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Propagation of States in Dilation Analytic Potentials and Asymptotic Completeness

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Abstract. We estimate the space-time behavior of scattering states for two-body Schrödinger operators with smooth, dilation analytic potentials. We use our estimates to give a simple proof of asymptotic completeness for a class of long-range potentials, including the Coulomb potential plus a fairly general short-range perturbation.

Introduction

The goal of this paper is to present a simple proof of asymptotic completeness for the modified wave operators that describe two-body quantum scattering with certain long-range potentials. Modified wave operators were introduced by Dollard [6] to study scattering for the Coulomb potential. Spectral and scattering theory for general long-range potentials has since been studied by many authors. Spectral representations for such long-range Schrödinger operators have been studied by Ikebe [14, 15] and Saitō [31, 32]. Their results imply completeness of the stationary wave operators defined via the spectral representation. Isozaki [18] proved completeness of the stationary wave operator and Kitada [22–24] proved completeness of time-dependent modified wave operators by a stationary method. More recently Ikebe and Isozaki [16, 17] have also given a proof of completeness for the modified wave operators. Agmon [1] has also proved completeness results for Schrödinger operators with long-range potentials and Enss [8] has given a "geometric" proof of completeness for certain long-range potentials.

Here we would like to give a simple, "geometric" proof of completeness for Schrödinger operators $H_1 = H_0 + V + \bar{V}$ on $L^2(\mathbb{R}^n)$, where $H_0 = -\frac{1}{2}\Delta$, V is a long-range, dilation analytic potential, and $\bar{V}$ is a fairly general short range perturbation (we formulate precise hypotheses below). Our class of potentials thus includes the Coulomb potential plus a fairly general short-range perturbation. Our assumptions

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are more restrictive than those of the authors mentioned above in that we require the long-range potential to be dilation analytic, but on the other hand we can allow a more general short range part.

The modified wave operators we will study are given by:

$$\Omega_D^{\pm}(H_1,H_0) = \operatorname*{s-lim}_{t\to\mp\infty} e^{iH_1 t}\mathscr{U}_0(t,0), \tag{1}$$

where the "modified free evolution" $\mathscr{U}_0(t,s)$ is defined by

$$\mathscr{U}_0(t,s) = \exp - i\{H_0(t-s) + \int_s^t W'(\mathbf{p}\tau)\mathrm{d}\tau\}. \tag{2}$$

In (2) W' is a smooth function that closely approximates the long-range behavior of V (we choose W' in Proposition 1.2 below). $\mathbf{p}$ is the momentum operator and, with our choice of H_0, $\mathbf{p}=\mathbf{v}$, the velocity operator.

The class of dilation analytic potentials was introduced in [2]; see [4] for a characterization of dilation analytic potentials and [30] for discussion and further references. We denote by $\mathscr{U}(\theta)$ the group of dilations: $\mathscr{U}(\theta)$ acts on $L^2(\mathbb{R}^n)$ by $(\mathscr{U}(\theta)\psi)(x) = e^{n\theta/2}\psi(e^\theta x)$ for vectors $\psi \in L^2(\mathbb{R}^n)$. A symmetric, H_0-compact operator is a dilation analytic potential if the operator $\mathscr{U}(\theta)V\mathscr{U}(\theta)^{-1}(H_0+i)^{-1}$ extends to a bounded operator-valued analytic function of θ in some strip $S_\varphi = \{\theta: |\mathrm{Im}\,\theta| < \varphi\}$. We will assume that $0<\varphi<\pi/4$. If $H=H_0+V$ and V is dilation analytic, $H(\theta)=\mathscr{U}(\theta)H\mathscr{U}(\theta)^{-1}$ extends to an analytic family of type (A) in S_φ. In [2] this analyticity is used to prove, among other results, that H has no singular spectrum and that eigenvalues of H can accumulate only at 0.

We are now ready to state our result.

Theorem 1. *Let* $H_0 = -\frac{1}{2}\Delta$ *and* $H_1 = H_0 + V + \bar{V}$ *on* $L^2(\mathbb{R}^n)$. *Suppose that*:

(i) V *is dilation analytic in some strip* S_φ *and* $(1+|x|)^{1+\alpha}(\nabla V)(x)$ (*distributional derivative*) *is uniformly locally* L^2 *for some* $\alpha > \frac{1}{2}$.

(ii) $(H_1+i)^{-1} - (\tilde{H}+i)^{-1} \in \mathscr{I}_\infty$, *the ideal of compact operators, where* $\tilde{H} = H_0 + V$.

(iii) *For some integers* $\beta, \gamma \geqq 1$ *and some* $\varepsilon > 0$, *the bounded, monotone decreasing function* $\bar{h}(R) = \|(H_1+i)^{-\beta}\bar{V}(H_0+i)^{-\gamma}F(|x| \geqq R^{1-\varepsilon})\|$ *is integrable on* $(0,\infty)$. (*Here and elsewhere,* $F(x\in S)$ *denotes multiplication by the characteristic function of the set* S.)

Then the modified wave operators $\Omega_D^{\pm}(H_1,H_0)$ *exist and are complete*, i.e., $\mathrm{Ran}\,\Omega_D^+ = \mathrm{Ran}\,\Omega_D^- = \mathscr{H}_{\mathrm{a.c.}}(H_1)$ *and* H_1 *has empty singular spectrum. Eigenvalues of* H_1 *can accumulate only at* 0.

Remarks. 1. To treat the Coulomb potential $|x|^{-1}$, we write it as $(1+|x|)^{-1} + [|x|^{-1} - (1+|x|)^{-1}]$ and group the term in square brackets with the short range potential $\bar{V}$. We can similarly treat power potentials $|x|^{-\alpha}$ for $\alpha > \frac{1}{2}$. 2. The existence theory of modified wave operators with our choice (2) of modified free evolution [3,5] breaks down at $\alpha = \frac{1}{2}$, so our hypothesis (i) is necessary.

Below in Sect. 1, we will prove that the potential V in Theorem 1 can be written as $V = W + \bar{W}$ where W is C^∞ and dilation analytic, $|\nabla W(x)| \leqq C(1+|x|)^{-(1+\alpha)}$, and $\bar{W}$ is short-range (Proposition 1.1). Since W is smooth, the operator $H = H_0 + W$

has several domain properties which are technically convenient: we collect them in Proposition 1.3.

Given Proposition 1.1, it is very natural to break up the proof of completeness of $\Omega_D^{\pm}(H_1, H_0)$ into two steps: (1) prove that the ordinary wave operators $\Omega^{\pm}(H_1, H)$ exist and are complete, and (2) prove that the modified wave operators $\Omega_D^{\pm}(H, H_0)$ are complete.

The heart of our method is a "geometrical" estimate on the space-time behavior of scattering states propagating under $\exp(-iHt)$. To state it, let $D = \frac{1}{2}(\mathbf{x}\cdot\mathbf{p} + \mathbf{p}\cdot\mathbf{x})$ be the generator of dilations and let P_+ (respectively P_-) project onto the positive (respectively negative) spectral subspace for D. Let g be a smooth function with compact support in $(0, \infty)$ away from eigenvalues of H. In Sect. 2 we prove:

$$\| F(|x| \leqq |t|^{1-\varepsilon}) e^{-itH} g(H) P_{\pm} \| \leqq C_{N,\varepsilon}(1 + |t|)^{-N} \tag{3}$$

for any integer N, any $\varepsilon > 0$, and $\pm t \in (0, \infty)$. Estimate (3) is proved by extending Mourre's technique in [26], where he proves a similar estimate for $H = H_0 = -\frac{1}{2}\Delta$. "Local decay" estimates similar to our estimate (3) have been proven for certain dilation-analytic potentials in [19] and for a larger class of potentials in [20, 21]. These authors do not apply their estimates to prove asymptotic completeness. Together with any formulation of Enss's method [7] for short-range scattering, (3) immediately implies:

Theorem 2. *Let $H = H_0 + W$, where:*

(i)′ *W is dilation analytic in S_φ and C^∞ with bounded derivatives. Suppose H_1 is another self-adjoint operator so that*

(ii)′ $(H_1 + i)^{-1} - (H + i)^{-1} \in \mathscr{I}_\infty$.

