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New Approach to the Semiclassical Limit of Quantum Mechanics

I. Multiple Tunnelings in One Dimension

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Abstract. We propose a new approach for the estimate of the rate of degeneracy of the lowest eigenvalues of the Schrödinger operator in the presence of tunneling based on the theory of diffusion processes. Our method provides lower and upper bounds for the energy splittings and the rates of localization of the wave functions and enables us to discuss cases which, as far as we know, have never been treated rigorously in the literature. In particular we give an analysis of the effect on eigenvalues and eigenfunctions of localized deformations of 1) symmetric double well potentials 2) potentials periodic and symmetric over a finite interval. Theses situations are characterized by a remarkable dependence on such deformations. Our probabilistic techniques are inspired by the theory of small random perturbations of dynamical systems.

1. Introduction

The estimate of the semiclassical rate of degeneracy of the lowest eigenvalues of the Schrödinger operator H in the presence of tunneling is not a new problem and has been solved in special situations, for example in connection with the theory of phase transitions in statistical mechanics [1]. More recently Harrell has produced two papers [2, 3] in which a rather complete analysis of the above problem for the case of symmetric double wells is given and where one can find a wide list of references. The methods employed in these papers require in general a detailed analysis of the eigenfunctions of H as $\hbar \to 0$ and their generalization to non-symmetric cases does not appear so easy. Here we propose a different approach based on the theory of diffusion processes which requires only an estimate of the

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log-derivative of the ground state wave function in the semiclassical limit in order to extract lower and upper bounds for the splittings of the lowest eigenvalues and for the rates of localization of the corresponding eigenfunctions. Our method enables us to discuss cases which, as far as we know, have never been treated rigorously in the literature.

In particular we give an analysis of the effect produced on the splitting of the eigenvalues and on the eigenfunctions by localized deformations of 1) symmetric double well potentials (Fig. 1); 2) potentials symmetric and periodic over a finite interval (Fig. 2). As we shall see, tunneling in these situations is very unstable under deformations.

The techniques presented here were inspired by the theory of small random perturbations of dynamical systems [4]. On the side of physics however, it is reasonable to see the origin of our approach in the stochastic quantization in the form developed by Nelson some years ago [5].

If for illustrative purposes we restrict ourselves to one dimensional systems, the scheme of stochastic quantization for stationary states goes as follows: the position of a particle in a potential $V(x)$ obeys the stochastic differential equation

$$dx_t = b\left(x_t, \frac{\hbar}{m}\right) dt + \left(\frac{\hbar}{m}\right)^{1/2} dW_t, \tag{1.1}$$

where W_t is the Wiener process with unit variance and $b(x)$ a drift term determined by

$$b^2(x) + \frac{\hbar}{m}\frac{d}{dx} b(x) = \frac{2}{m}(V(x) - E). \tag{1.2}$$

The connection with the Schrödinger equation is straightforward as we note that (1.2) is a Riccati equation which can be linearized in the usual way by putting

$$b(x) = \frac{\hbar}{2m}\frac{d}{dx}\ln \psi^2 .$$

From this the Schrödinger equation for ψ follows and one realizes that ψ^2 is the density of the invariant measure of the process described by (1.1). The description of time dependent states is more complicated but shall not be needed here.

The connection between stochastic processes and quantum mechanics has a long history. The relationship among the Schrödinger equation, the heat equation and brownian was pointed out a long time ago by Kac [6]. After the work of Nelson [5], Guerra and Ruggiero [7] remarked that the process corresponding to the ground state according to stochastic quantization is essentially the same as the process described by the imaginary time functional integral, i.e. the process generated by the heat equation. More recently this aspect has received a general mathematical formulation [8, 9] in terms of equivalence between the quadratic form associated to the Hamiltonian and the Dirichlet form (or "energy" form in the terminology of [8]) constructed with the ground state measure $\psi_0^2 dx$. The interesting fact is that the differential operator associated to this Dirichlet form

$$L^\varepsilon = \frac{\varepsilon^2}{2}\frac{d^2}{dx^2} + b^\varepsilon(x)\frac{d}{dx}, \tag{1.3}$$

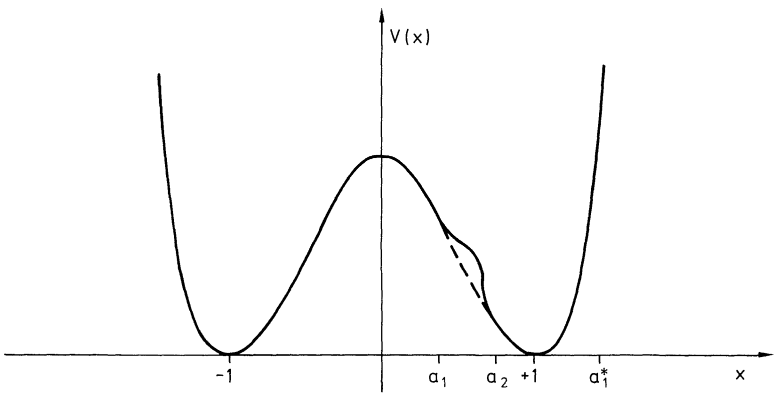

Fig. 1

with $\varepsilon=\left(\frac{\hbar}{m}\right)^{1/2}$, $b^\varepsilon(x)=\frac{\varepsilon^2}{2}\frac{d}{dx}\ln\psi_0^2$, can be studied as an operator on $L^2(\psi_0^2 dx)$ by direct probabilistic techniques. These techniques show that the study of the lower part of the spectrum of $-L^\varepsilon$ can be reduced as $\varepsilon\to 0$ to the study of the spectrum of a finite stochastic matrix which approximates, in the same limit, the transition probabilities between the absolute minima of the potential $V(x)$ of the process with infinitesimal generator L^ε.

We now come to a crucial question. In order that the general approach we have described may work, we need the drift $b^\varepsilon(x)$ of L^ε which is equivalent to the knowledge of the ground state wave function ψ_0. We must therefore construct the solution of the Riccati equation (1.2) corresponding to the ground state in the limit $\varepsilon\to 0$. This is a nontrivial matter and Sect. 2 of this paper is entirely devoted to such a problem[1].

In Sect. 2 besides establishing various properties of $b^\varepsilon(x)$ for smooth positive potentials which increase sufficiently rapidly at infinity (the precise hypotheses will be given there), we prove that for ε small enough $|b^\varepsilon(x)\pm\sqrt{2V(x)}|<C\varepsilon^\gamma$ in the complement of ε^γ-neighborhoods of finitely many points, $0<\gamma<2$. In those neighborhoods $b^\varepsilon(x)$ can exhibit a rapid increase from $-\sqrt{2V(x)}$ to $+\sqrt{2V(x)}$.The most difficult part of the job consists then in determining the location of such points of rapid increase. This problem is solved completely in several situations with the aid of an integral equation connecting the symmetric part of $b^\varepsilon(x)$, $b^\varepsilon(x)+b^\varepsilon(-x)$, with respect to an arbitrary reflection point, with the antisymmetric part $b^\varepsilon(x)-b^\varepsilon(-x)$. The method appears to be of general applicability at least in one dimension.

