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Some Twisted Self-Dual Solutions for the Yang-Mills Equations on a Hypertorus*

Gerard 't Hooft**

California Institute of Technology, Pasadena, CA 91125, USA

Abstract. The $SU(N)$ Yang-Mills equations are considered in a four-dimensional Euclidean box with periodic boundary conditions (hypertorus). Gauge-invariant twists can be introduced in these boundary conditions, to be labeled with integers $n_{\mu\nu}(=-n_{\nu\mu})$, defined modulo N. The Pontryagin number in this space is often fractional. Whenever this number is zero there are solutions to the equations $G_{\mu\nu}=0$. Here $G_{\mu\nu}$ is the covariant curl. When this number is not zero we find a set of solutions to the equations $G_{\mu\nu}=\tilde{G}_{\mu\nu}$, provided that the periods a_μ of the box satisfy certain relations.

1. Introduction

Understanding quantized gauge theories in the strong-interaction region is made difficult by severe infrared divergences. It is therefore useful to consider gauge models enclosed in a box with sides of variable lengths. As for the boundary conditions at the sides periodic boundary conditions are the most natural choice [1]. Indeed, computer simulations have been made of gauge theories in such boxes and taught us much about their phase structure [2].

After having dealt with the vacuum in the box one may consider studying some of the first excited states, such as those corresponding to a hadronic particle trapped in the box. But it is perhaps of more fundamental importance to look at a trapped amount of electric or magnetic flux in the box. The first of these would correspond to a string connecting two opposite sides of the box. (This is the string which in the infinite volume limit is believed to confine quarks inside hadrons.) The energy of such a state corresponds directly to the string constant. In [1] it is explained how this state is described in terms of field configurations in a box where the periodic boundary conditions have

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** On leave from the Institute for Theoretical Physics, University of Utrecht, P.O. Box 80.006, NL-3508 TA Utrecht, The Netherlands

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been "twisted." One must temporarily introduce also periodic boundary conditions in the Euclidean time direction and then the string state is obtained by handling twists in the space-time direction in a certain way. We will explain in Sect. 2 how the twist is defined and how it is related to the center $Z(N)$ in the case that the gauge group is SU(N). We will see that the twist is labeled by six integers $n_{\mu\nu}$ defined modulo N.

Field configurations with any twist are not difficult to write down, and it was soon realized that they do not always have integer Pontryagin number [1]:

$$\frac{g^2}{16\pi^2}\int_{\text{box}} \operatorname{Tr} G_{\mu\nu}\tilde{G}_{\mu\nu}\, d^4x = \nu - \frac{\kappa}{N}, \tag{1.1}$$

where ν is any integer and

$$\kappa = \tfrac{1}{4} n_{\mu\nu}\tilde{n}_{\mu\nu} = n_{12}n_{34} + n_{13}n_{42} + n_{14}n_{23}, \tag{1.2}$$

is also integer.

If κ is not divisible by N then the total action of the field configurations with twists $n_{\mu\nu}$ is bounded:

$$\tfrac{1}{4}\int G^a_{\mu\nu}G^a_{\mu\nu} \equiv \tfrac{1}{2}\int \operatorname{Tr} G_{\mu\nu}G_{\mu\nu} \geqq \left|\tfrac{1}{2}\int \operatorname{Tr} G\tilde{G}\right| \geqq \min_{\nu} \frac{8\pi^2}{g^2}\left|\nu - \frac{\kappa}{N}\right|. \tag{1.3}$$

The question is: can we always saturate the bound, or equivalently, is there a solution to the equation

$$G_{\mu\nu} = \tfrac{1}{2}\varepsilon_{\mu\nu\alpha\beta}G_{\alpha\beta} = \tilde{G}_{\mu\nu}, \tag{1.4}$$

for all integer values of ν and $n_{\mu\nu}$? In particular if only the space-space components of $n_{\mu\nu}$ are non-zero, then $\kappa=0$ and there should be a solution of $G_{\mu\nu}=0$. Indeed, such field configurations were found [3] and that marks a difference between the non-Abelian SU(N) theory and the Abelian theory, because in the latter all twists must carry a finite amount of action. In Sect. 3 we show in a simple way, that whenever $\nu-\kappa/N=0$ there is such a field configuration.