(iii)′ *For some integers $\beta, \gamma \geqq 1$ and some $\varepsilon > 0$, the bounded, monotone decreasing function $h(R) = \|(H_1 + i)^{-\beta}(H_1 - H)(H + i)^{-\gamma} F(|x| \geqq R^{1-\varepsilon})\|$ is integrable on* $(0, \infty)$.

Then *$\Omega^{\pm}(H_1, H)$ exist and are complete*, i.e., $\operatorname{Ran}\Omega^+(H_1, H) = \operatorname{Ran}\Omega^-(H_1, H) = \mathscr{H}_{\text{a.c.}}(H_1)$ *and H_1 has empty singular spectrum. Eigenvalues of H_1 can accumulate only at* 0.

Remark. To show that (i)–(iii) of Theorem 1 $\Rightarrow$ (i)′–(iii)′ of Theorem 2 when $H_1 - H = \overline{W} + \overline{V}$, one uses Propositions 1.1 and 1.3(a). Proposition 1.3 (a) enters in showing that (iii)′ holds given (iii).

The next step:

Theorem 3. *Let $H = H_0 + W$, where:*

(i)″ W is C^∞ with bounded derivatives and dilation analytic in S_φ and $|\nabla W(x)| \leqq C(1 + |x|)^{-(1+\alpha)}$ for some $\alpha > \frac{1}{2}$.

Then the modified wave operators $\Omega_D^{\pm}(H, H_0)$ are complete, i.e., $\operatorname{Ran}\Omega_D^+ = \operatorname{Ran}\Omega_D^- = \mathscr{H}_{\text{a.c.}}(H)$.

To prove Theorem 3, we will prove directly that the inverse modified wave operators $\Omega_D^{\pm}(H, H_0)^*$ exist as strong limits. Just as the usual "Cook's method" proof for the existence of $\Omega_D^{\pm}$ depends on the asymptotic equality of $\mathbf{x}$ and $\mathbf{p}t$ under the free evolution $\exp(-itH_0)$, so our proof depends on the same fact with H_0

replaced by H. In Sect. 3 we combine the estimate (3) with ideas of Enss [9] to prove that $\mathbf{x}$ and $\mathbf{p}t$ are asymptotically equal under $\exp(-itH)$. We use this result, a result of Enss on the operator $D(t) = e^{iHt} D e^{-iHt}$ [9], and Mellin transform estimates [27] to prove Theorem 3 in Sect. 4.

In an Appendix, we prove a result on the invariance of operator domains used in Sect. 2.

1. Regularization of the Potential V

Proposition 1.1. *Let V satisfy hypothesis* (i) *of Theorem 1. Then $V = W + \bar{W}$, where*:
(a) *W is dilation analytic in S_φ and C^∞ with bounded derivatives,*
(b) $|\nabla W(x) \leq C(1+|x|)^{-(1+\alpha)}$, *and*
(c) $(1+|x|)^{(1+\alpha-\varepsilon)} \bar{W}(x)$ *is uniformly locally L^2 for any $\varepsilon > 0$.*

Remarks. 1. Conclusion (c) implies that $\|\bar{W}(\mathrm{H}_0 + i)^{-\gamma} F(|x| < R^{1-\eta})\|$ is an integrable function of R on $(0, \infty)$ for some $\eta > 0$ and γ large enough [33, Ex. 2.1]. 2. $\bar{W}$ is obviously H_0-compact since it is the difference of two H_0-compact operators.

Proof. We set

$$W(x) = (4\pi)^{-n/2} \int d^n y\, e^{-(x-y)^2/4} V(y) \tag{1.1}$$

(W is the Weierstrass transform of V; see [11, 25]). The integral in (1.1) converges absolutely since, by a result of Strichartz [34], any H_0-bounded multiplication operator is uniformly locally L^2. W is obviously C^∞ with bounded derivatives by the smoothness and decay of $\exp(-(x-y)^2/4)$. To see that W is dilation analytic, first note that W is H_0-compact. For, letting $C = V(H_0 + i)^{-1}$ and $T(y) =$ translation by y, we can write:

$$W(H_0 + i)^{-1} = (4\pi)^{-n/2} \int d^n y\, T(y)^{-1} C T(y) \exp(-y^2/4).$$

The integrand is compact and norm-continuous since C is compact and $T(y)$ is strongly continuous: since the integral converges in operator norm, we conclude that $W(H_0 + i)^{-1}$ is compact. Next note that, for real θ,

$$W(\theta)(x) \equiv W(e^\theta x) = (4\pi)^{-n/2} e^{-n\theta/2} \int d^n y\, V(e^\theta(x-y)) \exp\{-e^{-2\theta} y^2/4\},$$

so that as an operator (again $T(y)$ denotes translation by y):

$$W(\theta) = (4\pi)^{-n/2} e^{-n\theta/2} \int d^n y\, T(y)^{-1} V(\theta) T(y) \exp\{-e^{-2\theta} y^2/4\}.$$

Hence if $C(\theta) = V(\theta)(H_0 + i)^{-1}$,

$$W(\theta)(H_0 + i)^{-1} = (4\pi)^{-n/2} e^{-n\theta/2} \int d^n y\, T(y)^{-1} C(\theta) T(y) \exp\{-e^{-2\theta} y^2/4\}.$$

Now $C(\theta)$ is an analytic bounded operator valued function in S_φ and the kernel $\exp\{-e^{-2\theta} y^2/4\}$ is analytic in θ and rapidly decaying for $|\mathrm{Im}\,\theta| < \pi/4$, so the integral converges absolutely. The integrand is norm continuous and analytic in θ: hence $W(\theta)\,(H_0 + i)^{-1}$ extends to an analytic bounded operator valued function in S_φ. This proves (a).

To prove (b), we estimate:

$$|(1+|x|)^{(1+\alpha)}\nabla W(x)| \leqq (4\pi)^{-n/2}(1+|x|)^{(1+\alpha)} \cdot \left\{ \int_{|y|<\frac{|x|}{2}} + \int_{|y|>\frac{|x|}{2}} \right\} |(\nabla V)(x-y)| e^{-y^2/4} d^n y.$$

The first term is bounded since $(1+|x|)^{(1+\alpha)}\nabla V$ is uniformly locally L^2 and the second is bounded owing to the rapid decay of $\exp(-y^2/4)$. This gives (b).

Finally, (c) is proved as follows. Pick $\varepsilon > 0$. Let χ_c be the characteristic function of the unit cube centered at $c \in \mathbb{Z}^n$; we want to show that $\sup_c \|(1+|x|)^{(1+\alpha-\varepsilon)}\bar{W}\chi_c\|_2 < \infty$. Write

$$(1+|x|)^{(1+\alpha-\varepsilon)}\bar{W} = (4\pi)^{-n/2}(1+|x|)^{(1+\alpha-\varepsilon)} \int d^n y [V(x-y) - V(x)] e^{-y^2/4},$$

and split the region of y-integration into $|y| < |x|^\delta$ and $|y| > |x|^\delta$ for some $\delta < \varepsilon$. The integral over $|y| > |x|^\delta$ decays rapidly in $|x|$. The L^2 norm of the other term is given by

$$\left[\int d^n x \left(\chi_c (1+|x|)^{(1+\alpha-\varepsilon)} \int_{|y|<|x|^\delta} [V(x-y) - V(x)] e^{-y^2/4} d^n y \right)^2 \right]^{1/2}. \tag{1.2}$$

Write

$$V(x-y) - V(x) = \int_0^1 y \cdot \nabla V(x-ty) dt$$

true in distributional sense. Putting this in (1.2), we can dominate (1.2) by

$$|2c|^\delta \sup_{t\in(0,1)} \sup_{|y|<|2c|^\delta} \left(\int |\chi_c (1+|x|)^{(1+\alpha-\varepsilon)} \nabla V(x-ty)|^2 d^n x \right)^{1/2}.$$

Since we have chosen $\delta < \varepsilon$, this is bounded uniformly in c. □

We note for later use (cf. Sect. 4, especially Lemma 4.2) that, by Hormander's construction ([12], Lemma 3.3), we can further regularize the C^∞ potential W:

Proposition 1.2. *Let W be a C^∞ function with bounded derivatives and suppose that $|\nabla W(x)| \leqq C(1+|x|)^{-(1+\alpha)}$ for some $\alpha > \frac{1}{2}$. Then for any δ with $0 < \delta < \alpha$, we can write $W = W' + W''$, where*

(a) *W' is C^∞ and $|(D^\beta W)(x)| \leqq C_{|\beta|}(1+|x|)^{-m(|\beta|)}$, where $m(j) = 1 + j\delta$, $j = 1, 2, \ldots$.*

(b) *$|W''(x)| \leqq C(1+|x|)^{-(1+\varepsilon)}$ for some $\varepsilon > 0$*, i.e., *W'' is a short-range potential.*

For the proof see ([12], Lemma 3.3).