We now describe the main results of the paper. First of all, as a test for our method, we calculate the level splitting for the symmetric double well potential and we recover in a simple way well-known upper and lower bounds [10]. We then consider localized deformations of the previous case (see Fig. 1) and compute

$$E_1-E_0,\quad \psi_0(1)/\psi_0(-1),\quad \psi_1(-1)/\psi_1(1)$$

1 There are points of contact between our Sect. 2 and the work of Harrell in [2] which however was brought to our attention only after completion of the present work

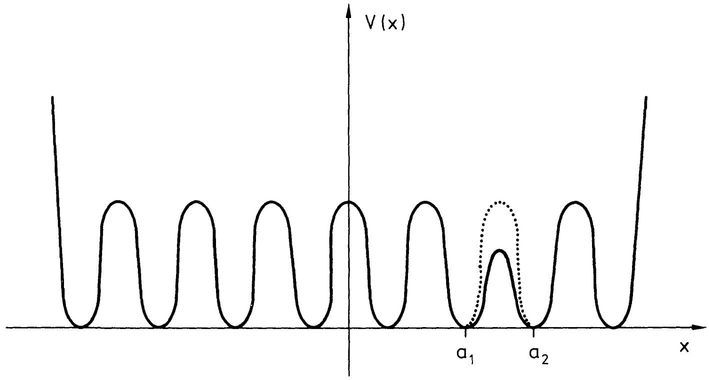

Fig. 2

in the new situation. E_0, E_1, ψ_0, ψ_1 are of course the energies and the wave functions of the ground and the first excited state respectively.

The interesting conclusion is that only the part of the barrier between $x=a_2$ and $x=1$ determines the leading logarithmic term of the splitting E_1-E_0, in the sense that we prove bounds of the form

$$\exp\left\{\frac{-4\int_{a_2}^{1}\sqrt{2V(x)}\,dx-h}{2\varepsilon^2}\right\}<E_1-E_0<\exp\left\{\frac{-4\int_{a_2}^{1}\sqrt{2V(x)}\,dx+h}{2\varepsilon^2}\right\}. \qquad (1.4)$$

On the other hand for the eigenfunctions ψ_0 and ψ_1 we get

$$\exp\left\{\frac{-2\int_{-a_2}^{a_2}\sqrt{2V(x)}\,dx-h''}{2\varepsilon^2}\right\}<\frac{\psi_0(1)}{\psi_0(-1)}\cong\left|\frac{\psi_1(-1)}{\psi_1(1)}\right|$$

$$<\exp\left\{\frac{-2\int_{-a_2}^{a_2}\sqrt{2V(x)}\,dx+h'}{2\varepsilon^2}\right\}, \qquad (1.5)$$

i.e. small local deformations are enough to produce exponential localization of the wave functions. h, h', h'' can be made as small as we like (compared with the integral) for ε sufficiently small. We show in addition that if the deformation is moved, for example, to the right beyond the point $x=1$ and beyond a critical distance a_1^*, the situation approaches for $\varepsilon\to 0$ the symmetric case in the sense that in (1.4) and (1.5) one can take $a_2=0$. In other words if the deformation is moved sufficiently far it does not influence the tunneling.

We next consider the effect of deformations of a potential which is symmetric and periodic over a finite interval (see Fig. 2). In the undeformed situation the particle tunnels through all the barriers and the wave function has equal maxima in correspondence of the minima of the potential. However, as soon as we lower one barrier only tunneling through the lower barrier is effective and the wave function is localized in the two wells at the sides of it.

As for the splitting of the levels we have a transition from a situation where a number of levels exponentially near to the ground state is present (the number of such levels is of the order of the number of barriers) to a situation which is close to the symmetric double well. We have therefore only one level at exponentially small distance from the ground state energy while the others are pushed up. We think that this type of results may be interesting in connection with problems in solid state physics. These results are discussed in Sects. 5 and 6.

We finally give some indications on the structure of the other parts of the paper where all the necessary techniques are developed. In Sect. 2 we discuss the ground state solution of the Riccati equation (1.2) as $\varepsilon \to 0$ for a certain class of potential functions. In Sect. 3 we adapt to our case a theorem of Ventzel [11] which in one dimension will permit us to reduce the study of the lowest part of the spectrum of $-L^\varepsilon$ to the study of a finite matrix q_{ij}^ε. In Sect. 4 we give estimates of the elements q_{ij}^ε as $\varepsilon \to 0$ using well-known explicit formulas for one-dimensional diffusion processes.

Applications of these techniques to excited states and multidimensional situations will be discussed in subsequent papers.

2. The Drift $b^\varepsilon(x)$

We begin by stating some hypotheses on the potential $V(x)$. These will certainly not be the most general and have been chosen so as to simplify as much as possible the exposition. The reason for doing so is that one of the main purposes of this paper is to illustrate in some nontrivial cases and in the clearest possible way the possibilities of a new interesting technique.

The potential V is assumed to be a real valued function on $\mathbb{R}$ such that:

1) $V(x) \geqq 0$ and $V \in C^\infty$.

2) V has a finite number of zeros: x_i, $i = 1, \ldots, N$ and $\lim\limits_{x \to x_i} \dfrac{V(x)}{(x - x_i)^2} = \omega_i^2$, that is V has quadratic minima.

3) There exists a real number $1 < \alpha < \infty$ such that $\lim\limits_{x \to +\infty} \dfrac{V'(x)}{x^\alpha}$ exists and it is finite and positive. This implies that there exists a positive constant c and a point x_0 such that: $V(x) > Cx^{\alpha+1}$ for $x > x_0$ and $\lim\limits_{x \to \infty} \dfrac{V'(x)}{\sqrt{2V(x)}} = +\infty$.

Similarly we assume that there exists a real number $1 < \beta < \infty$ such that $\lim\limits_{x \to -\infty} \dfrac{V'(x)}{x^\beta}$ exists and it is finite and negative.

Calling H the Friedrichs extension of the operator on $L^2(\mathbb{R}, dx)$, $H\varphi = \left(-\dfrac{\hbar^2}{2m} \cdot \dfrac{d^2}{dx^2} + V\right)\varphi$ for $\varphi \in C_0^\infty(\mathbb{R})$, from 1), 2), 3) it follows that:

a) The eigenfunctions of H are C^∞ functions and the spectrum of H is discrete and positive (see e.g. [10, 12]).

b) The ground state wave function ψ_0 is strictly positive and has at least an exponential fall off at infinity (see [10]).

c) If $E_0(\hbar)$ is the energy of the ground state, then there exists a constant $C > 0$ such that $\lim\limits_{\hbar \to 0} \dfrac{E(\hbar)}{\hbar} < C$; this follows from the mini-max principle using as a trial function the ground state wave function of one of the harmonic oscillators $\omega_i^2(x-x_i)^2$, $i=1,2,\dots N$.

d) Let ψ_n be an eigenfunction of H then ψ_n/ψ_0 grows at infinity at most like a power law: this result will be proved in Appendix A, as it will be needed only in Appendix B.

e) The operator $\frac{1}{\varepsilon^2}(H-E_0)$ on $L^2(\mathbb{R}, dx)$ is unitarily equivalent to the Friedrichs extension of the operator on $L^2(\mathbb{R}, \psi_0^2 dx)$, $-L^\varepsilon f=\left(\frac{\varepsilon^2}{2}\frac{d^2}{dx^2}+b^\varepsilon(x)\frac{d}{dx}\right)f$, $f\in C_0^\infty(\mathbb{R})$, where $\varepsilon^2=\hbar/m$, $\psi_0(x)$ is the ground state of H with eigenvalue $E_0(\hbar)$ and $b^\varepsilon(x)=\frac{\varepsilon^2}{2}\frac{d}{dx}\ln|\psi_0(x)|^2$. The unitary operator realizing the equivalence $U: L^2(\mathbb{R}, dx)\to L^2(\mathbb{R}, \psi_0^2(x)dx)$ is given by: $Uf=f/\psi_0$. We remark that this important result holds under much more general hypotheses on $V(x)$ (see [8, 13]).

In this section we give a qualitative analysis of the solution of the Riccati equation corresponding to the ground state ψ_0:

$$\frac{(b^\varepsilon(x))^2}{2}+\frac{\varepsilon^2}{2}\frac{db^\varepsilon(x)}{dx}=V(x)-E_0(\varepsilon) \tag{2.1}$$

for $\varepsilon\to 0$, where the potential satisfies the above hypotheses. Although some of the results of this section could be reduced to known facts in the theory of ordinary differential equations (see [14]), we prefer to give a direct proof of the main statements in order to keep the paper as self contained as possible.

