system of coordinates similar to the cylindrical and spherical ones by the equations:

$$
\begin{aligned}
&x^1 = \varrho_1 \cos\omega_1\,, \qquad x^2 = \varrho_1 \sin\omega_1\,,\\
&\ldots\\
&x^{2k-1} = \varrho_k \cos\omega_k\,, \qquad x^{2k} = \varrho_k \sin\omega_k\,,\\
&\left.\begin{aligned}
&\varrho_1 = r\sin\vartheta_1 \ldots \sin\vartheta_{s-k-2}\sin\vartheta_{s-k-1}\,,\\
&\varrho_2 = r\sin\vartheta_1 \ldots \sin\vartheta_{s-k-2}\cos\vartheta_{s-k-1}\,,\\
&\ldots\\
&\varrho_k = r\sin\vartheta_1 \ldots \sin\vartheta_{s-2k}\quad \cos\vartheta_{s-2k+1}\,,
\end{aligned}\right\}\ \text{valid for } k>1 \text{ only}\\
&x^{2k+1} = r\sin\vartheta_1 \ldots \sin\vartheta_{s-2k-1}\cos\vartheta_{s-2k}\,,\\
&\ldots\\
&x^s = r\cos\vartheta_1\,,
\end{aligned}
$$

where $r \geqq 0$, $0 \leqq \vartheta_i \leqq \pi$ $(i=1,\ldots,s-2k)$, $0 \leqq \vartheta_j \leqq \frac{\pi}{2}$ $(j=s-2k+1,\ldots,s-k-1)$, $0 \leqq \omega_l \leqq 2\pi$ $(l=1,\ldots,k)$. For $k=1$ (this is certainly the case for $s=3$) we obtain the usual spherical coordinates. In the following, all real values of ω_l are allowed (or the formulas should be understood as valid by $\bmod 2\pi$). The rotation by matrix $\mathscr{R}$ is the transformation

$$\omega_l \mapsto \omega_l + \alpha_l \qquad (l=1,\ldots,k)\,. \tag{2.8}$$

Let us denote

$$\delta = \frac{1}{r}\left[\sum_{l=1}^{k} \varrho_l^2 \alpha_l^2\right]^{1/2} > 0$$

(if at least one $\varrho_l > 0$). There exists an orthogonal matrix $\mathcal{O}$ (dependent on angles ϑ) of dimension $k \times k$ such that

$$\frac{1}{r}\begin{pmatrix} \varrho_1\alpha_1 \\ \varrho_2\alpha_2 \\ \vdots \\ \varrho_k\alpha_k \end{pmatrix} = \mathcal{O}\begin{pmatrix} \delta \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

We shall use the variables ω_l' defined by the equation

$$\begin{pmatrix} \omega_1' \\ \vdots \\ \omega_k' \end{pmatrix} = \frac{1}{r}\mathcal{O}^{-1}\begin{pmatrix} \varrho_1\omega_1 \\ \vdots \\ \varrho_k\omega_k \end{pmatrix}$$

instead of ω_l. The transformation (2.8) becomes a translation in one variable

$$\omega_1' \mapsto \omega_1' + \delta\,.$$

Let us denote

$$N_1 = \mathscr{R}^{-1}(L \cap \mathscr{R}N)\,.$$

Now

$$N_1 \subset N, \quad \mathscr{R} N_1 \subset L, \quad \mu(N_1) > 0.$$

There exist $\omega_{10}, \ldots, \omega_{k0}$ such that $\mu(N_2) > 0$, where N_2 is the part of N_1 contained in the region described by the inequalities

$$|\omega_l - \omega_{l0}| < \tfrac{\pi}{2} \quad (l = 1, \ldots, k).$$

There exists ω'_{10} such that

$$\mu_{s-2}(N_2^*) = \int_{N_2^*} \sigma(\vartheta)\, d^{s-k-1}\vartheta\, d\omega'_2 \ldots d\omega'_k > 0,$$

where

$$N_2^* = \{(\vartheta_1, \ldots, \vartheta_{s-k-1}, \omega'_2, \ldots, \omega'_k) \mid \xi(\vartheta_1, \ldots, \vartheta_{s-k-1}, \omega'_{10}, \omega'_2, \ldots, \omega'_k) \in N_2\}$$

[$\xi(\vartheta, \omega')$ is the unit vector described by the variables ϑ, ω'] and

$$\sigma(\vartheta) = \sin^{s-k-2}\vartheta_1 \ldots \sin\vartheta_{s-k-2}.$$

Using our coordinates, we have

$$\int_{\mathbb{R}^s} (\nabla\varphi)^2\, d^s x \geq \int_{N_2^*} \left\{ \int_0^\infty \left[\int_{\omega'_{10}}^{\omega'_{10}+\delta} \left(\frac{\partial \varphi}{\partial \omega'_1} \right)^2 d\omega'_1 \right] r^{s-3}\, dr \right\} \sigma(\vartheta)\, d^{s-k-1}\vartheta\, d\omega'_2 \ldots d\omega'_k.$$

It can be shown that the last integral is infinite similarly as in the case $s=2$. Proposition 4 is completely proved now.