The operator $H = H_0 + W$ has several nice domain properties that follow from the smoothness of W. We collect them in:

Proposition 1.3. *Let $H = H_0 + W$, where W is C^∞ with bounded derivatives. Then*

(a) *$D(H^\alpha) = D(H_0^\alpha)$ for all positive integers α,*

(b) *$\exp(isH)$ and $(H+i)^{-1}$ preserve $\mathscr{S}(\mathbb{R}^n)$,*

(c) *For any $g \in C_0^\infty(\mathbb{R})$, $g(H)$ preserves $\mathscr{S}(\mathbb{R}^n)$.*

Proof. Part (a) follows by calculating the difference $H^\alpha - H_0^\alpha$ in the operator sense on vectors in $\mathscr{S}(\mathbb{R}^n)$. The difference consists of a sum of lower powers of H_0 times derivatives of W; such terms are H_0-bounded and since $\mathscr{S}(\mathbb{R}^n)$ is a core for H_0^α, it follows that $D(H^\alpha)=D(H_0^\alpha)$, proving (a). To see that $\exp(isH)$ preserves $\mathscr{S}(\mathbb{R}^n)$, introduce the seminorms $\|u\|_k = \sup_{\substack{j\le k \\ m\le |k|-|j|}} \|x^j H_0^m u\|$ for multi-indices k (these seminorms generate the usual topology on $\mathscr{S}$). By a result of Hunziker [13], for any $u\in\mathscr{S}(\mathbb{R}^n)$,

$$\|\exp(isH)u\|_k \leqq C_{|k|}(1+|s|)^{|k|}\|u\|_k. \tag{1.3}$$

Hence $\exp(isH)$ preserves $\mathscr{S}(\mathbb{R}^n)$. $(H+i)^{-1}$ preserves $\mathscr{S}(\mathbb{R}^n)$ since

$$(H+i)^{-1} = -i\int_0^\infty e^{-s}e^{iHs}ds,$$

so by (1.3),

$$\|(H+i)^{-1}u\|_k \leqq D_{|k|}\|u\|_k.$$

This proves (b). (c) follows similarly by writing

$$g(H)=(2\pi)^{-1/2}\int_{-\infty}^{+\infty} ds\,\hat{g}(s)\exp(isH)$$

and using the bound (1.3). □

2. The Basic Estimate

In what follows, we will denote by $\sigma_{\text{p.p.}}(H)$ the pure point spectrum of the operator H, i.e., the set of eigenvalues of H. We will prove:

Theorem 2.1. *Let $H=H_0+W$, where W is C^∞ with bounded derivatives and dilation analytic in S_φ. Let $g\in C_0^\infty((0,\infty)\setminus\sigma_{\text{p.p.}}(H))$. Then for any positive integer N, any $\varepsilon>0$, and any t with $\pm t\in(0,\infty)$,*

$$\|F(|x|\leqq|t|^{1-\varepsilon})e^{-itH}g(H)P_\pm\| \leqq C_{N,\varepsilon}(1+|t|)^{-N}. \tag{2.1}$$

Remarks. 1. Since eigenvalues of H accumulate only at 0, the set of vectors $g(H)\varphi$ with $\varphi\in L^2$ and g as above is dense in $\mathscr{H}_{\text{a.c.}}(H)$. 2. Remark 1 and estimate (2.1) imply that $P_\mp e^{-itH}P_{\text{a.c.}}(H)\xrightarrow{s}0$ as $t\to\pm\infty$. 3. The proof of Theorem 2.1 depends on the analyticity of $U(\theta)P_\pm = e^{i\theta D}P_\pm$ for $\pm\operatorname{Im}\theta>0$. Given an analytic vector ψ for D, $e^{i\theta D}\psi$ is a vector-valued analytic function of θ for $\pm\operatorname{Im}\theta<\delta$ for some $\delta>0$. By mimicking the proof of Theorem 2.1 below, we can show that the estimate

$$\|F(|x|\leqq|t|^{1-\varepsilon})e^{-itH}g(H)\psi\| \leqq C_{N,\varepsilon,\psi}(1+|t|)^{-N}$$

holds for analytic vectors ψ for D. The constant $C_{N,\varepsilon,\psi}$ depends on ψ through $\sup_{0<\theta<\delta'}\|e^{-\theta D}\psi\|$ for some $\delta'<\delta$.

Theorem 2.1 follows immediately from:

Theorem 2.2. *With the hypothesis and notation of Theorem 2.1, for any positive integer N and $\varepsilon'>0$,*

$$\|(1+|x|)^{-N}e^{-itH}g(H)P_\pm\| \leqq C_{N,\varepsilon}(1+|t|)^{-N+\varepsilon'}. \tag{2.2}$$

Our approach to proving Theorem 2.2 follows Mourre's proof of Lemma 1 in [26], where a similar estimate is proven for $H = H_0 = -\frac{1}{2}\Delta$. To extent his approach to our case, we need Proposition 2.5 below and the results of the Appendix (see Lemma 2.4). For the reader's convenience, we repeat the arguments of [26]. We begin with several reductions.

Lemma 2.3. *Suppose that H, g obey the hypotheses of Theorem 2.2, that $\pm t \in (0, \infty)$, and that for every positive integer N,*

$$\| |D+i|^{-(N+2)} e^{-iHt} g(H) P_\pm \| \leqq C_N (1+|t|)^{-N}. \tag{2.3}$$

Then the conclusion of Theorem 2.2 holds.

Proof. By a simple interpolation, (2.3) implies that

$$\| |D+i|^{-N} e^{-iHt} g(H) P_\pm \| \leqq C_{N,\varepsilon'} (1+|t|)^{-N+\varepsilon'} \tag{2.3'}$$

for any $\varepsilon' > 0$. By writing

$$(1+|x|)^{-N} e^{-iHt} g(H) P_\pm = (1+|x|)^{-N} (H+i)^{-N} e^{-iHt} [(H+i)^N g(H)] P_\pm ,$$

we are reduced to showing that the operator $(1+|x|)^{-N}(H+i)^{-N}|D+i|^N$ is bounded. To do this we need only show that terms of the form

$$(1+|x|)^{-N} (H+i)^{-N} x_{i_1} p_{i_1} \cdots x_{i_N} p_{i_N} \tag{2.4}$$

are bounded. By Proposition 1.3 (a), $D(H_0^N) = D(H^N)$ for all positive integers N and $(H+i)^{-1}$ preserves $\mathscr{S}$. By commutation, one can rewrite (2.4) as a sum of bounded terms plus terms of the form $(1+|x|)^{-N} x_{i_1} \cdots x_{i_N} (H+i)^{-N} p_{i_1} \cdots p_{i_N}$. The factor involving the x_{i_k} is obviously bounded for all N; the factor involving the p_{i_k} is bounded for N even and hence for all N by interpolation. □

To estimate $\| |D+i|^{-(N+2)} e^{-iHt} g(H) P_\pm \|$, we reexpress e^{-iHt} in terms of the resolvent of H and prove a resolvent bound using the dilation analyticity of H. The first step is