Proposition 2.1. *If we define $x_N\equiv\max\{x_i;\ V(x_i)=0\}$ then on every compact interval $[c,d]\subset(x_N,+\infty)$ the solution $b^\varepsilon(x)$ of Eq. (2.1) tends uniformly to $-\sqrt{2V(x)}$ as $\varepsilon\to 0$. Analogously, if we define $x_1\equiv\min\{x_i;\ V(x_i)=0\}$ on every compact interval contained in $(-\infty, x_1)$, $b^\varepsilon(x)$ tends uniformly to $+\sqrt{2V(x)}$ as $\varepsilon\to 0$.*

Proof. First of all we observe that:

$$b^\varepsilon(x)=\varepsilon^2\frac{\psi_0'(x)}{\psi_0(x)}<0,\ \forall x\in[C,\infty) \tag{2.2}$$

for ε sufficiently small. In fact given $C>x_N$ there exists an $\varepsilon_0>0$ such that $V(x)>E_0(\varepsilon)$, $\forall x\in[C,\infty)$, $\forall\varepsilon<\varepsilon_0$; from the Schrödinger equation and from the positivity of ψ_0, if $V(x)-E_0(\varepsilon)>0$ then $\psi_0''(x)>0$, hence ψ_0' is monotonically increasing and therefore has to tend to 0^-.

We now turn to the proof of our first statement. It is obvious that $-\sqrt{2V(x)}$ is a first solution of the equation

$$(b^0(x))^2+\varepsilon^2\frac{db^0(x)}{dx}=2V(x)-\varepsilon^2\frac{V'(x)}{\sqrt{2V(x)}},\qquad x\in[C,+\infty). \tag{2.3}$$

Exploiting the quadratic structure of the Riccati equation it is possible to study the difference $b^\varepsilon(x)-b^0(x)$ between a solution of (2.1) and a solution of (2.3), in terms of their sum $b^\varepsilon(x)+b^0(x)$.

We have in fact that

$$[b^\varepsilon(x)-b^0(x)][b^\varepsilon(x)+b^0(x)]=(b^\varepsilon(x))^2-(b^0(x))^2$$
$$=-2E_0(\varepsilon)+\varepsilon^2\frac{V'(x)}{\sqrt{2V(x)}}+\varepsilon^2\frac{d}{dx}[b^0(x)-b^\varepsilon(x)]\,. \tag{2.4}$$

Solving with respect to $b^0(x)-b^\varepsilon(x)$, we find

$$b^0(x)-b^\varepsilon(x)=F(x)\exp\left\{-\int_{x_0}^{x}(b^0(x')+b^\varepsilon(x'))dx'/\varepsilon^2\right\}, \tag{2.5}$$

where

$$F(x)-F(x_0)=\int_{x_0}^{x}\left(\frac{2E_0}{\varepsilon^2}-\frac{V'(x')}{\sqrt{2V(x')}}\right)\exp\left\{\int_{x_0}^{x'}(b^0(x'')+b^\varepsilon(x''))dx''/\varepsilon^2\right\}dx'\,. \tag{2.6}$$

By the assumption 3) on the derivative $V'(x)$, we have that $F'(x)$ for $x\to+\infty$ becomes eventually negative, that is $\lim\limits_{x\to+\infty}F(x)$ exists, possibly $-\infty$.

Let us now choose $b^0(x)=-\sqrt{2V(x)}$ and use (2.2). It is easy to see that in this case we must impose $\lim\limits_{x\to+\infty}F(x)=0$, otherwise from (2.5) $|\sqrt{2V(x)}+b^\varepsilon(x)|$ would diverge at infinity at least exponentially, but this is forbidden by the Riccati equation (2.1).

In fact if we put in (2.1) $b^\varepsilon(x)=-\exp\{f(x,\varepsilon)\}$, $x\in[C,\infty)$ with $f(x,\varepsilon)$ such that $\lim\limits_{x\to+\infty}\frac{\exp\{f(x,\varepsilon)\}}{x^n}=+\infty$, $\forall n<\infty$ we obtain for x sufficiently large and fixed ε:

$$\varepsilon^2\frac{df}{dx}(x,\varepsilon)\exp\{f(x,\varepsilon)\}$$
$$=\exp\{2f(x,\varepsilon)\}-2[V(x)-E_0]>\exp\{2f(x,\varepsilon)\}(1-\bar{C})$$

with $\bar{C}<1$; by an integration

$$\exp\{-f(x,\varepsilon)\}<\exp\{-f(x_0,\varepsilon)\}-\frac{C'}{\varepsilon^2}(x-x_0)\,,\qquad C'>0\,.$$

But for large x this inequality cannot clearly be satisfied.

We have thus that: $\lim\limits_{x\to+\infty}F(x)=0$ that is, taking now $x_0=C$,

$$F(C)=-\int_{C}^{+\infty}\left(\frac{2E_0(\varepsilon)}{\varepsilon^2}-\frac{V'(x')}{\sqrt{2V(x')}}\right)\exp\left\{\int_{C}^{x'}(b^\varepsilon(x'')+b^0(x''))dx''/\varepsilon^2\right\}dx'\,. \tag{2.7}$$

With this choice of $F(C)$, Eq. (2.5) becomes:

$$\sqrt{2V(x)}+b^\varepsilon(x)=\int_{x}^{+\infty}\left(\frac{2E_0(\varepsilon)}{\varepsilon^2}-\frac{V'(x')}{\sqrt{2V(x')}}\right)\exp\left\{\int_{x}^{x'}(b^\varepsilon(x'')+b^0(x''))dx''/\varepsilon^2\right\}dx'\,. \tag{2.8}$$

If we use the a priori estimate (2.2) on $b^\varepsilon(x)$ we obtain:

$$|\sqrt{2V(x)}+b^\varepsilon(x)|<\int_{x}^{\infty}\left|\frac{2E_0(\varepsilon)}{\varepsilon^2}-\frac{V'(x')}{\sqrt{2V(x')}}\right|\exp\left\{-\int_{x}^{x'}\sqrt{2V(x'')}dx''/\varepsilon^2\right\}dx'\,. \tag{2.9}$$

Some simple computations now show that

$$|\sqrt{2V(x)}+b^{\varepsilon}(x)|<C_{cd}\varepsilon^{\gamma},\ \forall\gamma;\quad 0<\gamma<2$$

uniformly in [c, d]. The second part of the proposition goes in the same way.

Remark. Equation (2.8) gives an estimate on b^{ε} stronger than (2.2):

$$b^{\varepsilon}(x)<-\sqrt{2V(x)},\ \forall x\in(a,+\infty), \tag{2.10}$$

where a is such that

$$\frac{V'(x)}{\sqrt{2V(x)}}>\frac{2E_0}{\varepsilon^2},\ \forall x\in(a,+\infty).$$

This point a always exists for the hypothesis 3) on $V'(x)$.

We now prove some properties of $b^{\varepsilon}(x)$, which allow us to study its behaviour between the points x_1 and x_N.

Proposition 2.2. (i) *Let $[p,q]$ be an interval such that $V'(x)>0(<0)$, $\forall x\in[p,q]$; then in $[p,q]$ there is at most one minimum (maximum) of $b^{\varepsilon}(x)$.*

(ii) *Let $[p,q]\subset\mathbb{R}$ be a finite closed interval containing in its interior the points x_1, x_N; then given $\eta>0$ for ε sufficiently small, $|b^{\varepsilon}(x)|$ is bounded by:*

$$\max_{y\in[p,q]}\sqrt{2\mathrm{V}(\mathrm{y})}+\eta,\ \forall x\in[\mathrm{p},\mathrm{q}].$$

(iii) *Between two consecutive zeros of $V(x)$ there exists at most one point y^{ε} such that $b^{\varepsilon}(y^{\varepsilon})=0$ and $\left.\dfrac{db^{\varepsilon}(x)}{dx}\right|_{x=y^{\varepsilon}}>0$.*

(iv) *Let us fix ε and suppose that between two consecutive zeros of $V(x)$, say x_i, x_{i+1}, there is a point y_i^{ε} defined as in* (iii); *then for any $x\in[y_i^{\varepsilon}+\varepsilon^{\gamma},x_{i+1}]$:*

$$|b^{\varepsilon}(x)-\sqrt{2V(x)}|<C\varepsilon^{\gamma}$$

for some constant C and any $0<\gamma<2$. Analogously for all $x\in[x_i,y_i^{\varepsilon}-\varepsilon^{\gamma}]$:

$$|b^{\varepsilon}(x)+\sqrt{2V(x)}|<C'\varepsilon^{\gamma}$$

for some constant $C'>0$ and any γ, $0<\gamma<2$.

