A Note on the Power Spectrum of the Iterates of Feigenbaum's Function[*]

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Abstract. We give rigorous bounds on the scaling behaviour of the power spectrum for Feigenbaum's map.

Recently, estimates have been given concerning the scaling behaviour of the power spectrum for the Feigenbaum map, Feigenbaum [1, 2], Nauenberg and Rudnick [3]. Since these estimates are based on somewhat heuristic arguments, it might be useful to give *rigorous bounds* on this scaling behavior. The purpose of this note is to give such bounds.

We denote by φ the Feigenbaum function, defined [4] by the relations

$$\varphi(\varphi(\lambda x)) = -\lambda \varphi(x), \quad \lambda = -\varphi(1), \quad \varphi(0) = 1, \tag{1}$$

φ analytic. For every diadic rational $q = k/2^N$, k odd, we define the amplitudes

$$a(q) = \lim_{p \to \infty} \frac{1}{p} \sum_{j=0}^{p-1} \exp(2\pi i \, kj/2^N) \, \varphi^j(0).^1 \tag{2}$$

(The limit exists since $\{\varphi^j(0)\}$ is almost periodic.) We define the averaged square amplitude A_N at level N as

$$A_N = \frac{1}{2^{N-1}} \sum_{k=0}^{2^{N-1}-1} |a((2k+1)/2^N)|^2. \tag{3}$$

Theorem. *For all $M > 0$, one has the bound*

$$\left(\frac{\lambda^2}{4}\right)^M \leq \frac{A_{N+M}}{A_N} < \left(\frac{\lambda^2}{4}\right)^M X Y^M,$$

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1 $\varphi^j(x) = \varphi(\varphi^{j-1}(x))$, $j > 0$, $\varphi^0(x) = x$

with $X < 14.9$, $Y < 1.109$. When $M = 1$, the upper bound can be replaced by $2.52 \cdot (\lambda^2/4)$.

We give analytic expressions for X, Y in the proof.

Remarks. Successive averaged square amplitudes decrease asymptotically by between 13.45 and 14 decibels (i.e. a factor F, $22.3 < F < 25.1$). This result should be compared with that of Feigenbaum [1, 2] who considers a family $\{\varphi_N\}$ of maps, φ_N having a stable period of period 2^N, $\lim_{N \to \infty} \varphi_N = \varphi$, and satisfying $\varphi_N(\varphi_N(\lambda x)) = -\lambda \varphi_{N-1}(x)$. By assuming the existence of a scaling function relating $\varphi_N^j(0) - \varphi_N^{j+2^{N-1}}(0)$ to $\varphi_{N+1}^j(0) - \varphi_{N+1}^{j+2^N}(0)$, he obtains a scaling relation between successive Fourier amplitudes $a_N(\cdot)$, $a_{N+1}(\cdot)$ which leads to a decrease of about 16.3 db for successive averaged square amplitudes. (We emphasize that our result is a statement about the function φ only.) Nauenberg and Rudnick [3] have investigated a sequence analogous to A_N defined above. They find a decrease of about 13.2 db both by numerical means, and by exploiting a self-similarity property in the shape of φ and its iterates.

Proof. Using the definitions (2), (3), it is easy to see that

$$A_N = \lim_{p \to \infty} \frac{1}{4^N p^2} \sum_{j=0}^{2^{N-1}-1} \left(\sum_{s=0}^{p-1} \varphi^{j+s2^N}(0) - \varphi^{j+s2^N+2^{N-1}}(0) \right)^2. \tag{4}$$

If we neglect all terms in (4) with odd j, and use the identity (1) then we see that

$$A_N \geq \frac{\lambda^2}{4} A_{N-1},$$

which is the lower bound in the theorem.

The upper bound is more delicate. In order to analyze the squares in (4), we first assert that, for fixed $j < 2^{N-1}$, the sign of $\varphi^{j+s2^N}(0) - \varphi^{j+s2^N+2^{N-1}}(0)$ is *independent* of s. To prove this, we observe first that φ^j has no zero on $(0, \lambda^{N-1})$ for $j = 1, ..., 2^{N-1} - 1$. For, if $0 < x < \lambda^{N-1}$ and if $\varphi^j(x) = 0$, we would find that $\varphi^{2^{N-1}}(\lambda^{N-1}t) = (-\lambda)^{N-1}\varphi(t)$ would have a zero derivative at $t \in (0, 1)$, which contradicts $\varphi'(t) \neq 0$ if $t \neq 0$. Using the chain rule of differentiation, we see that φ^j is monotone on $[0, \lambda^{N-1}]$. The assertion follows now from the identities

$$\varphi^{j+s2^N}(0) - \varphi^{j+s2^N+2^{N-1}}(0) = \varphi^j(\lambda^{N-1}\varphi^{2s}(0)) - \varphi^j(\lambda^{N-1}\varphi^{2s}(1)),$$

and $\varphi^{2s}(1) > |\varphi^{2s}(0)|$ for all $s \geq 0$, a general property of the solution to Eq. (1).