Lemma 2.4. (2.3′) *holds if for any compact subset K of $(0,\infty) \setminus \sigma_{\text{p.p.}}(H)$,*

$$\sup_{\substack{\lambda \in K \\ \varepsilon > 0}} \| |D+i|^{-(N+2)} (H - \lambda \mp i\varepsilon)^{-(N+1)} P_\pm \| < \infty . \tag{2.5}$$

Proof. We first note that for $\pm t \in (0,\infty), g \in C_0^\infty$,

$$e^{-iHt} g(H) = \lim_{\varepsilon \downarrow 0} \frac{N!}{2\pi i} \frac{(-1)}{(it)^N} \int_{-\infty}^{+\infty} d\lambda (H - \lambda \mp i\varepsilon)^{-(N+1)} e^{-i\lambda t} g(H). \tag{2.6}$$

(2.6) follows from the functional calculus if we apply the Cauchy integral formula for the N[th] derivative to the function $f_\varepsilon(x) = e^{-\varepsilon|t|} e^{-itx}$. For $f_\varepsilon(x) \to e^{-itx}$ in sup norm as $\varepsilon \downarrow 0$, and by Cauchy formula

$$f_\varepsilon(x) = \frac{N!}{2\pi i} \frac{(-1)}{(it)^N} \int_{-\infty}^{+\infty} d\lambda (x - \lambda \mp i\varepsilon)^{-(N+1)} e^{-i\lambda t}$$

for x in a fixed compact subset of $\mathbb{R}$. Hence to show that (2.3′) holds, it suffices by (2.6) to show that

$$\| \, |D+i|^{-(N+2)}(H-\lambda \mp i\varepsilon)^{-(N+1)} g(H) P_{\pm} \| \tag{2.7}$$

is an integrable function of λ. Since g has compact support, (2.7) decays rapidly outside any compact subset K of $(0,\infty)$ containing supp$g$. We can find such a compact K away from eigenvalues of H. Hence we need only show that (2.7) is bounded uniformly in $\lambda \in K$ and $\varepsilon > 0$. Furthermore, we show in Corollary A.6 of the Appendix that if $g \in C_0^\infty$, then $g(H)$ preserves the domain of D^N for all positive integers N. Hence $|D+i|^{-N} g(H) |D+i|^N$ is a bounded operator, so (2.7) is bounded if (2.5) holds. □

To prove (2.5) we consider the operator-valued function

$$F(\theta) = |D+i|^{-(N+2)}(H(\theta)-\lambda \mp i\varepsilon)^{-(N+1)} e^{i\theta D} P_{\pm}, \tag{2.8}$$

which by hypothesis extends to an analytic bounded operator-valued function in the strip $0 < \pm \operatorname{Im}\theta < \varphi$. We will derive the following differential inequality on its restriction $G(s) = F(is)$ to the imaginary axis:

$$\|G'(s)\| \leqq C_{K,\delta} \|G(s)\|^{(N+1)/(N+2)} |s|^{-(N+1)/(N+2)}, 0 < \pm s < \delta, \tag{2.9}$$

for some positive $\delta < \varphi$ and $C_{K,\delta}$ independent of ε. We can integrate (2.9) directly and conclude that $G(s)$ is uniformly bounded in $(0, \pm \delta)$. In fact, $G(s)$ is Hölder continuous in s! So it clearly suffices to prove (2.9). We first need an a priori estimate on the resolvent of $H(\theta)$.

Proposition 2.5. *Let K be any compact subset of $(0,\infty)$ not containing eigenvalues of H. Then there is a $\delta > 0$ so that, uniformly in $0 < \pm \operatorname{Im}\theta < \delta$,*

$$\sup_{\substack{\lambda \in K \\ \varepsilon > 0}} \|(H(\theta)-\lambda \mp i\varepsilon)^{-1}\| \leqq C_{K,\delta} |\operatorname{Im}\theta|^{-1}. \tag{2.10}$$

Proof. We will show that for any $\lambda_0 \in (0,\infty) \setminus \sigma_{\text{p.p.}}(H)$, there is some interval $\left(\lambda_0 - \frac{\eta}{2}, \lambda_0 + \frac{\eta}{2}\right) \subset (0,\infty) \setminus \sigma_{\text{p.p.}}(H)$ and a $\delta > 0$ for which (2.10) holds. The proposition then follows by a covering argument. Further, we will only estimate $\|(H(\theta)-\lambda-i\varepsilon)^{-1}\|$, since the other estimate follows by taking adjoints. Finally, since $(H(\theta_0+i\theta_1)-z)^{-1}$ and $(H(i\theta_1)-z)^{-1}$ are unitarily equivalent, we will suppose that $\operatorname{Re}\theta = 0$ without loss.

By the spectral theorem, for any $\lambda_0 > 0$ there is a neighborhood N of λ_0 contained in $(0,\infty)$ so that

$$\|(e^{-2i\theta_1} H - \lambda - i\varepsilon)^{-1}\| \leqq C |\theta_1|^{-1} \tag{2.10′}$$

for $\lambda \in N$, where C is uniform in $\varepsilon > 0$, $\lambda \in N$ and $0 < \theta_1 < \varphi$. Furthermore,

$$(H(\theta)-z)^{-1} = (e^{-2\theta} H - z)^{-1} [\mathbb{1} + X(\theta)(e^{-2\theta} H - z)^{-1}]^{-1}, \tag{2.11a}$$

where

$$X(\theta) = (1-e^{-2\theta}) W + (W(\theta) - W) \tag{2.11b}$$

whenever

$$\|X(\theta)(e^{-2\theta} H - z)^{-1}\| < 1. \tag{2.11c}$$

By the estimate (2.10′), we need only prove that $\|X(i\theta_1)(e^{-2i\theta_1}H-\lambda-i\varepsilon)^{-1}\|<1$ uniformly in $0<\theta_1<\delta$, $\lambda\in(\lambda_0-\eta,\lambda_0+\eta)$ and $\varepsilon>0$ for some numbers $\delta>0,\eta>0$. Equation (2.11b) and the hypotheses on W show that $X(\theta)=\theta\cdot Y(\theta)$, where $Y(\theta)(H+i)^{-1}$ is an analytic compact operator-valued function of θ. Since there are no eigenvalues of H in a neighborhood of λ_0, $E_{(\lambda_0-\eta,\lambda_0+\eta)}(H)\xrightarrow{s}0$ as $\eta\to 0$ so $\|Y(\theta)E_{(\lambda_0-\eta,\lambda_0+\eta)}(H)\|\to 0$ as $\eta\to 0$ (by analyticity this holds uniformly for $|\mathrm{Im}\,\theta|<\varphi/2$). So we insert $\mathbb{1}=E_{(\lambda_0-\eta,\lambda_0+\eta)}(H)+E_{\mathbb{R}\setminus(\lambda_0-\eta,\lambda_0+\eta)}(H)$ in

$$\begin{aligned}\|X(\theta)(e^{-2\theta}H-\lambda-i\varepsilon)^{-1}\| \leqq |\theta|\,\|Y(\theta)E_{(\lambda_0-\eta,\lambda_0+\eta)}(H)\|\;\|(e^{-2\theta}H-\lambda-i\varepsilon)^{-1}\| \\ +|\theta|\,\|Y(\theta)(H+i)^{-1}\|\;\|(H+i)(e^{-2\theta}H-\lambda-i\varepsilon)^{-1} \\ \cdot E_{\mathbb{R}\setminus(\lambda_0-\eta,\lambda_0+\eta)}(H)\|.\end{aligned}$$

Put $\theta=i\theta_1$. In the first term, $|\theta|=|\theta_1|$ cancels the singularity of the resolvent up to a constant factor that can be made small by choosing η small enough. If we then restrict λ to the interval $\left(\lambda_0-\frac{\eta}{2},\lambda_0+\frac{\eta}{2}\right)$, the second term is bounded by a constant times $|\theta|=|\theta_1|$, so it can be made small by restricting θ_1 to $0<\theta_1<\delta$ for some $\delta>0$. Hence $\|X(\theta)(e^{-2i\theta_1}H-\lambda-i\varepsilon)^{-1}\|<1$, uniformly in $\lambda\in\left(\lambda_0-\frac{\eta}{2},\lambda_0+\frac{\eta}{2}\right)$, $0<\theta_1<\delta$, and $\varepsilon>0$, and the proposition is proved. □