(v) *If there is no such point y_i^{ε} between x_i and x_{i+1} then:*

$$|b^{\varepsilon}(x)\pm\sqrt{2V(x)}|<C''\varepsilon^{\gamma},$$

for any $x\in[x_i+\varepsilon^{\gamma},x_{i+1}-\varepsilon^{\gamma}]$, some constant $C''>0$ and any $0<\gamma<2$. The choice of the sign of the square root of $V(x)$ is determined by the specific form of the function $V(x)$.

Proof. (i) By the derivative of the Riccati equation:

$$b^{\varepsilon}(x)b^{\varepsilon\prime}(x)+\frac{\varepsilon^2}{2}b^{\varepsilon\prime\prime}(x)=V'(x), \tag{2.11}$$

if $b^{\varepsilon\prime}(x)=0$, then $b^{\varepsilon\prime\prime}(x)=\dfrac{2V'(x)}{\varepsilon^2}$. The statement follows if in $[p,q]$ the sign of $V'(x)$ is constant.

(ii) From the Riccati equation (2.1) we know that whenever $b^\varepsilon(x)$ has an extremum then $|b^\varepsilon(x)|=\sqrt{2V(x)-2E_0}$. Besides $b^\varepsilon(p)\to+\sqrt{2V(p)}$, $b^\varepsilon(q)\to-\sqrt{2V(q)}$ as $\varepsilon\to 0$ (by Proposition 2.1).

(iii) We first remark that by the Riccati equation $b^\varepsilon(x)$ can have a zero with negative derivative only for those x such that $V(x)<E_0(\varepsilon)$. Let us now take two consecutive zeros of $V(x)$, say x_i and x_{i+1}, and the corresponding two turning points $\bar{x}_i^\varepsilon$, $\bar{x}_{i+1}^\varepsilon$ [2] between them, and suppose that there exists a point $y_i^\varepsilon\in(x_i,x_{i+1})$ such that $b^\varepsilon(y_i^\varepsilon)=0$, $b^{\varepsilon\prime}(y_i^\varepsilon)>0$. By the previous remark $b^\varepsilon(x)>0$, $\forall x\in(y_i^\varepsilon,\bar{x}_{i+1}^\varepsilon)$. Analogously $b^\varepsilon(x)<0$, $\forall x\in(\bar{x}_i^\varepsilon,y_i^\varepsilon)$.

(iv) We compare, as in Proposition 2.1, the solutions of the two equations:

$$(b^\varepsilon(x))^2+\varepsilon^2 b^{\varepsilon\prime}(x)=2(V(x)-E_0)$$

$$(b^0(x))^2+\varepsilon^2 b^{0\prime}(x)=2V(x)+\frac{\varepsilon^2 V'(x)}{\sqrt{2V(x)}},$$

with initial conditions $b^\varepsilon(y_i^\varepsilon)=0$, $b^0(y_i^\varepsilon)=+\sqrt{2V(y_i^\varepsilon)}$, in the interval $(y_i^\varepsilon,x_{i+1})$. From the above equations we obtain:

$$\begin{aligned} b^\varepsilon(x)-b^0(x)= & -\sqrt{2V(y_i^\varepsilon)}\exp\left\{-\int_{y_i^\varepsilon}^{x}(b^\varepsilon(x')+b^0(x'))dx'/\varepsilon^2\right\}\\ & +-\int_{y_i^\varepsilon}^{x}\left(\frac{E_0(\varepsilon)}{\varepsilon^2}+\frac{V'(x')}{\sqrt{2V(x')}}\right)\exp\left\{-\int_{x'}^{x}(b^\varepsilon(x'')+b^0(x''))dx''/\varepsilon^2\right\}dx'. \end{aligned} \tag{2.12}$$

Using now the explicit expression, for $b^0(x)$, $b^0(x)=+\sqrt{2V(x)}$, the estimate $b^\varepsilon(x)>0$, $\forall x\in(y_i^\varepsilon,\bar{x}_i^\varepsilon)$ (see iii)] we get:

$$|b^\varepsilon(x)-\sqrt{2V(x)}|<C\varepsilon^\gamma,\ \forall x\in[y_i^\varepsilon+\varepsilon^\gamma,x_{i+1}]$$

for some $C>0$ and any $0<\gamma<2$.

Similarly:

$$|b^\varepsilon(x)+\sqrt{2V(x)}|<C'\varepsilon^\gamma,\ \forall x\in[x_i,y_i^\varepsilon-\varepsilon^\gamma]$$

for some $C'>0$ and any $0<\gamma<2$

(v) In this case the proof follows the previous one iv) with the difference that the proper intial condition for $b^\varepsilon(x)$ is unknown.

But this is not important for the estimate we need because from ii) we know that in any case $b^\varepsilon(x_i)$ is bounded by some ε-independent constant.

Summarizing from Propositions 2.1 and 2.2 we can conclude that for ε small enough, $|b^\varepsilon(x)\pm\sqrt{2V(x)}|<\bar{C}\varepsilon^\gamma$ in the complement of ε^γ-neighborhoods of the points y_i^ε with $i\leqq N-1$, for some constant $\bar{C}$ and any $0<\gamma<2$. Proposition 2.2 however, does not exclude the possibility that, in general situations the limit for $\varepsilon\to 0$ of the points y_i^ε may not exist. We now describe a technique which solves completely this problem in situations of practical interest and allows us to compute explicitly the limit y_i^0. In other words, as we shall see in Sects. 5 and 6, one is able to prove that $\lim_{\varepsilon\to 0} b^\varepsilon(x)=\pm\sqrt{2V(x)}$ in the complement of the points y_i^0.

2 $\bar{x}_i^\varepsilon$ is called a turning point if $V(\bar{x}_i^\varepsilon)=E_0(\varepsilon)$

We choose an arbitrary point x_0 and we consider the symmetric and the antisymmetric parts of $b^\varepsilon(x)$ with respect to the point x_0. For simplicity we put the origin of the x-axis in x_0.