Next, if $n_{12}=n_{34}=1$ and all other $n_{\mu\nu}$ vanish, then the lower bound (1.3) amounts to $8\pi^2/g^2N$. Saturating this bound means that we have non-trivial field configurations surviving in the usual $N\to\infty$ limit, because in that limit g^2N is kept fixed [4]. We show in Sect. 4 how to construct configurations with such an action. All our solutions will be represented in a suitably chosen gauge that makes them look essentially translationally invariant and Abelian. However, considering the difficulty we had in finding them it looked worth-while to publish the result.

2. The Boundary Conditions

We have four coordinates x_μ with

$$0 \leqq x_\mu \leqq a_\mu, \qquad \mu = 1, \ldots, 4, \tag{2.1}$$

and in this space we have a vector field

$$A_\mu(x),$$

which for every x and μ is a Hermitian traceless matrix with N rows and columns. The coupling constant g occurring in the introduction will from now on be put equal to one, since it is irrelevant for our discussion. For discussing the boundary conditions we need a short-hand notation for functions defined on the walls of the box:

$$f(x_1=a_1) \quad \text{stands for } f(a_1,x_2,x_3,x_4), \tag{2.2}$$

etc. Actually, on the four walls $x_\mu=a_\mu$ we have unitary matrices $\Omega_\mu(x_\mu=a_\mu)$. Also, we will write

$$\Omega_\mu A_\lambda \quad \text{instead of } \Omega_\mu\left(A_\lambda - i\frac{\partial}{\partial x_\lambda}\right)\Omega_\mu^{-1}, \tag{2.3}$$

where $\frac{\partial}{\partial x_\mu}\Omega_\mu$ is defined to be zero for each μ.

In each μ direction we now require the boundary condition to be

$$A_\lambda(x_\mu=a_\mu)=\Omega_\mu A_\lambda(x_\mu=0), \tag{2.4}$$

which just means that we have periodicity modulo gauge transformations.

When functional integrals are considered in this system we must keep the Ω_μ fixed, but vary the fields $A_\mu(x)$. This implies that at the corners of the box the conditions (2.4) must not give rise to mutual incompatibilities for any choice of A_μ. For example,

$$\begin{aligned} A_\lambda(x_1=a_1,x_2=a_2) &= \Omega_1(x_2=a_2)\,\Omega_2(x_1=0)\,A_\lambda(x_1=x_2=0) \\ &= \Omega_2(x_1=a_1)\,\Omega_1(x_2=0)\,A_\lambda(x_1=x_2=0), \end{aligned} \tag{2.5}$$

and the two different transformations on any chosen $A_\lambda(x_1=x_2=0)$ should give the same result. Therefore, the periodicity conditions on Ω are

$$\Omega_1(x_2=a_2)\,\Omega_2(x_1=0)=\Omega_2(x_1=a_1)\,\Omega_1(x_2=0)\,Z_{12}, \tag{2.6}$$

where Z_{12} is one member of the center group $Z(N)$ of SU(N). A similar condition holds in all pairs of directions μ, ν. We write

$$Z_{\mu\nu}=\exp(2\pi i\, n_{\mu\nu}/N); \qquad n_{\mu\nu}=-n_{\nu\mu}. \tag{2.7}$$

Clearly, these exponents cancel out in (2.3). Because of continuity, the $n_{\mu\nu}$ must be x-independent. Because of that, we find no further periodicity conditions.

What happens when we perform gauge transformations? Consider an arbitrary continuous and differentiable gauge transformation $\Omega(x)$. Then,

$$A_\mu \to \Omega A_\mu, \tag{2.8a}$$

$$\Omega_\mu \to \Omega(x_\mu=a_\mu)\,\Omega_\mu\,\Omega^{-1}(x_\mu=0). \tag{2.8b}$$

This allows us to transform any set of Ω_μ into any other set except that the $Z_{\mu\nu}$ in (2.6) and the numbers $n_{\mu\nu}$ remain the same. Therefore, if we wish to find a field configuration with a given twist combination, we may just pick any convenient set of Ω_μ satisfying (2.6). It is as good as any other. Indeed, most of the structure of the solutions described in the following sections will be absorbed in the Ω_μ, so that they look rather complicated. The fields $A_\mu(x)$ on the other hand are very simple.