To prove the differential inequality (2.9), we note that if $\theta=\theta_0+i\theta_1$, then by (2.8)

$$F(\theta)=|D+i|^{-(N+2)}e^{i\theta_0D}(H(i\theta_1)-\lambda\mp i\varepsilon)^{-(N+1)}e^{-\theta_1D}P_\pm\,.$$

Taking the derivative along the real direction, we find

$$F'(\theta)=|D+i|^{-(N+2)}iD\,e^{i\theta_0D}(H(i\theta_1)-\lambda\mp i\varepsilon)^{-(N+1)}e^{-\theta_1D}P_\pm\,.$$

But $G'(s)=iF'(s)$ so we have

$$\|G'(s)\|\leqq\|\,|D+i|^{-(N+1)}(H(i\theta_1)-\lambda\mp i\varepsilon)^{-(N+1)}e^{-\theta_1D}P_\pm\|. \tag{2.12}$$

To obtain (2.9) we estimate $\|A(z)\|=\|\,|D+i|^{-z}(H(is)-\lambda\mp i\varepsilon)^{-(N+1)}\cdot e^{-sD}P_\pm\|$ by interpolating between $\mathrm{Re}\,z=0$ and $\mathrm{Re}\,z=N+2$. For $\mathrm{Re}\,z=0$ we have $\|A(z)\|\leqq C_{K,\delta}|s|^{-(N+1)}$ by Proposition 2.5, while for $\mathrm{Re}\,z=N+2$, $\|A(z)\|=\|G(s)\|$. Inequality (2.9) follows since $N+1=0\cdot\frac{1}{N+2}+(N+2)\cdot\frac{N+1}{N+2}$.

We have thus proven:

Lemma 2.6. *The differential inequality* (2.9) *holds.*

Collecting Lemmas 2.6, 2.4, and 2.3, Theorem 2.2 is proved.

3. Evolution of Observables Under $\exp(-itH)$

In this section we use ideas of Enss [9] to study the Heisenberg operators $\mathbf{x}(t)$, $\mathbf{p}(t)$, and $D(t)$, where $A(t)\equiv e^{iHt}Ae^{-iHt}$. We will prove:

Theorem 3.1. *Let $H = H_0 + W$ where W satisfies* (i)″ *of Theorem 3. Then*

(a) $\frac{D(t)}{2t} \to H$ *in strong resolvent sense as* $t \to \pm\infty$ *on* $\mathscr{H}_{\text{a.c.}}(H)$.

(b) *Let* $0 < \delta < \alpha$. *Then* $\frac{\mathbf{x}(t) - t\mathbf{p}(t)}{|t|^{1-\delta}} \to 0$ *in strong resolvent sense as* $t \to \pm\infty$ *on* $\mathscr{H}_{\text{a.c.}}(H)$.

Remark. Theorem 3.1 (a) and its proof below are due to Enss [9]; Theorem 3.1 (b) is new.

Theorem 3.1 implies:

Theorem 3.2. *Let* $\psi \in \mathscr{H}_{\text{a.c.}}(H)$ *and suppose that* $\psi = E_{(a,b)}(H)\psi$, *where* $(a,b) \subset (0,\infty) \setminus \sigma_{\text{p.p.}}(H)$. *Let* $\psi_t = e^{-itH}\psi$. *Then*:

$$\psi_t - F_0(|\mathbf{x} - \mathbf{p}t| < |t|^{1-\delta}) E_{(a,b)}\left(\frac{D}{2t}\right) g(H_0)\psi_t \to 0 \text{ as } t \to \pm\infty,$$

where $g \in C_0^\infty(0,\infty)$ *satisfies* $g = 1$ *on* (a,b).

Remark. For an n-tuple $\mathbf{A}$ of commuting self-adjoint operators and a subset S *of* $\mathbb{R}^n$, the "smooth" projection $F_0(\mathbf{A} \in S)$ is defined as follows. Let χ_S denote the characteristic function of S and let ξ satisfy $\int d^n y\, \xi(y) = 1$ and $\xi \in C_0^\infty(\mathbb{R}^n)$. $F_0(\mathbf{A} \in S)$ is the operator associated to the convolution $\chi_S * \xi$ by the functional calculus for $\mathbf{A}$.

Proof of Theorem 3.2 given Theorem 3.1. It is enough to show that $\psi_t - E_{(a,b)}\left(\frac{D}{2t}\right)\psi_t \to 0$ as $t \to \pm\infty$ and $\psi_t - F_0(|\mathbf{x} - \mathbf{p}t| < |t|^{1-\delta})\psi_t \to 0$ as $t \to \pm\infty$ separately, since the result then follows by the uniform boundedness in t of the projections. By Theorem 3.1 (a) and Theorem VIII. 24 (b) of [28],

$$E_{(a,b)}\left(\frac{D(t)}{2t}\right) \xrightarrow{s} E_{(a,b)}(H)$$

as $t \to \pm\infty$. Write

$$\left\| \psi_t - E_{(a,b)}\left(\frac{D}{2t}\right)\psi_t \right\| = \left\| \left[E_{(a,b)}(H) - E_{(a,b)}\left(\frac{D}{2t}\right) \right] \psi_t \right\|$$
$$= \left\| \left[E_{(a,b)}(H) - E_{(a,b)}\left(\frac{D(t)}{2t}\right) \right] \psi \right\| \to 0,$$

where in the last step we have used the unitarity of e^{itH}. A similar argument using Theorem 3.1 (b) shows that $\|(1 - F_0(|x - pt| < |t|^{1-\delta})\psi_t\| \to 0$. Finally, since $g = 1$ on (a,b), $(1 - g(H_0))\psi_t = (g(H) - g(H_0))\psi_t$, which goes to zero by the compactness of $(H + i)^{-1} - (H_0 + i)^{-1}$ and a standard argument [33, Lemma 2.4]. □

To prove Theorem 3.1, we first recall a standard criterion [28, Theorem VIII. 25 (a)] for strong resolvent convergence: $A_n \to A$ in strong resolvent sense if $A_n \to A$ on a core for A contained in $D(A_n)$ for each n. Hence our first step is to find a nice set of vectors on which to study the Heisenberg operators $D(t)$ and $\mathbf{x}(t) - t\mathbf{p}(t)$.

Proposition 3.3. *Let* $N=p^2+x^2+1$ *and let* $\mathscr{D}$ *be the set of all vectors of the form* $g(H)\,e^{-\theta N}\varphi$ *for* $\varphi\in L^2$, $\theta>0$, *and* $g\in C_0^\infty((0,\infty)\setminus\sigma_{\text{p.p.}}(H))$. *Then*:

(a) $\mathscr{D}$ *is a core for* $H\upharpoonright\mathscr{H}_{\text{a.c.}}(H)$.

(b) $\mathscr{D}\subset\mathscr{S}(\mathbb{R}^n)$ *and* $\exp(-itH)\,\mathscr{D}\subset\mathscr{S}(\mathbb{R}^n)$.

(c) $\|F(|x|<|t|^{1-\varepsilon})e^{-itH}\psi\|\leqq C_N(1+|t|)^{-N}$ *for any* $\varepsilon>0$, $t\in\mathbb{R}$, *and* $\psi\in\mathscr{D}$.