We fix now K and M, $K > M$ and compare A_K to A_M. We use the notation $\sum_j^L$ for $\sum_{j=0}^{2^L-1}$. We have, from (4),

$$A_K = \lim_{p \to \infty} \frac{1}{4^K p^2} \sum_k^M \sum_\ell^{K-M-1} \left(\sum_{s=0}^{p-1} \varphi^{k+\ell 2^M + s2^K}(0) - \varphi^{k+\ell 2^M + (s+\frac{1}{2})2^K}(0) \right)^2$$

$$= \lim_{p \to \infty} \frac{1}{4^K p^2} \sum_k^M \sum_\ell^{K-M-1} \left(\sum_{s=0}^{p-1} \left(\frac{d}{dx}\varphi^k \right)(z_{k,\ell,s}^{K,M}) \right. \tag{5}$$

$$\left. \cdot \left[\varphi^{\ell 2^M + s2^K}(0) - \varphi^{\ell 2^M + (s+\frac{1}{2})2^K}(0) \right] \right)^2,$$

where

$$z_{k,\ell,s}^{K,M} \in [\varphi^{\ell 2^M + s 2^K}(0); \varphi^{\ell 2^M + (s+\frac{1}{2}) 2^K}(0)]$$

$$= [\lambda^M \varphi^{\ell + s 2^{K-M}}(0); \lambda^M \varphi^{\ell + (s+\frac{1}{2}) 2^{K-M}}(0)], \tag{6}$$

and $[a; b]$ denotes the interval with endpoints a, b (any order). Using the constancy of sign discussed above, and (1), we can bound (5) by

$$A_K \leq \left(\frac{\lambda^2}{4}\right)^M D_M A_{K-M}, \tag{7}$$

where

$$D_M = \sum_k^M \sup_{\ell,s} \left| \left(\frac{d}{dx} \varphi^k\right) (z_{k,\ell,s}^{K,M}) \right|^2$$

$$\leq \sum_k^M \sup_{t \in [-1, 1]} \left| \left(\frac{d}{dx} \varphi^k\right) (\lambda^M t) \right|^2 =: \Delta'_M.$$

We have used (6) in the last inequality.

We shall now estimate Δ'_M. Using repeatedly Eq. (1) and the binary decomposition of k, $1 \leq k < 2^M$,

$$k = \sum_{j=0}^{r-1} 2^{\sum_{q=0}^{j} \sigma_q}, \qquad r = r(k),$$

$\sigma_q \geq 1$ if $q \geq 1$, $\sigma_0 \geq 0$, we find

$$\left| \left(\frac{d}{dx} \varphi^k\right) (\lambda^M t) \right| = \lambda^{-M} \left| \frac{d}{dt} \varphi^k (\lambda^M t) \right|$$

$$= \lambda^{-M} \left| \frac{d}{dt} \lambda^{\sigma_0} \varphi(\lambda^{\sigma_1} \dots \varphi(\lambda^{\sigma_r} t) \dots) \right|,$$

where $\sigma_0 + \dots + \sigma_r = M$, with $\sigma_r \geq 1$. By the chain rule of differentiation, we get

$$\left| \left(\frac{d}{dx} \varphi^k\right) (\lambda^M t) \right| = |\varphi'(\lambda^{\sigma_r} t)| \prod_{j=1}^{r-1} |\varphi'(\lambda^{\sigma_j} \varphi(\lambda^{\sigma_{j+1}} v_j))|$$

with $|v_j| \leq 1$. From the known coefficients of φ, see [5], [4] we conclude that sign $\varphi'''(x) = \text{sign } x$ if $|x| \leq \lambda$. This, and the concavity of φ imply $-\varphi''(0) = \sup_{|x| < \lambda} |\varphi''(x)|$. It follows that for $|v| \leq 1$,

$$|\varphi'(\lambda^\sigma \varphi(\lambda^\tau v))| \leq \gamma \lambda^\sigma |\varphi(\lambda^\tau)|,$$

where $\gamma = |\varphi''(0)| < 3.056$. It is now convenient to define b_σ as follows

$$b_\sigma = \begin{cases} \gamma^2 & \text{if } \sigma > 1 \\ \gamma^2 \varphi(\lambda)^2 & \text{if } \sigma = 1. \end{cases}$$