3. Generalized Topological Charges for a σ-Model

We introduce generalized definition of the topological charges for a σ-model with simple homotopy properties. Let us consider a field of unit vectors with $n = s+1$ components on the s-dimensional space. The field satisfies the constraint

$$|\varphi(x, t)| = 1.$$

The Lagrangian of the model is

$$\mathscr{L} = \tfrac{1}{2}(\partial_\mu \varphi)(\partial^\mu \varphi)$$

and the corresponding Hamiltonian is

$$\mathscr{H} = \frac{1}{2}\left(\frac{\partial \varphi}{\partial t}\right)^2 + \frac{1}{2}(\nabla\varphi)^2. \tag{3.1}$$

The smooth field with constant uniform limit

$$\psi_t(\xi) = \lim_{r \to +\infty} \varphi(r\xi, t) \tag{3.2}$$

can be extended on the space $\mathbb{R}^s$ compactified by adding a "point at infinity" (topologically equivalent to the sphere $\mathbb{S}^s$) and defines a smooth map $\mathbb{S}^s \to \mathbb{S}^s$. The topological charge defined as the degree of mapping is then expressed by the formula

$$\deg(\varphi) = \frac{1}{\mu(\mathbb{S}^s)} \int_{\mathbb{R}^s} \det(\varphi)\, d^s x\,, \tag{3.3}$$

where $\mu(\mathbb{S}^s)$ is the Lebesgue measure (surface) of the unit sphere $\mathbb{S}^s$ and we denote

$$\det(\varphi) = \begin{vmatrix} \varphi_1 & \dots & \varphi_{s+1} \\ \frac{\partial \varphi_1}{\partial x^1} & \dots & \frac{\partial \varphi_{s+1}}{\partial x^1} \\ \dots & \dots & \dots \\ \frac{\partial \varphi_1}{\partial x^s} & \dots & \frac{\partial \varphi_{s+1}}{\partial x^s} \end{vmatrix}.$$

The finiteness of energy implies the existence of the constant limit (3.2) for almost all directions in the space of dimension $s \geqq 3$ according to the results of Sect. 2 and the form of the Hamiltonian (3.1). It is therefore desirable to generalize the definition of the topological charge to some fields without constant uniform limit (3.2) at all directions. We study this possibility for the fields which can be approximated by a sequence of auxiliary fields having the necessary limit (3.2).

Let us consider a field $\varphi : \mathbb{R}^s \times \mathbb{R} \to \mathbb{S}^s$ for which the following assumptions are valid. There is a system of open intervals in $\mathbb{R}$ covering $\mathbb{R}$. For every interval I from this system there exists a sequence of fields $\varphi_m : \mathbb{R}^s \times I \to \mathbb{S}^s$ $(m = 1, 2, \dots)$ with continuous first derivatives and the following properties:

(i) the constant $\lim_{r \to +\infty} \varphi_m(r\xi, t) = a$ exists uniformly with respect to $\xi \in \mathbb{S}^{s-1}$ and $t \in I$ for every $m = 1, 2, \dots$;

(ii) at all $t \in I$

$$\lim_{m \to \infty} \varphi_m(x, t) = \varphi(x, t)\,, \qquad \lim_{m \to \infty} \nabla \varphi_m(x, t) = \nabla \varphi(x, t)$$

for almost all $x \in \mathbb{R}^s$;

(iii) at all $t \in I$ there exists a function $g_t \in L^s(\mathbb{R}^s)$ such that

$$|\nabla \varphi_m(x, t)| \leqq g_t(x)$$

for almost all $x \in \mathbb{R}^s$ and all $m = 1, 2, \dots$.

Then there exists

$$\lim_{m \to \infty} \deg(\varphi_m) = \deg(\varphi)\,,$$

where $\deg(\varphi)$ is given by Eq. (3.3). It is a conserved integer number as a limit of such numbers[2], independent of the choice of the sequence $\{\varphi_m\}_{m=1}^{\infty}$ of the properties (i)–(iii). The number $\deg(\varphi)$ can therefore be defined as a generalized topological charge of the field φ although it may not always have a simple interpretation as the degree of a mapping.