Proof. Since $e^{-\theta N}\psi\to\psi$ as $\theta\to 0$, $\mathscr{D}$ is dense in $\bigcup_g \operatorname{rang}(H)$, which is obviously dense in $D(H\upharpoonright\mathscr{H}_{\text{a.c.}}(H))$ in graph norm, proving (a). (b) holds since for $\theta>0$, $e^{-\theta N}\psi\in C^\infty(N)=\mathscr{S}(\mathbb{R}^n)$ [28] and by Proposition 1.3, $g(H)$ and $\exp(-itH)$ both preserve $\mathscr{S}(\mathbb{R}^n)$. Finally (c) holds since, for $\theta>0$, $e^{-\theta N}\psi$ is an analytic vector for N and N analytically dominates D (e.g. by Faris [10, Theorem 16.4]); hence any $\psi\in\mathscr{D}$ is of the form $g(H)\chi$ where χ is an analytic vector for D, and by Remark 3 after Theorem 2.1, such vectors obey the estimate of (c). □

Proof of Theorem 3.1. Following the method of [9], we consider the Heisenberg equations of motion for $D(t)$ and $\mathbf{x}(t)-\mathbf{p}(t)\cdot t$. Weakly on $\mathscr{D}\times\mathscr{D}$,

$$\frac{d}{dt}D(t)=e^{iHt}[2H_0-(x\cdot\nabla)W]\,e^{-iHt}, \tag{3.1}$$

and by Proposition 3.3 (b), (3.1) holds in the operator sense on $\mathscr{D}$. Write the quantity in brackets as $2H+I$; by Proposition 1.1, I is H-compact. Integrate (3.1) and divide by $2t$ to obtain

$$\frac{D(t)}{2t}=\frac{D(0)}{2t}+H+\frac{1}{2t}\int_0^t ds\, e^{isH}\,I\,e^{-isH}. \tag{3.1$'$}$$

Applied to vectors $\psi\in\mathscr{D}$, the first term in (3.1′) vanishes by the RAGE theorem [29, Theorem XI. 115] since $I(H+i)^{-1}$ is compact and $(H+i)\psi$ is bounded, if $\psi\in\mathscr{D}$. This proves (a). To prove (b) we compute, weakly on $\mathscr{D}\times\mathscr{D}$,

$$\frac{d}{dt}(\mathbf{x}(t)-t\cdot\mathbf{p}(t))=e^{itH}(\nabla W)(x)\,e^{-itH}. \tag{3.2}$$

Again, (3.2) actually holds in the operator sense on $\mathscr{D}$. Integrate (3.2) and divide by $|t|^{1-\delta}$ to obtain

$$\frac{\mathbf{x}(t)-t\cdot\mathbf{p}(t)}{|t|^{1-\delta}}=\frac{\mathbf{x}(0)}{|t|^{1-\delta}}+\frac{1}{|t|^{1-\delta}}\int_0^t e^{itH}(\nabla W)(x)\,e^{-itH}\,ds. \tag{3.2$'$}$$

The first term vanishes as $t\to\pm\infty$ when applied to $\psi\in\mathscr{D}$. The integrand of the second term applied to $\psi\in\mathscr{D}$ is estimated using Proposition 3.3 (c) and the estimate on ∇W in Proposition 1.1 (b):

$$\|(\nabla W)(x)\,e^{-isH}\psi\|\leqq \text{cst.}\times\|F(|x|<|s|^{1-\varepsilon})\,e^{-isH}\psi\|+\|(\nabla W)(x)\,F(|x|\leqq|s|^{1-\varepsilon})\|$$

$$\leqq C_N(1+|s|)^{-N}+\text{const}\,(1+|s|)^{-(1+\alpha-\varepsilon)} \tag{3.3}$$

for any $\varepsilon>0$. On integrating the right hand side of (3.3) and dividing by $|t|^{1-\varepsilon}$, we obtain an estimate for the second term in (3.2′) that vanishes as $t\to\pm\infty$, since $\varepsilon>0$ is arbitrary and $\delta<\alpha$. Hence $\dfrac{\mathbf{x}(t)-\mathbf{p}(t)\cdot t}{|t|^{1-\delta}}\to 0$ on $\mathscr{D}$, proving (b).

4. Proof of Theorem 3

To prove Theorem 3, we will show that the inverse modified wave operators $\Omega_D^{\pm}(H,H_0)^*$ exist as strong limits on a dense subset of $\mathscr{H}_{\text{a.c.}}(H)$. We will only give the proof for Ω_D^{-*} since the proof for Ω_D^{+*} is similar. Ω_D^{-*} exists if

$$\lim_{s\to\infty}\sup_{t\geqq s}\|[e^{-iH(t-s)}-\mathscr{U}_0(t,s)]\,e^{-iHs}\psi\|=0, \tag{4.1}$$

for ψ in a dense subset of $\mathscr{H}_{\text{a.c.}}(H)$. Consider the set of vectors ψ with $E_{(a,b)}(H)\psi=\psi$ for some $(a,b)\subset(0,\infty)\setminus\sigma_{\text{p.p.}}(H)$. For such ψ, (4.1) holds if

$$\lim_{s\to\infty}\sup_{t\geqq s}\left\|[e^{-iH(t-s)}-\mathscr{U}_0(t,s)]\,F_0(|\mathbf{x}-\mathbf{p}s|<|s|^{1-\delta})g(H_0)E_{(a,b)}\left(\frac{D}{2s}\right)\right\|=0, \tag{4.2}$$

where $g\in C_0^\infty(0,\infty)$ satisfies $g=1$ on (a,b), by Theorem 3.2. By a "Cook's method" argument, (4.2) holds if

$$\lim_{s\to\infty}\sup_{t\geqq s}\int_s^t\left\|[W'(\mathbf{x})+W''(\mathbf{x})-W'(\mathbf{p}s')]\,\mathscr{U}_0(s',s)\right.$$
$$\left.\cdot F_0(|\mathbf{x}-\mathbf{p}s|<|s|^{1-\delta})g(H_0)E_{(a,b)}\left(\frac{D}{2s}\right)\right\|ds'=0. \tag{4.3}$$

We will prove (4.3). We first collect some estimates on the modified free evolution $\mathscr{U}_0(s',s)$. The modified free evolution is dominated by the free evolution $e^{-iH_0(s'-s)}$; the first Lemma is a simple extension of the estimate on e^{-iH_0t} proven in [27] by Mellin transform methods.

Lemma 4.1. *Let $g\in C_0^\infty(0,\infty)$. Then there is a $c>0$ so that, for all s' with $s'>s>0$ and any integer N,*

$$\left\|F(|x|<cs')\,e^{-i(s'-s)H_0}g(H_0)E_{(a,b)}\left(\frac{D}{2s}\right)\right\|\leqq C_N(1+|s'|)^{-N}.$$

We omit the proof.

The next lemma shows that the corrections to the free evolution introduced by the factor $\exp\left[-i\int_s^{s'}W'(\mathbf{p}\tau)\mathrm{d}\tau\right]$ are small.

Lemma 4.2 [8]. *Let $\bar{g}\in C_0^\infty(0,\infty)$ and let $K(s',s)=\exp\left[-i\int_s^{s'}W'(\mathbf{p}\tau)\,d\tau\right]\bar{g}(H_0)$. Let S_1, S_2 be subsets of $\mathbb{R}^n$ with* $\operatorname{dist}(S_1,S_2)\geqq d>0$. *Then for any integer l,*

(a) $\|F(\mathbf{x}\in S_1)K(s',s)F(\mathbf{x}\in S_2)\|\leqq D_l(1+|s'|)^{(1-\delta')(n+l+1)}(1+d)^{-l}$, *where δ' is defined in Proposition 1.2.*

(b) *The same estimate holds with* $\mathbf{x}$ *replaced by* $\mathbf{x}-\mathbf{p}s'$.

(c) (a) *and* (b) *hold with F replaced by F_0, where F_0 is defined as in the remark after Theorem 3.2.*

Proof. (b) follows from (a) since $\exp(-is'H_0)$ commutes with $K(s',s)$ and $\exp(-is'H_0)f(\mathbf{x})=f(\mathbf{x}-\mathbf{p}s')\exp(-is'H_0)$ for Borel functions f. (c) follows from (a) (respectively (b)) by using the rapid decay of $F_0(\mathbf{x}\in S)$ [respectively

$F_0(\mathbf{x}-\mathbf{p}s\in S)]$ outside of S. To prove (a), we note that in x-space, $K(s',s)$ is the operator of convolution with a rapidly decaying kernel. (a) then follows by using the estimates of Proposition 1.2 on derivatives of W' together with Young's inequality (cf. [8, Eq. (41) ff.]). □

Next, we note some useful properties of the "smooth" projections F_0 introduced in Sect. 3.