Exploiting again the quadratic structure of the Riccati equation we can express the symmetric part of $b^\varepsilon(x)$, $b^\varepsilon(x)+\bar{b}^\varepsilon(x)$ $(\bar{b}^\varepsilon(x)=b^\varepsilon(-x))$, in terms of the antisymmetric one $b^\varepsilon(x)-\bar{b}^\varepsilon(x)$. In fact the following nonhomogeneous differential equation holds:

$$\frac{d}{dx}[b^\varepsilon(x)+\bar{b}^\varepsilon(x)] = -\frac{1}{\varepsilon^2}[b^\varepsilon(x)+\bar{b}^\varepsilon(x)][b^\varepsilon(x)-\bar{b}^\varepsilon(x)] + \frac{2}{\varepsilon^2}\Delta V(x), \tag{2.13}$$

where $\Delta V(x) \equiv V(x) - V(-x)$. Equation (2.13) can be solved explicitly, and we get:

$$b^\varepsilon(x)+\bar{b}^\varepsilon(x) = F(x)\exp\left\{-\int_0^x (b^\varepsilon(x')-\bar{b}^\varepsilon(x'))dx'/\varepsilon^2\right\} \tag{2.14}$$

with

$$F(x)-F(0) = \int_0^x \frac{2}{\varepsilon^2}\Delta V(x')\exp\left\{\int_0^{x'}(b^\varepsilon(x'')-\bar{b}^\varepsilon(x''))dx''/\varepsilon^2\right\}dx'.$$

From the previous general discussion on the behaviour of $b^\varepsilon(x)$, as $|x|\to\infty$, it is clear that $b^\varepsilon(x)+\bar{b}^\varepsilon(x)$ cannot go to infinity exponentially and this implies that the constant $F(0)$ must be chosen equal to

$$-\int_0^\infty \frac{2}{\varepsilon^2}\Delta V(x')\exp\left\{\int_0^{x'}(b^\varepsilon(x'')-\bar{b}^\varepsilon(x''))dx''/\varepsilon^2\right\}dx'.$$

We have thus the following form for Eq. (2.14)

$$b^\varepsilon(x)+\bar{b}^\varepsilon(x) = -\int_x^\infty \frac{2}{\varepsilon^2}\Delta V(x') \cdot\exp\left\{\int_x^{x'}(b^\varepsilon(x'')-\bar{b}^\varepsilon(x''))dx''/\varepsilon^2\right\}dx'. \tag{2.15}$$

By using this formula we prove the following:

Proposition 2.3. a) *Let us suppose that the potential $V(x)$ satisfies the following additional assumptions:*

(i) *there exists a point $x_0\in(x_1,x_N)$ (x_i is the i^{th} zero of $V(x)$), such that*

$$V(x)-V(2x_0-x)\geqq 0, \forall x\in(x_0,+\infty).$$

(ii) *The open set $I\equiv\{x>x_0;\ (V(x)-V(2x_0-x))>0\}$ is nonempty; I will be in general a (possibly infinite) union of disjoint open intervals (a_i,b_i), $x_0\leqq a_i\leqq b_i\leqq +\infty$.*

Then $\lim_{\varepsilon\to 0} b^\varepsilon(x) = -\sqrt{2V(x)}$, $\forall x\in I$.

b) *Analogously if $V(x)-V(2x_0-x)\leqq 0$ $\forall x>x_0$ and if $\bar{I}\equiv\{x<x_0;\ (V(x)-V(2x_0-x))>0\}$ is nonempty, then* $\lim_{\varepsilon\to 0} b^\varepsilon(x) = +\sqrt{2V(x)}$, $\forall x\in\bar{I}$.

Proof. a) We consider the function $b^\varepsilon(x)+b^\varepsilon(2x_0-x)$, e.g. the symmetric part of b^ε with respect to x_0. By Eq. (2.15) with inversion point x_0 we get

$$b^\varepsilon(x)+b^\varepsilon(2x_0-x)<0,\ \forall x\in I.$$

We will show that this implies

$$\lim_{\varepsilon\to 0} b^\varepsilon(x)=-\sqrt{2V(x)},\ \forall x\in I.$$

Let us begin by examining $b^\varepsilon(x)$ in the interval (a_1,b_1). If $a_1>x_N$ then the proposition follows from Proposition 2.1. If $b_1<x_N$ and $V(x)>0$, $\forall x\in(a_1, b_1)$, then it is easy to show that, if $(a^\varepsilon, b_1)\subsetneqq(a_1,b_1)$ is the set where $b^\varepsilon(x)$ is strictly positive, then $a^\varepsilon\to b_1$ as $\varepsilon\to 0$. In fact if for any $\varepsilon_0>0$ there exists an $\bar\varepsilon<\varepsilon_0$ such that $(b_1-a^{\bar\varepsilon})>C$ for some ε_0-independent constant C, then by Proposition 2.2 (v), in the interval $(a^{\bar\varepsilon}, b_1)$ there exists a point such that $b^{\bar\varepsilon}(x)>\sqrt{2V(x)}+0(\bar\varepsilon)$. From this we would get for such points x:

$$b^\varepsilon(x)+b^\varepsilon(2x_0-x)>\sqrt{2V(x)}-\sqrt{2V(2x_0-x)}+O(\varepsilon_0)>0$$

for ε_0 sufficiently small, and this is forbidden by (2.15). We have thus proved that $\lim_{\varepsilon\to 0} a^\varepsilon=b_1$ and this, again with the help of Proposition 2.2, is sufficient to conclude that:

$$\lim_{\varepsilon\to 0} b^\varepsilon(x)=-\sqrt{2V(x)},\ \forall x\in(a_1,b_1).$$

The proof for the remaining intervals $(a_i,b_i)\subset I$ is just the same. Part b) is proved in an analogous way.

3. Reduction of the Study of L^ε to the Study of a Finite Matrix

This section follows essentially the paper by Ventzel [11] with the difference that Ventzel treats processes with their semigroup acting on the Banach space of bounded continuous functions, while we are dealing with a diffusion process possessing an invariant measure $\psi_0^2(x)dx$ with its semigroup P^t acting on $L_2(\psi_0^2 dx)$. We recall that the action of the semigroup generated by $-L^\varepsilon$ can be defined by:

$$(\exp(L^\varepsilon t)f)(x)\equiv(P^t f)(x)=M_x^\varepsilon f(x_t),$$

where M_x^ε denotes the expectation over the diffusion process X_t^ε starting at x generated by L^ε, i.e. the solution of the stochastic differential equation (1.1). We are interested in the eigenvalues and eigenfunctions of $-L^\varepsilon$ defined by the equation:

$$(P^t f_a^\varepsilon)(x)=\exp(-at)\,f_a^\varepsilon(x),$$

where f_a^ε is in $L_2(\psi_0^2 dx)$. We will refer to f_a^ε as an a-eigenfunction of $-L^\varepsilon$.

Let E and D be two finite unions of points such that $E\cap D=\emptyset$. Define $\tau_E(\tau_D)$ to be the hitting time of the diffusion process X_t^ε to the set $E(D)$. Clearly both τ_E and τ_D are Markov times [15]. Now let τ_1 be the Markov time defined as follows:

$$\tau_1=\tau_D+\theta_{\tau_D}(\tau_E),$$

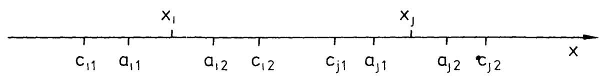

Fig. 3

where θ_t is the usual shift over the process X_t^ε [15]. τ_1 is the time the process X_t^ε completes the cycle of motion taking it for the first time to D and from there subsequently to E.

Let M be the number of points of E; we then define the $M \times M$ matrix:

$$(q_a^\varepsilon)_{\bar{x},\bar{y}} = M_{\bar{x}}^\varepsilon\{\exp(a\tau_1)\,\chi(X_{\tau_1}=\bar{y})\},\ \bar{x},\bar{y}\in E\,,$$

where $\chi(A)$ is the characteristic function of the set A, provided

$$M_x^\varepsilon\{\exp(a\tau_1)\} < \infty, \forall x \in E\,.$$

Theorem. *Let $a > 0$ be such that $\|M_x^\varepsilon \exp(2a\tau_1)\|_{L_2(\psi_0^2)} < \infty$. Then there is a one-to-one correspondence between the a-eigenfunctions of $-L^\varepsilon$ and the eigenvectors of $(q_a^\varepsilon)_{ij}$ with eigenvalue* 1.

The proof, given in Appendix B for completeness, follows step by step that given by Ventzel in [11]. The interest of the above theorem resides in the fact that one can take advantage of the arbitrariness of the sets E and D. Of course there is a price one has to pay in going from L^ε to $(q_a^\varepsilon)_{ij}$ due to the condition $\|M_x^\varepsilon \exp(2a\tau_1)\|_{L_2(\psi_0^2)} < \infty$ which implies an upper bound for a.