3. Zero Pontryagin Number

Now we will look for zero-action solutions for the case that

$$\tfrac{1}{4}\, n_{\mu\nu}\tilde{n}_{\mu\nu}=0 \pmod N). \tag{3.1}$$

We will show that the solution of [3] is explicitly possible but we must assume that N is not divisible by a prime number squared. One asks for a solution of

$$G_{\mu\nu}=0, \tag{3.2}$$

where

$$G_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu+i[A_\mu, A_\nu]. \tag{3.3}$$

Equation (3.2) implies that there must be a gauge transformation $\Omega(x)$ that transforms all A_μ to zero. This Ω will transform the Ω_μ into some special set. Now from the boundary condition (2.4) and (2.3) one easily reads off that the new Ω_μ must be completely x-independent. The boundary condition (2.6) for Ω_μ now becomes simply

$$\Omega_\mu\,\Omega_\nu=\Omega_\nu\,\Omega_\mu\exp(2\pi i\, n_{\mu\nu}/N). \tag{3.4}$$

We will now show how to construct these Ω_μ for any set of $n_{\mu\nu}$ satisfying (3.1). We then have the solution of (3.2) in the gauge where $A_\mu=0$, and the Ω_μ are constant. If the original Ω_μ were not constant it is a trivial exercise to find a gauge transformation $\Omega(x)$ that satisfies (2.8b).

So how do we solve Eq. (3.4)? Let us define the SU(N) matrices P and Q as follows:

$$P=\begin{bmatrix} 0 & 1 & & & 0\\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1\\ 1 & & & & 0\end{bmatrix};\qquad Q=e^{\frac{\pi i(1-N)}{N}}\begin{bmatrix} 1 & & & 0\\ & e^{2\pi i/N} & & \\ & & \ddots & \\ 0 & & & e^{2\pi i(N-1)/N}\end{bmatrix}, \tag{3.5}$$

so that they satisfy

$$PQ=QP\exp(2\pi i/N). \tag{3.6}$$

Theorem. *Whenever* $n_{\mu\nu}$ *satisfies* (3.1) *and* N *is not divisible by a prime number squared, there is a set of integer numbers* s_μ, t_μ *such that*

$$\Omega_\mu = P^{s_\mu} Q^{t_\mu} \tag{3.7}$$

satisfy (3.4).

Proof. Equation (3.4) for these Ω reduces to

$$n_{\mu\nu} = s_\mu t_\nu - t_\mu s_\nu \pmod N. \tag{3.8}$$

We must therefore show that one can solve (3.8) for any set of $n_{\mu\nu}$ that satisfies (3.1). That is, for any $n_{\mu\nu}$ with

$$\tfrac{1}{8}\varepsilon^{\mu\nu\alpha\beta} n_{\mu\nu} n_{\alpha\beta} = 0 \pmod N. \tag{3.9}$$

The most elegant way to prove this is by observing that Eq. (3.8) and (3.9) have a large invariance group, namely $SL(4, Z(N))$. This is the group of 4×4 matrices with coefficients in $Z(N)$ and determinant one. Using these transformations it is easy to bring $n_{\mu\nu}$ in a standard form:

$$n_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & n_{14} \\ 0 & 0 & n_{23} & n_{24} \\ 0 & -n_{23} & 0 & 0 \\ -n_{14} & -n_{24} & 0 & 0 \end{pmatrix}. \tag{3.10}$$

To see (3.10) it is sufficient to consider $n_{\mu\nu}$ as a combination of one covariant and one contravariant three-vector in $SL(3, Z(N))$, which can be brought in a standard form by applying successive $SL(2, Z(N))$ rotations. If N is a prime then Eq. (3.9) corresponds to $n_{14}=0$. In that case the solution of (3.8) has become trivial:

$$s_\mu = (0,1,0,0);\ t_\mu = (0,0,n_{23},n_{24}). \tag{3.11}$$

If N is the product of *different* primes $P_1 \dots P_k$ then we first solve (3.8) for $N = P_1, P_2$, etc. and then combine the results.

4. Field Configurations with Non-zero Pontryagin Number

Some field configurations with non-vanishing Pontryagin number are easy to construct. Those easy configurations are obtained by assuming the fields $A_\mu(x)$ and the matrices $\Omega_\mu(x)$ all to correspond to the same Abelian subgroup of $SU(N)$. But then the total action, if non-trivial, never descends below $8\pi^2(N-1)/g^2 N$. If we want a configuration with non-trivial twist and an action decreasing as $1/g^2 N$ for large N, then we must search for a non-Abelian solution. We will now describe a successful approach.