Lemma 4.3 [8]. *Let* $F_0(\mathbf{x}-\mathbf{p}s\in S)$ *be defined as in* Sect. 3.

(a) (*small momentum transfer*) *Let* $g\in C_0^\infty(0,\infty)$. *Then for* $\operatorname{supp}\xi$ *small enough there is a* $\bar{g}\in C_0^\infty(0,\infty)$ *with* $\bar{g}=1$ *on* $\operatorname{supp} g$ *so that* $F_0(\mathbf{x}-\mathbf{p}s\in S)g(H_0) = \bar{g}(H_0)F_0(\mathbf{x}-\mathbf{p}s\in S)g(H_0)$ *and similarly for s replaced by* s'.

(b) (*small position transfer*) *For* $c>0$ *and* $\operatorname{supp}\xi$ *small enough,*

$$F\left(|x|<\frac{cs'}{2}\right)F_0(\mathbf{x}-\mathbf{p}s'\in S)F(|x|>cs')=0.$$

Proof. Let $f_S=\chi_S*\xi$. Then

$$F_0(\mathbf{x}-\mathbf{p}s\in S)=\int d^n\lambda \hat{f_S}(\lambda)\exp[i\lambda\cdot(\mathbf{x}-\mathbf{p}s)].$$

By the Baker-Campbell-Hausdorff formula, $\exp[i\lambda\cdot(\mathbf{x}-\mathbf{p}s)]=[\exp i\lambda\cdot\mathbf{x}]\times[\exp -i\lambda\cdot\mathbf{p}s]\times\exp\frac{1}{2}i\lambda^2 s$. Using this fact along with the compact support of $\hat{f_S}$, it follows that $F_0(\mathbf{x}-\mathbf{p}s\in S)$ has a momentum transfer of at most a where $\operatorname{supp}\xi\subset\{\lambda:|\lambda|<a\}$. This shows (a). (b) is proved similarly. □

Next we note a formula for the operator difference $W'(\mathbf{x})-W'(\mathbf{p}s')$ that occurs in (4.3).

Lemma 4.4.

$$W'(\mathbf{x})-W'(\mathbf{p}s')=\int_0^1 d\theta\,\{(\nabla W')(\theta\mathbf{x}+(1-\theta)\mathbf{p}s')\cdot(\mathbf{x}-\mathbf{p}s')+is'(\Delta W')(\theta\mathbf{x}+(1-\theta)\mathbf{p}s')\}.$$

Lemma 4.4 is proved by writing W' as the integral of its Fourier transform and using the formula

$$e^{iq\cdot\mathbf{x}}-e^{iq\cdot\mathbf{p}s'}=i\int_0^1 d\theta\, e^{i\theta q\cdot\mathbf{x}}\,iq\cdot(\mathbf{x}-\mathbf{p}s')\,e^{i(1-\theta)q\cdot\mathbf{p}s'}$$

together with the Baker-Campbell-Hausdorff formula. We omit details. Finally, we note:

Lemma 4.5. *For any positive integer k and* $s'>0$,

$$\sup_{\theta\in[0,1]}\left\|F\left(|\theta\mathbf{x}+(1-\theta)\mathbf{p}s'|<\frac{cs'}{8}\right)F_0(|\mathbf{x}-\mathbf{p}s'|<2s'^{1-\delta})F\left(|\mathbf{x}|>\frac{cs'}{4}\right)\right\|<C_k(1+s')^{-k}.$$

The proof is very similar to the proof of Lemma 1 in [8] and is omitted.

We now carry out the

Proof of Theorem 3. We will show that the integrand of (4.3) is estimated by $C(1+s')^{-(1+\eta)}$ for some $\eta>0$ and C independent of s,s'. We first note that, by Lemma 4.3, $F_0(|\mathbf{x}-\mathbf{p}s|<|s|^{1-\delta})g(H_0)=\bar{g}(H_0)F_0(|\mathbf{x}-\mathbf{p}s|<|s|^{1-\delta})g(H_0)$ for some $\bar{g} \in C_0^\infty(0,\infty)$. Writing $B = W'(x) + W''(x) - W'(\mathbf{p}s')$, we see that the integrand of (4.3) equals

$$\left\| BK(s',s)e^{-i(s'-s)H_0} F_0(|\mathbf{x}-\mathbf{p}s| < |s|^{1-\delta}) g(H_0) E_{(a,b)}\left(\frac{D}{2s}\right) \right\|,$$

which, by the identity

$$e^{-i(s'-s)H_0} f(\mathbf{x}-\mathbf{p}s) = f(\mathbf{x}-\mathbf{p}s') e^{-i(s'-s)H_0},$$

equals

$$\left\| BK(s',s) F_0(|\mathbf{x}-\mathbf{p}s'| < |s|^{1-\delta}) e^{-i(s'-s)H_0} g(H_0) E_{(a,b)}\left(\frac{D}{2s}\right) \right\|. \tag{4.4}$$

Writing $\mathbb{1} = F(|x| < cs') + F(|x| > cs')$ and using Lemma 4.1, we conclude that (4.4) is estimated by a term decaying rapidly in s' plus

$$\| BK(s',s) F_0(|\mathbf{x}-\mathbf{p}s'| < |s|^{1-\delta}) F(|\mathbf{x}| > cs') \|. \tag{4.5}$$

By choosing $\delta' < \delta$ in Proposition 1.2 (so $\frac{1}{2} < \delta' < \delta < \alpha$), we can use Lemma 4.2 to see that $\| F_0(|\mathbf{x}-\mathbf{p}s'| > 2s'^{1-\delta}) K(s',s) F_0(|\mathbf{x}-\mathbf{p}s'| < s^{1-\delta}) \|$ decays rapidly in s' for $s' > s$. Hence we can estimate (4.5) by terms that decay rapidly in s' plus

$$\| B F_0(|\mathbf{x}-\mathbf{p}s'| < 2s'^{1-\delta}) K(s',s) F_0(|\mathbf{x}-\mathbf{p}s'| < s^{1-\delta}) F(|\mathbf{x}| > cs') \|. \tag{4.6}$$

By Lemma 4.3,

$$F_0(|\mathbf{x}-\mathbf{p}s'| < s'^{1-\delta}) F(|\mathbf{x}| > cs') = F\left(|\mathbf{x}| > \frac{cs'}{2}\right) F_0(|\mathbf{x}-\mathbf{p}s'| < s'^{1-\delta}) F(|\mathbf{x}| > cs')$$

for supp $\hat{\xi}$ small enough, and by Lemma 4.2, $\left\| F\left(|\mathbf{x}| < \frac{cs'}{4}\right) K(s',s) F\left(|\mathbf{x}| > \frac{cs'}{2}\right) \right\|$ decays rapidly in s'. Hence, finally, we can dominate (4.6) by terms decaying rapidly in s' plus

$$\begin{aligned} &\left\| B F_0(|\mathbf{x}-\mathbf{p}s'| < 2s'^{1-\delta}) F\left(|\mathbf{x}| > \frac{cs'}{4}\right) \right\| \\ &\quad \leqq \left\| [W'(\mathbf{x}) - W'(\mathbf{p}s')] F_0(|\mathbf{x}-\mathbf{p}s'| < 2s'^{1-\delta}) F\left(|\mathbf{x}| > \frac{cs'}{4}\right) \right\| \\ &\qquad + \left\| W''(\mathbf{x}) F_0(|\mathbf{x}-\mathbf{p}s'| < 2s'^{1-\delta}) F\left(|\mathbf{x}| > \frac{cs'}{4}\right) \right\|. \end{aligned} \tag{4.7}$$