In Appendix C we will show that the upper bound for a is large enough for our purposes.

4. Probabilistic Estimates

Our main purpose is to provide an estimate as $\varepsilon \to 0$ of the elements of the matrix q_a^ε defined in the previous section for a particular choice of the sets E and D. These sets are defined as follows: for each zero of the potential $V(x)$, x_i, $i = 1, \ldots, N$, we take two closed neighborhoods E_i and D_i such that $x_i \in E_i \subset D_i$, $D_i \cap D_j = \emptyset$ if $i \neq j$; let γ_i and Γ_i be the boundaries of E_i and D_i respectively, $\gamma_i = a_{i1} \cup a_{i2}$, $\Gamma_i = c_{i1} \cup c_{i2}$, then we define $E = \bigcup_{i=1}^N \gamma_i$, $D = \bigcup_{i=1}^N \Gamma_i$.

The situation is summarized in Fig. 3.

With the above definition of E and D we have

$$(q_a^\varepsilon)_{\bar{x},\bar{y}} = M_{\bar{x}}^\varepsilon(\exp(a\tau_1)\,\chi(x_{\tau_1}=\bar{y})\,, \tag{4.1}$$

where $\bar{x} = a_{ik}$, $\bar{y} = a_{j\ell}$, $i, j = 1, \ldots, N$; $k, \ell = 1, 2$ and the parameter $a \geqq 0$ must be such that:

$$\|M_x^\varepsilon(\exp(2a\tau_1))\|_{L_2(\psi_0^2 dx)} < +\infty\,.$$

Using now the Hölder inequality we have:

$$p_{\bar{x}}^\varepsilon(x_{\tau_1}=\bar{y}) \leqq (q_a^\varepsilon)_{\bar{x}\bar{y}} \leqq [M_{\bar{x}}^\varepsilon \exp(a\kappa\tau_1)]^{1/\kappa} \cdot [p_{\bar{x}}^\varepsilon(x_{\tau_1}=\bar{y})]^{\frac{\kappa-1}{\kappa}}\,, \tag{4.2}$$

where κ is an arbitrary constant > 1. To compute $p_{\bar{x}}^\varepsilon(x_{\tau_1}=\bar{y})$ we have to distinguish different cases. First of all if the points $\bar{x} = a_{ik}$, $\bar{y} = a_{j\ell}$ are such that $|i-j| > 1$ then by the definition of τ_1 $p^\varepsilon_{\bar{x}}(x_{\tau_1}=\bar{y})=0$. We suppose then that $\bar{x}$ and $\bar{y}$ are nearest neighbours that is $|i-j|\leqq 1$, $k \neq \ell$. If $i=j$ using the strong Markov property we have the decomposition:

$$p^\varepsilon_{a_{i1}}(x_{\tau_1}=a_{i2})=p^\varepsilon_{a_{i1}}(c_{i2},c_{i1})\,p^\varepsilon_{c_{i2}}(a_{i2},a_{i+1,1})$$
$$p^\varepsilon_{a_{i2}}(x_{\tau_1}=a_{i1})=p^\varepsilon_{a_{i2}}(c_{i1},c_{i2})\,p^\varepsilon_{c_{i1}}(a_{i1},a_{i-1,2}),$$

where $p^\varepsilon_x(y,z)$ is the probability that x^ε_t starting at $x\in[y,z]$ exits at y.

If $\bar{x}=\bar{y}$ the analogous decomposition reads:

$$p^\varepsilon_{a_{i1}}(x_{\tau_1}=a_{i1})=p^\varepsilon_{a_{i1}}(c_{i1},c_{i2})p^\varepsilon_{c_{i1}}(a_{i1},a_{i-1,2})$$

and similarly for a_{i2}. If $i=1$: $p^\varepsilon_{a_{1k}}(x_{\tau_1}=a_{11})=p^\varepsilon_{a_{1k}}(c_{11},c_{12})$, analogously if $i=N$: $p^\varepsilon_{a_{Nk}}(x_{\tau_1}=a_{N2})=p^\varepsilon_{a_{Nk}}(c_{N2},c_{N1})$. Finally if $j-i=1$: $p^\varepsilon_{a_{i2}}(x_{\tau_1}=a_{j1})=p^\varepsilon_{a_{i2}}(c_{i2},\ c_{i1})$ $p^\varepsilon_{c_{i2}}(a_{j1},\ a_{i2})$, and similarly if $i-j=1$. This decomposition is useful because the probabilities $p^\varepsilon_x(y,z)$ are solutions of the differential equation: $\varepsilon^2/2\dfrac{d^2}{dx^2}p^\varepsilon_x+b^\varepsilon(x)\dfrac{d}{dx}$ $p^\varepsilon_x=0$ with boundary conditions $p^\varepsilon_y(y,z)=1$, $p^\varepsilon_z(y,z)=0$ and are given by the simple formula [16]:

$$p^\varepsilon_x(y,z)=\int_x^z \phi^\varepsilon(x')\,dx'\Big/\int_y^z \phi^\varepsilon(x')dx', \tag{4.3}$$

where

$$\phi^\varepsilon(x')=\exp\left\{-\int_y^{x'} 2/\varepsilon^2 b^\varepsilon(x'')dx''\right\}.$$

We now assume that $\lim_{\varepsilon\to 0} b^\varepsilon(x)=\pm\sqrt{2V(x)}$ in the complement of finitely many points y_i^0, where $b^0(x)$ jumps from $+\sqrt{2V(y_i^0)}$ to $+\sqrt{2V(y_i^0)}$. This allows estimation of the integrals appearing in (4.3). It is enough to make the computation for $j-i=1$ as all the other cases are obtained taking complementary probabilities. By taking now the width of the intervals $[a_{i1},a_{i2}]$, $[c_{i1},c_{i2}]$ equal to $\delta/2$ and δ respectively, it is not difficult to verify (see Appendix D) that:

$$\exp\{(-V_{ij}-h(\varepsilon,\delta))/2\varepsilon^2\}\leqq p^\varepsilon_{a_{i2}}(x_{\tau_1}=a_{j1})\leqq\exp\{(-V_{ij}+h(\varepsilon,\delta))/2\varepsilon^2\} \tag{4.4}$$

with $V_{ij}=4\int_{x_i}^{y_i^0}\sqrt{2V(x)}\,dx\equiv 4\int_{x_i}^{y_i^0}|b^0(x)|\,dx$ where y_i^0 is the jump point between x_i and x_j, i.e. the integral has to be extended to the region between x_i and x_j where $b^0(x)\leqq 0$; $h(\varepsilon,\delta)$ is an error arbitrarily small for ε and δ sufficiently small; $h(\varepsilon,\delta)$ includes the error coming both from the approximation on $b^\varepsilon(x)$ and from the estimate of the integrals. We now need estimates for $\varepsilon\to 0$ of $M^\varepsilon_{\bar{x}}\exp(a\kappa\tau_1)$, $\bar{x}\in E$, appearing in (4.2). First of all we have the obvious inequality:

$$M^\varepsilon_{\bar{x}}(\exp(a\kappa\tau_1))\geqq 1+a\kappa M^\varepsilon_{\bar{x}}\tau_D\geqq 1+a\kappa t_0 p^\varepsilon_{\bar{x}}(\tau_D>t_0),\ \bar{x}\in E,$$

where τ_D is the hitting time defined in the previous section, t_0 being an arbitrary time. From Appendix E, we have:

$$p^\varepsilon_{\bar{x}}(\tau_D>t_0)\geqq 1-\exp\{-h'(\varepsilon,\delta,t_0)/2\varepsilon^2\},$$

where $h'(\varepsilon, \delta, t_0)$ is again as small as we like for ε and δ sufficiently small. Therefore:

$$M_{\bar{x}}^{\varepsilon}\{\exp(a\kappa\tau_1)\} \geqq 1 + a\kappa(1 - \exp\{-h'(\varepsilon, \delta, t_0)/2\varepsilon^2\})t_0 . \tag{4.5}$$