We choose two positive integers k and ℓ, such that $k+\ell=N$, and split all rows and columns of the matrices in two parts. We work in the subgroup $SU(k)\otimes SU(\ell)\otimes U(1) \subset SU(N)$. Let ω be the traceless matrix

$$\omega = 2\pi \left[\begin{array}{c|c} \begin{matrix} \ell & & \\ & \ddots & \\ & & \ell \end{matrix} & 0 \\ \hline 0 & \begin{matrix} -k & & \\ & \ddots & \\ & & -k \end{matrix} \end{array}\right], \tag{4.1}$$

corresponding to the $U(1)$ generator. Then we define $P_{1,2}$ and $Q_{1,2}$ as in Eq. (3.5) but now acting in the two subgroups SU(k) and SU(ℓ). Their commutation rules are

$$\begin{aligned} P_1 Q_1 &= Q_1 P_1 \exp(2\pi i/N + \omega i/N k), \\ P_2 Q_2 &= Q_2 P_2 \exp(2\pi i/N - \omega i/N \ell), \end{aligned} \tag{4.2}$$

and all other pairs commute:

$$[P_1 Q_2] = 0, \text{ etc.} \tag{4.3}$$

In making an ansatz we now must realize that the Ω_μ must have on the one hand an explicit x-dependence and on the other hand satisfy simple commutation rules. We try

$$\Omega_\mu(x) = P_1^{s_\mu} Q_1^{t_\mu} P_2^{u_\mu} Q_2^{v_\mu} \exp(i\omega\alpha_{\mu\lambda} x_\lambda/a_\lambda), \tag{4.4}$$

summed only over λ in the exponent. The numbers s_μ, t_μ, u_μ, and v_μ are as yet arbitrary integers and $\alpha_{\mu\nu}$ any real matrix with vanishing diagonal components: $\alpha_{\mu\mu} = 0$ for all μ.

Inserting the twisted boundary condition (2.6) and using (4.2) we find

$$\begin{aligned} &\left(\frac{2\pi i}{N} + \frac{\omega i}{N k}\right)(s_\mu t_\nu - s_\nu t_\mu) + \left(\frac{2\pi i}{N} - \frac{\omega i}{N \ell}\right)(u_\mu v_\nu - u_\nu v_\mu) \\ &\quad + i\omega\alpha_{\mu\nu} = i\omega\alpha_{\nu\mu} + 2\pi i n_{\mu\nu}/N, \qquad \text{modulo } 2\pi i. \end{aligned} \tag{4.5}$$

This equation only contains the matrix ω and the identity matrix in $U(N)$, so we really get for each (μ, ν) two equations, one at the upper elements and one at the lower elements of the matrix ω:

$$\begin{aligned} &\frac{1}{k}(s_\mu t_\nu - s_\nu t_\mu) + \ell(\alpha_{\mu\nu} - \alpha_{\nu\mu}) = n_{\mu\nu}/N + A_{\mu\nu}, \\ &\frac{1}{\ell}(u_\mu v_\nu - u_\nu v_\mu) - k(\alpha_{\mu\nu} = \alpha_{\nu\mu}) = n_{\mu\nu}/N + B_{\mu\nu}, \end{aligned} \tag{4.6}$$

where $A_{\mu\nu}$ and $B_{\mu\nu}$ are integers. We can rewrite this as

$$n_{\mu\nu} = n_{\mu\nu}^{(1)} + n_{\mu\nu}^{(2)}, \tag{4.7a}$$

$$\alpha_{\mu\nu} - \alpha_{\nu\mu} = n_{\mu\nu}^{(2)}/N\ell - n_{\mu\nu}^{(1)}/N k, \tag{4.7b}$$

$$n^{(1)}_{\mu\nu}=s_\mu t_\nu - s_\nu t_\mu + k A_{\mu\nu}, \tag{4.8a}$$

$$n^{(2)}_{\mu\nu}=u_\mu v_\nu - u_\nu v_\mu + \ell B_{\mu\nu}. \tag{4.8b}$$

Let us assume that neither k nor ℓ are divisible by a prime squared. From the theorem of the previous sector we derive that given any set of $n^{(1)}_{\mu\nu}$, $n^{(2)}_{\mu\nu}$, with

$$n^{(1)}_{\mu\nu}\,\tilde{n}^{(1)}_{\mu\nu}=0 \pmod{k}$$

and

$$n^{(2)}_{\mu\nu}\,\tilde{n}^{(2)}_{\mu\nu}=0 \pmod{\ell}, \tag{4.9}$$

one can find numbers A, B, s, t, u, v that satisfy (4.8).