To estimate the first term on the right hand side of (4.7), we use Lemmas 4.4 and 4.5 and bound it by rapidly decaying terms plus

$$\begin{aligned} \sup_{\theta \in [0,1]} \Bigg\{ &\left\| \nabla W'(\theta\mathbf{x} + (1-\theta)\mathbf{p}s') F\left(|\theta\mathbf{x} + (1-\theta)\mathbf{p}s'| \leqq \frac{cs'}{8}\right) \right\| \\ &\cdot \| (\mathbf{x}-\mathbf{p}s') F_0(|\mathbf{x}-\mathbf{p}s'| < 2s'^{1-\delta}) \| \\ &+ s' \left\| \Delta W(\theta\mathbf{x} + (1-\theta)\mathbf{p}s') F\left(|\theta\mathbf{x} + (1-\theta)\mathbf{p}s'| > \frac{cs'}{8}\right) \right\| \Bigg\}. \end{aligned}$$

The first term in brackets is bounded by a constant times $(1+s')^{-(1+\delta')}(1+s')^{1-\delta} = (1+s')^{-(1+\eta)}$ for $\eta>0$, since $\delta, \delta' > \frac{1}{2}$. The second term in brackets is bounded by a constant times $(1+s')^{-2\delta'}$, and since $\delta' > \frac{1}{2}$, $2\delta' = 1+\eta$ for some $\eta > 0$. The second term in (4.7) is dominated by $\left\| W'(\mathbf{x}) F\left(|\mathbf{x}| > \frac{cs'}{8}\right)\right\|$ since, by Lemma 4.3,

$$F_0(|\mathbf{x}-\mathbf{p}s'| < 2s'^{1-\delta}) F\left(|\mathbf{x}| > \frac{cs'}{4}\right)$$
$$= F\left(|\mathbf{x}| > \frac{cs'}{8}\right) F_0(|\mathbf{x}-\mathbf{p}s'| > 2s'^{1-\delta}) F\left(|\mathbf{x}| > \frac{cs'}{4}\right)$$

for suitable choice of supp ξ. $\left\| W''(\mathbf{x}) F\left(|\mathbf{x}| > \frac{cs'}{8}\right)\right\|$ is estimated by a constant times $(1+s')^{-(1+\eta)}$ for some $\eta > 0$ by Proposition 1.2(b). This shows that (4.3) holds, proving Theorem 3. □

Appendix. On the Invariance of Operator Domains

Let A be a self-adjoint operator and let $U(\alpha) = \exp(i\alpha A)$. For any self adjoint operator B, $U(\alpha)$ induces a family of self-adjoint operators $B(\alpha) = U(\alpha) B U(\alpha)^{-1}$ unitarily equivalent to B. We wantt to show that if the map $\alpha \to B(\alpha)$ is smooth, nice functions of B preserve $D(A^k), k=1,\dots,n$, where n depends on the smoothness of the map $\alpha \to B(\alpha)$. The following Proposition is central:

Proposition A.1. *$\varphi \in D(A^k)$ if and only if the vector-valued function $\varphi(\alpha) = U(\alpha)\varphi$ is C^k at 0.*

Remark. If $\varphi(\alpha)$ is differentiable at 0, then by translating with the unitary group, it is differentiable everywhere. Hence the phrase "C^k at 0" makes sense.

Proof. For $k=1$, this is Theorem VIII.7 of [28]. Suppose the proposition holds for $\varphi \in D(A^{k-1})$. If $\varphi(\alpha)$ is C^k, certainly $\varphi \in D(A^{k-1})$ and $\varphi^{(k)}(0) = \psi'(0)$, where $\psi = A^{k-1}\varphi$. But then $\psi \in D(A)$, i.e., $\varphi \in D(A^k)$. By a similar argument, any $\varphi \in D(A^k)$ is C^k at zero. □

Suppose g is a smooth function; then $g(B)\varphi \in D(A^k)$ if $U(\alpha) g(B)\varphi$ is C^k at 0. But $U(\alpha) g(B)\varphi = g(B(\alpha)) U(\alpha)\varphi$ so $U(\alpha) g(B)\varphi$ is C^k, if $\varphi \in D(A^k)$ and $g(B(\alpha))$ is norm C^k as a function of α. Hence:

Corollary A.2. *Suppose that $B(\alpha) = U(\alpha) B U(\alpha)^{-1}$ as above and that $g(B(\alpha))$ is norm-C^n as a function of α. Then $g(B(\alpha))$ preserves $D(A^k)$ for $k=1,\dots,n$.*

We now find sufficient conditions on the map $\alpha \to B(\alpha)$ and the function g for $g(B(\alpha))$ to be norm-C^n. We first consider bounded operators B and then make an easy extension to semibounded self-adjoint operators.

Since the operator $B(\alpha)$ are self-adjoint, we can write

$$g(B(\alpha)) = (2\pi)^{-1/2} \int \hat{g}(t) \exp(itB(\alpha))\, dt. \tag{A.1}$$

We are then motivated to consider the operator $\exp itB(\alpha)$:

Lemma A.3. *Let $B(\alpha)$ be a family of bounded, self-adjoint operators and suppose that the map $\alpha \to B(\alpha)$ is norm-C^k in some interval I containing* 0. *Let* $\|B(\alpha)\|_k = \sup_{\substack{0 \leqq j \leqq k \\ \alpha \in I}} \|(D_\alpha^j B)(\alpha)\|$. *Then*

$$\|\exp(itB(\alpha))\|_k \leqq C_k(1+|t|)^k \|B(\alpha)\|_k. \tag{A.2}$$

Proof. By the Duhamel formula,

$$\exp itB(\alpha+\varepsilon) - \exp itB(\alpha) = it \int_0^1 ds \exp(istB(\alpha+\varepsilon)) \cdot [B(\alpha+\varepsilon) - B(\alpha)] \times \exp i(1-s)tB(\alpha),$$

so $\exp itB(\alpha)$ is norm-continuous in α for fixed t. Dividing by ε and taking norm limits, we get

$$\frac{d}{d\alpha}(\exp itB(\alpha)) = it \int_0^1 ds \exp(istB(\alpha))B'(\alpha)\exp(i(1-s)tB(\alpha)).$$

Repeated application of this formula gives (A.2). □

Combining (A.1.) and Lemma A.3, and using Corollary A.2, we obviously have:

Proposition A.4. *Let $\alpha \to B(\alpha)$ be norm-C^n and let $g \in C_0^\infty$. Then $g(B(\alpha))$ is norm-C^n. If $B(\alpha) = U(\alpha)BU(\alpha)^{-1}$ with B and $U(\alpha)$ as above, then $g(B(\alpha))$ preserves $D(A^k)$ for $k = 1, \ldots, n$.*

Now let B be a semibounded self-adjoint operator and let $B(\alpha) = U(\alpha)BU(\alpha)^{-1}$. Suppose that for some suitable c, $R(\alpha) = (B(\alpha)+c)^{-1}$ is norm-C^n. If $g \in C_0^\infty(-c, \infty)$, $f(y) = g\left(\frac{1}{y} - c\right)$ is a C_0^∞ function so $g(B(\alpha)) = f(R(\alpha))$ is norm-C^n. We have therefore proved:

Theorem A.5. *Let $B(\alpha) = U(\alpha)BU(\alpha)^{-1}$, where B is a semibounded self-adjoint operator and $U(\alpha)$ is a unitary group generated by the self-adjoint operator A. Suppose that $R(\alpha) = (B(\alpha)+c)^{-1}$ is norm-C^n for suitable c. Then for any $g \in C_0^\infty(-c, \infty)$, $g(B(\alpha))$ preserves $D(A^k)$, $k = 1, \ldots, n$.*

If $H = H_0 + W$ with W dilation analytic and $H(\theta) = \mathcal{U}(\theta)H\mathcal{U}(\theta)^{-1}$, $R(\theta) = (H(\theta)+c)^{-1}$ is analytic. Clearly:

Theorem A.6. *Let $H = H_0 + W$ with W dilation analytic and $H + c > 0$ for some c. Let $g \in C_0^\infty(-c, \infty)$. Then $g(H)$ preserves the domain of D^n for all positive integers n.*

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