Note that $0.7589 < |\varphi(\lambda)| < 0.759$. Then we obtain

$$\left| \left(\frac{d}{dx} \varphi^k\right) (\lambda^M t) \right|^2 \leq \lambda^{2(M-\sigma_0)} \varphi(\lambda)^{-2} \prod_{j=1}^{r} b_{\sigma_j}, \tag{8}$$

and hence

$$D_M \leqq \Delta_M'$$

$$\leqq 1 + \varphi(\lambda)^{-2} \sum_{\substack{r=1}}^{M} \sum_{\substack{\sigma_0 + \ldots + \sigma_r = M \\ \sigma_i \geqq 1 \text{ if } i \geqq 1 \\ \sigma_0 \geqq 0}} \lambda^{2(M-\sigma_0)} \prod_{j=1}^{r} b_{\sigma_j}$$

$$= 1 + \varphi(\lambda)^{-2} \sum_{\sigma_0 = 0}^{M-1} \lambda^{2(M-\sigma_0)} \sum_{r=1}^{M-\sigma_0} \sum_{\substack{\sigma_1 + \ldots + \sigma_r = M - \sigma_0 \\ \sigma_i \geqq 1 \forall i}} \prod_{j=1}^{r} b_{\sigma_j} =: \Delta_M. \tag{9}$$

In order to estimate Δ_M we introduce

$$C_L = \sum_{r=1}^{L} \sum_{\substack{\sigma_1 + \ldots + \sigma_r = L \\ \sigma_i \geqq 1 \forall i}} \prod_{j=1}^{r} b_{\sigma_j}, \quad L > 0,$$

$C_0 = 1$, $C_L = 0$ if $L < O$. Then we have

$$C_{L+1} = \gamma^2 \varphi(\lambda)^2 C_L + \gamma^2 \sum_{j=1}^{\infty} C_{L-j}.$$

It follows that the generating function $F(x) = \sum_{j=0}^{\infty} C_j x^j$ equals

$$F(x) = \frac{1-x}{1 - x(1 + \gamma^2 \varphi(\lambda)^2) - x^2 \gamma^2 (1 - \varphi(\lambda)^2)}. \tag{10}$$

Thus

$$F(x) = \frac{\alpha_+}{x - x_+} + \frac{\alpha_-}{x - x_-}, \tag{11}$$

where $x_\pm$ are the roots of the denominator of (10), $x_+ > x_-$ and

$$\alpha_\pm = \frac{x_\pm - 1}{x_\pm - x_\mp} \cdot \frac{1}{\gamma^2 (1 - \varphi(\lambda))}.$$

We find from (11),

$$C_L = -\frac{\alpha_+}{x_+^{L+1}} - \frac{\alpha_-}{x_-^{L+1}}. \tag{12}$$

Substituting back into (9), we see that

$$D_M < \Delta_M$$

$$= 1 + \varphi(\lambda)^{-2} \sum_{\sigma_0 = 0}^{M-1} \lambda^{2(M-\sigma_0)} C_{M-\sigma_0} \tag{13}$$

$$= 1 - \frac{\lambda^2}{\varphi(\lambda)^2} \left\{ \frac{\alpha_+}{x_+^2 \left(\frac{\lambda^2}{x_+} - 1 \right)} \left(\left(\frac{\lambda^2}{x_+} \right)^M - 1 \right) + \frac{\alpha_-}{x_-^2 \left(\frac{\lambda^2}{x_-} - 1 \right)} \left(\left(\frac{\lambda^2}{x_-} \right)^M - 1 \right) \right\}.$$

Substituting the bounds for λ, $\varphi(\lambda)$ and $\varphi''(0)$, we see that

$$D_M < 1 - \frac{\lambda^2}{\varphi(\lambda)^2} \frac{\alpha_+}{x_+^2 \left(\dfrac{\lambda^2}{x_+} - 1\right)} \left(\frac{\lambda^2}{x_+}\right)^M$$

$$< 1 + 13.996 \cdot (1.10883)^M. \tag{14}$$

This combined with Eq. (7) proves the theorem. For small M, one can get better bounds by using (13) directly; e.g. $D_1 < 2.493$. A slight further improvement can be obtained by replacing the right hand side of (8) by $\lambda^{2(M - \sigma_0)} \gamma^2 \prod\limits_{j=2}^{r} b_{\sigma_j}$, yielding 12.31 instead of 13.996 in Eq. (14).

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