It should be stressed that the existence of the sequence $\{\varphi_m\}_{m=1}^{\infty}$ is our assumption and we did not study the conditions warranting it. The sequence can

2 $\deg(\varphi_m)$ are conserved in time intervals I, $\deg(\varphi)$ is conserved for $t \in \mathbb{R}$ consequently

be easily constructed if $s \geqq 2$ and if the field φ has the following properties: φ has continuous first derivatives; $\nabla\varphi \in L^s(\mathbb{R}^s)$; there exists a constant $b \in \mathbb{S}^s$ and for any bounded interval $I \subset \mathbb{R}$ there exist positive numbers A_I, ε_I such that $|\varphi(x,t) - b| > \varepsilon_I$ for $|x| > A_I$ and $t \in I$. The maps φ_m are obtained from φ by modification of its behaviour (redefinition) at large $|x|$ to obtain a map constant outside a compact subset (ball specifically) of $\mathbb{R}^s$. More details of this construction are given in [7].

The assumptions (i)–(iii) can be somewhat varied, e.g. the uniformity of the limit at $r \to +\infty$ with respect to t can be replaced by the independence of the majorant g_t on t or directly by continuity of $\deg(\varphi)$ expressed by Eq. (3.3) in t. The fields φ_m do not need to satisfy the field equations. They even do not need to be continuous in time, supposing $\deg(\varphi)$ is continuous. However, the generalized solutions of the field equations φ defined by a sequence of smooth solutions φ_m might be of interest.

Appendix

We show the existence of the rotation $\mathscr{R}$ used in the proof of Proposition 4.

Lemma 1. *Let G be a locally compact Lie group of differentiable transformations transitive on the finite dimensional differentiable manifold S, μ be a measure on S invariant with respect to the transformations of G, $0 < \mu(S) < +\infty$, ν be the right invariant measure on the group G. Then for every function f integrable on S and every point $\xi \in S$*

$$\int_G f(\mathscr{R}\xi)\, d\nu(\mathscr{R}) = \frac{\nu(G)}{\mu(S)} \int_S f(\eta)\, d\mu(\eta).$$

Proof. By the substitution $\eta' = \mathscr{R}\eta$, $\mathscr{R}' = \mathscr{R}$ we obtain

$$\iint_{S \times G} f(\mathscr{R}\eta)\, d\mu(\eta)\, d\nu(\mathscr{R}) = \nu(G) \int_S f(\eta')\, d\mu(\eta')$$

(expressing integrals as ones over the parameters of S and G, the same Jacobian appears as in the equation

$$\int_S f(\mathscr{R}\eta)\, d\mu(\eta) = \int_S f(\eta')\, d\mu(\eta')$$

valid by the assumption; theorems on homogeneous spaces can also be used here [6]). Since G is transitive on S, to all $\xi, \eta \in S$, there exists a transformation $\mathscr{R}_{\xi\eta} \in G$ such that $\eta = \mathscr{R}_{\xi\eta}\xi$. Using the fact that ν is the right invariant measure on G, we obtain

$$\begin{aligned}\iint_{S \times G} f(\mathscr{R}\eta)\, d\mu(\eta)\, d\nu(\mathscr{R}) &= \int_S \int_G f(\mathscr{R}\mathscr{R}_{\xi\eta}\xi)\, d\nu(\mathscr{R})\, d\mu(\eta) \\ &= \mu(S) \int_G f(\mathscr{R}\xi)\, d\nu(\mathscr{R}).\end{aligned}$$

By comparison of the two expressions for the double integral, the statement of Lemma 1 follows.

Lemma 2. *Let the assumptions of Lemma 1 be valid,* $\nu(G)>0$, $L\subset S$, $N\subset S$, $\mu(L)>0$, $\mu(N)>0$. *Then there exists a transformation* $\mathscr{R}\in G$ *such that*

$$\mu(L\cap\mathscr{R}N)>0.$$

Proof. Let us denote by $f, g, \chi_{\mathscr{T}}$ the characteristic functions of the sets N, L, $L\cap\mathscr{T}^{-1}N$ for every $\mathscr{T}\in G$. Then

$$\chi_{\mathscr{T}}(\xi)=g(\xi)\,f(\mathscr{T}\xi)$$

for $\xi\in S$ and

$$\iint\limits_{S\times G}\chi_{\mathscr{T}}(\xi)\,d\mu(\xi)\,d\nu(\mathscr{T})=\frac{\nu(G)}{\mu(S)}\mu(L)\,\mu(N)>0$$

by Lemma 1. Therefore there exists $\mathscr{T}\in G$ such that

$$\mu(L\cap\mathscr{T}^{-1}N)=\int\limits_{S}\chi_{\mathscr{T}}(\xi)\,d\mu(\xi)>0$$

and it is sufficient to put $\mathscr{R}=\mathscr{T}^{-1}$.

In the proof of Proposition 4 we use Lemma 2 for $S=\mathbb{S}^{s-1}$ and $G=\mathrm{SO}(s)$.

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