Our ansatz for the vector field is

$$A_\mu(x)=\omega\, B_\mu(x), \tag{4.10}$$

where B_μ is just a real vector field. The boundary condition is then

$$A_\lambda(x_\mu=a_\mu)=A_\lambda(x_\mu=0)-\omega\,\alpha_{\mu\lambda}/a_\lambda. \tag{4.11}$$

A solution to the second-order field equations is certainly

$$A_\lambda(x)=-\omega\sum_\mu \alpha_{\mu\lambda}\, x_\mu/a_\mu\, a_\lambda, \tag{4.12}$$

$$G_{\mu\nu}=-\omega(\alpha_{\mu\nu}-\alpha_{\nu\mu})/a_\mu\, a_\nu. \tag{4.13}$$

Let us check the index theorem:

$$\begin{aligned}\operatorname{Tr} G_{\mu\nu}\,\tilde{G}_{\mu\nu}&=\frac{1}{V}\operatorname{Tr}\omega^2\cdot 2\varepsilon_{\mu\nu\alpha\beta}\,\alpha_{\mu\nu}\,\alpha_{\alpha\beta}\\ &=-\frac{16\pi^2}{V}\left(\frac{n_{\mu\nu}\tilde{n}_{\mu\nu}}{4N}+\text{integer}\right),\end{aligned} \tag{4.14}$$

where $V=\prod_\mu a_\mu$, and (4.7) and (4.9) were used. Notice that if s, t, u, and v were chosen to vanish then the index would always contain a factor $\operatorname{Tr}\omega^2/N^2 = k\ell/N$ instead of $1/N$. It is now also obvious that without loss of generality we could have taken $\alpha_{\mu\nu}$ antisymmetric.

Now our last point. We are interested in self dual (or anti-self dual) solutions:

$$G_{\mu\nu}=\pm\tilde{G}_{\mu\nu}. \tag{4.15}$$

This implies an equation between the coefficients $\alpha_{\mu\nu}$ and the periods a_μ: if

$$\beta_{\mu\nu}=\frac{\alpha_{\mu\nu}-\alpha_{\nu\mu}}{a_\mu\, a_\nu},$$

then

$$\beta_{\mu\nu}=\pm\tilde{\beta}_{\mu\nu}. \tag{4.16}$$

Since the $\alpha_{\mu\nu}$ are constrained to be some simple rational numbers we find that some or all ratios between the a_μ are restricted to simple rational numbers.

Let us consider more closely the case $-n_{12}=n_{34}=1$, rest$=0$. Let us search for solutions with

$$\operatorname{Tr} G_{\mu\nu} G_{\mu\nu} = \operatorname{Tr} G_{\mu\nu} \tilde{G}_{\mu\nu} = \frac{16\pi^2}{VN}, \tag{4.17}$$

that is, we have to choose $\beta_{\mu\nu}$ as small as possible. Therefore,

$$\begin{aligned} A_{\mu\nu} &= B_{\mu\nu} = 0, \\ s_1 &= -1, \\ t_2 &= u_3 = v_4 = 1, \\ \text{rest} &= 0. \end{aligned} \tag{4.18}$$

Note that here we could drop the restriction on N. We have

$$\begin{aligned} \beta_{12} &= 1/N\, k\, a_1 a_2 \\ \beta_{34} &= 1/N\, \ell\, a_3 a_4. \end{aligned}$$

Therefore we have a solution if

$$\frac{a_1 a_2}{a_3 a_4} = \frac{\ell}{k} = \frac{N-k}{k}. \tag{4.19}$$

This restriction is typical for this kind of solution. If more twists are non-zero we get more constraints on the periods a_μ. If all $n_{\mu\nu} \neq 0$ then the squares of all ratios are restricted to similar rational numbers.

5. Conclusion

We found solutions with non-vanishing Pontryagin number only if the ratios of the sides a_μ of the box satisfy certain relations, containing simple rational numbers. We have not gone through the labor of finding the most complete generalization of these solutions (for instance one might suggest to split N into three or more integers rather than two). More important is the observation that they exist. Many solutions will be gauge transformations of each other, but it is impossible to transform a solution with one set (k, ℓ) into one with different k and ℓ, because the eigenvalues of $G_{\mu\nu}$ are different. If solutions exist for all a_μ then it seems to us that the functional dependence on a_μ will be complicated.

It is important that we have solutions with total action decreasing as $1/g^2N$ for $N \to \infty$, so that they will certainly survive in the usual $N \to \infty$ limit [4]. The implications of that for the $N \to \infty$ theory are however not clear to the author.

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