To obtain an upper bound we apply the strong Markov property:

$$M_{\bar{x}}^{\varepsilon}(\exp(a\kappa\tau_1)) \leqq \sup_{\bar{x}\in E} M_{\bar{x}}^{\varepsilon}(\exp(a\kappa\tau_D)) \sup_{\bar{y}\in D} M_{\bar{y}}^{\varepsilon}(\exp(a\kappa\tau_E)) . \tag{4.6}$$

Let us estimate for example the first factor, the calculation for the other one being identical. Let t_0 be an arbitrary time, then:

$$\begin{aligned}
\sup_{\bar{x}\in E} M_{\bar{x}}^{\varepsilon}(\exp(a\kappa\tau_D)) &= \sup_{\bar{x}\in E} \int_0^\infty \exp(a\kappa t)\, dp_{\bar{x}}^{\varepsilon}(\tau_D \leqq t) \\
&= \sup_{\bar{x}\in E} \sum_{0=n}^{\infty} \int_{nt_0}^{(n+1)t_0} e^{a\kappa t} dp_{\bar{x}}^{\varepsilon}(\tau_D \leqq t) \\
&\leqq \sup_{\bar{x}\in E} \sum_{0=n}^{\infty} \exp(a(n+1)\kappa t_0) \\
&\quad \cdot \{p_{\bar{x}}^{\varepsilon}(\tau_D \leqq (n+1)t_0) - p_{\bar{x}}^{\varepsilon}(\tau_D \leqq nt_0)\} \\
&= \sup_{\bar{x}\in E} \sum_{0=n}^{\infty} \exp(a\kappa(n+1)t_0) \\
&\quad \cdot p_{\bar{x}}^{\varepsilon}(\tau_D > nt_0) - p_{\bar{x}}^{\varepsilon}(\tau_D > (n+1)t_0)\} \\
&\leqq \exp(a\kappa t_0) + \{\exp(a\kappa t_0) - 1\} \\
&\quad \cdot \sum_{1=n}^{\infty} \exp(n\kappa a t_0) \left[\sup_{\bar{x}\in E} p_{\bar{x}}^{\varepsilon}(\tau_D > t_0)\right]^n ,
\end{aligned}$$

and this sum converges provided $\exp\{a\kappa t_0\} \sup_{\bar{x}\in E} p_{\bar{x}}^{\varepsilon}(\tau_D > t_0) < 1$. By summing the series and doing some other obvious majorizations

$$\sup_{\bar{x}\in E} M_{\bar{x}}^{\varepsilon} \exp\{a\kappa\tau_D\} \leqq 1 + a\kappa t_0 \bigg/ \left\{1 - a\kappa t_0 - \sup_{\bar{x}\in E} p_{\bar{x}}^{\varepsilon}(\tau_D > t_0)\right\} ,$$

provided $a\kappa t_0 < \left\{1 - \sup_{\bar{x}\in E} p_{\bar{x}}^{\varepsilon}(\tau_D > t_0)\right\} C$, $C < 1$.

Using again the result of Appendix E

$$p_{\bar{x}}^{\varepsilon}(\tau_D > t_0) = 1 - p_{\bar{x}}^{\varepsilon}(\tau_D \leqq t_0) \leqq 1 - \exp\{-h''(\varepsilon, \delta)/2\varepsilon^2\},\ \forall \bar{x} \in E .$$

In conclusion we get:

$$\sup_{\bar{x}\in E} M_{\bar{x}}^{\varepsilon}(\exp\{a\kappa\tau_D\}) \leqq 1 + \frac{a\kappa t_0}{1 - C} \exp\{h''(\varepsilon, \delta)/2\varepsilon^2\} ,$$

if $a\kappa t_0 < C \exp\{-h''/2\varepsilon^2\}$.

The same holds for $\sup_{\bar{y}\in D} M_{\bar{y}}^{\varepsilon} \exp(a\kappa\tau_E)$. If we choose the Hölder constant κ appearing in (4.2) in such a way that $p_{\bar{x}}^{\varepsilon}(x_{\tau_1} = \bar{y})^{(\kappa-1)/\kappa}$ is still of the form (4.4), that is $\kappa \sim 1/h(\varepsilon, \delta)$ which in turn implies $a < h \exp\{-h''/2\varepsilon^2\}$, we finally have for the elements of the matrix q_a^ε the following inequalities:

$$\exp\{(-V_{ij}-\bar{h}(\varepsilon,\delta))/2\varepsilon^2\} \leqq (q_a^\varepsilon)_{a_{ik},a_{jl}} \leqq \exp\{(-V_{ij}+\bar{h}(\varepsilon,\delta))/2\varepsilon^2\}. \tag{4.7}$$

For $|i-j|=1$ where $\bar{h}(\varepsilon,\delta)$ is slightly bigger than $h(\varepsilon,\delta)$

$$\begin{aligned} 1+a\exp\{-\bar{h}'(\varepsilon,\delta)/2\varepsilon^2\} &\leqq \sum_{\ell=1}^{2}\sum_{j=1}^{N} q_a^\varepsilon(a_{ik},a_{j\ell}) \\ &= M_{a_{ik}}^\varepsilon e^{a\tau_1} < 1+a\exp\{\bar{h}''/2\varepsilon^2\}. \end{aligned} \tag{4.8}$$

Now using these inequalities it is possible to estimate in concrete cases the values of a for which 1 is in the spectrum of q_a^ε, i.e. to estimate the eigenvalues of $-L^\varepsilon$.

5. Double Well Potentials

In this section we consider some typical tunneling problems in which it is clear how the previous techniques work in view of the determination of the drift term $b^\varepsilon(x)$ and the calculation of the main logarithmic term of the splitting of the ground state. The simplest situation is provided by the symmetric double well potential. In this case, by symmetry reasons, the drift $b^\varepsilon(x)$ has a unique "jump" in the symmetry point of the potential $V(x)$. The precise form of the function $V(x)$ is not important, however, for definiteness, we shall refer to the function $V(x)=\varrho(1-x^2)^2$. In this case by Proposition 2.2 we have:

$$\lim_{\varepsilon\to 0} b^\varepsilon(x) = +\sqrt{2\varrho}(1-x^2),\ \forall x\in(0,1],\ \lim_{\varepsilon\to 0} b^\varepsilon(x) = -\sqrt{2\varrho}(1-x^2),\ \forall x\in[-1,0).$$

We then construct following Sect. 4 the 4×4 matrix q_a^ε which, according to (4.7), is determined by the quantities $V_{-1,+1}$, $V_{+1,-1}$. Due to the symmetry of the problem:

$$V_{+1,-1}=V_{-1,+1}=4\int_{-1}^{0}\sqrt{2V(x)}\,dx=\tfrac{8}{3}(2\varrho)^{1/2}.$$

By applying now the estimates (4.7), (4.8) to the equation $\det(q_a^\varepsilon-I)=0$ where I is the 4×4 identity matrix, we obtain for the values of a such that $0<a<\exp\{-h/2\varepsilon^2\}$ and $1\in\sigma(q_a^\varepsilon)$

$$\exp\{(-\tfrac{8}{3}(2\varrho)^{1/2}-h)/2\varepsilon^2\} \leqq a \leqq \exp\{(-\tfrac{8}{3}(2\varrho)^{1/2}+h)/2\varepsilon^2\}, \tag{5.1}$$

or

$$\varepsilon^2\exp\{(-\tfrac{8}{3}(2\varrho)^{1/2}-h)/2\varepsilon^2\} \leqq E_1-E_0 \leqq \varepsilon^2\exp\{(-\tfrac{8}{3}(2\varrho)^{1/2}+h)/2\varepsilon^2\} \tag{5.2}$$

for the splitting of the ground state energy. Here, as well as in the next examples, $h=h(\varepsilon,\delta)$ is a positive constant depending on ε and δ, where δ is the width of the sets γ_i, Γ_i from which the matrix q_a^ε was constructed (see Sect. 4), which is arbitrarily small for ε and δ sufficiently small.

A more interesting situation can be obtained if we consider an asymmetric double well potential obtained by a C^∞ deformation of the previous case such that

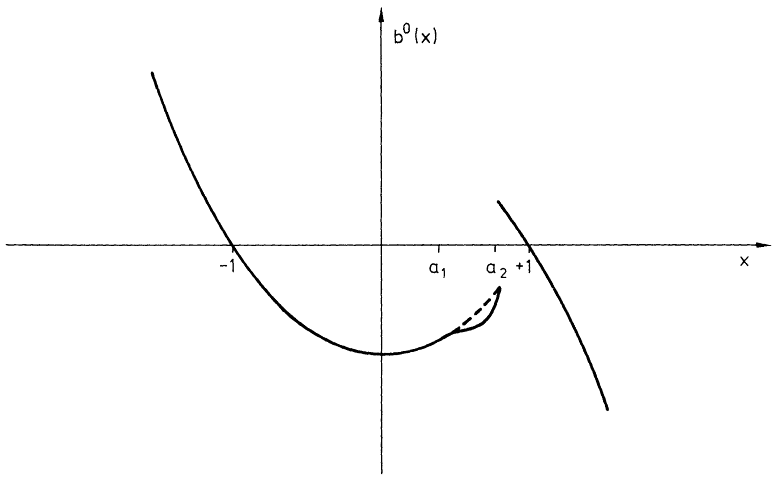

Fig. 4

the new potential still satifies all the assumptions of Sect. 2. We begin with the case in which

$$V(x)-V(-x)\equiv \Delta V(x)>0,\ \forall x\in(a_1,a_2)\subset(0,1),$$
$$\Delta V(x)=0, \forall x\notin(a_1,a_2)\cup(-a_2,\ -a_1).$$

In order to specify the sign of b^ε on the whole line we make use of the formula (2.15) connecting the antsymmetric part of $b^\varepsilon(x)$ with the symmetric one:

$$b^\varepsilon(x)+\bar{b}^\varepsilon(x)=-2/\varepsilon^2 \int_x^{+\infty} \Delta V(x')\exp\left\{\int_x^{x'}(b^\varepsilon(x'')-\bar{b}^\varepsilon(x''))dx''/\varepsilon^2\right\}dx', \qquad (5.3)$$

where

$$\bar{b}^\varepsilon(x)\equiv b^\varepsilon(-x).$$

From this formula it follows immediately that $b^\varepsilon(x)+\bar{b}^\varepsilon(x)=0, \forall x>a_2$; we remark that this property also follows directly from the Schrödinger equation for the ground state wave function. Besides by the positivity of $\Delta V(x), x\in(a_1,a_2)$,

$$b^\varepsilon(x)+\bar{b}^\varepsilon(x)<0,\ \forall x\in(a_1,a_2).$$

From Proposition 2.1

$$\lim_{\varepsilon\to 0} b^\varepsilon(x)=+\sqrt{2V(x)},\ \forall x\in(-\infty,-1),\ \lim_{\varepsilon\to 0} b^\varepsilon(x)=-\sqrt{2V(x)},\ \forall x\in[1,\infty)$$

and from Proposition 2.3 applied to the interval (a_1,a_2) we get:

$$\lim_{\varepsilon\to 0} b^\varepsilon(x)=-\sqrt{2V(x)}, \forall x\in(a_1,a_2);$$

by Proposition 2.2, this holds for the whole interval $[-1,a_2)$.

The situation is summarized in Fig. 4.

The case with $\Delta V(x)<0$, $\forall x\in(a_1,a_2)$ and $\Delta V(x)=0$, $\forall x\notin(a_1,a_2)\cup(-a_2,\ -a_1)$. can be treated analogously and the result is that the jump of $b^0(x)$ occurs at the point $x=-a_2$ (see Fig. 5).

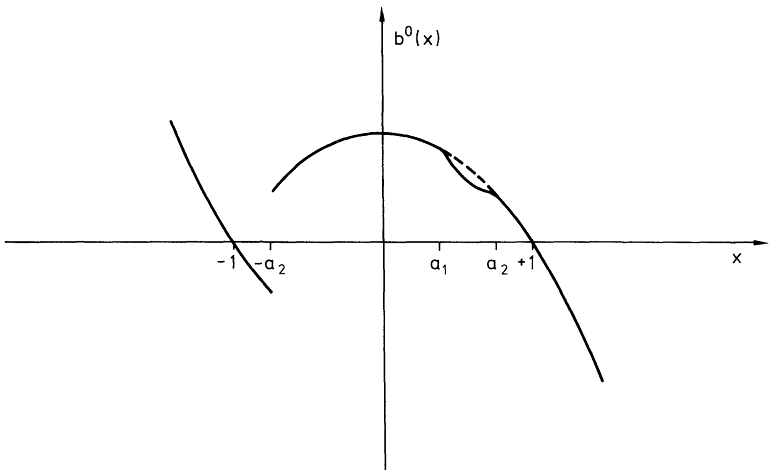

Fig. 5

As far as the splitting of the ground state level is concerned, in the first case ($\Delta V > 0$) we obtain:

$$\begin{aligned}\varepsilon^2 \exp\{(-V_{+1,-1} - h(\varepsilon,\delta))/2\varepsilon^2\} &< E_1 - E_0 \\ &< \varepsilon^2 \exp\{(-V_{+1,-1} + h(\varepsilon,\delta))/2\varepsilon^2\}, \end{aligned} \tag{5.4}$$

where $V_{+1,-1} = \min(V_{+1,-1}, V_{-1,+1}) = 4 \int\limits_{a_2}^{+1} \sqrt{2V(x)}\,dx$; using the identity $\psi_0(x)/\psi_0(-1) = \exp\left\{\int\limits_{-1}^{x} b^\varepsilon(x')dx'/\varepsilon^2\right\}$ we get immediately that, as $\varepsilon \to 0$, the invariant measure $|\psi_0(x)|^2$ concentrates on the left hand side well.

By computing now the right eigenvector of $q_a^\varepsilon(a = (E_1 - E_0)/\varepsilon^2)$ with eigenvalue 1, we can evaluate the first eigenfunction of $-L^\varepsilon$, $\varphi_1(x) = \psi_1(x)/\psi_0(x)$ in the points $x = -1$ and $x = +1$ and from this the ratio $\psi_1(+1)/\psi_1(-1)$. The result is the following

$$\begin{aligned}-\exp\left\{\left(2\int\limits_{-a_2}^{a_2} \sqrt{2V(x)}\,dx + \bar{h}(\varepsilon,\delta)\right)\Big/2\varepsilon^2\right\} &< \frac{\psi_1(+1)}{\psi_1(-1)} \\ &< -\exp\left\{\left(2\int\limits_{-a_2}^{a_2} \sqrt{2V(x)}\,dx - \bar{h}(\varepsilon,\delta)\right)\Big/2\varepsilon^2\right\},\end{aligned} \tag{5.5}$$

where $\bar{h}$ is the usual error term.

Analogous results can be obtained in the negative case $\Delta V < 0$. We now consider the situation in which the symmetry is broken in the region to the right hand side of the point $x = +1$, that is $\Delta V(x) \neq 0$ only if $x \in (a_1, a_2) \cup (-a_1, -a_2)$, where $(a_1, a_2) \subseteq (1, +\infty)$. We also make the technical assumption that $\lim\limits_{x \to a_1^+} \dfrac{\Delta V(x)}{(x - a_1)^{n_0}} = A$, $0 < n_0 < +\infty$ in order to simplify some estimates.

The two main results in this case are the following:

i) If $a_1=1$ then,

$$\lim_{\varepsilon\to 0} b^\varepsilon(x) = -\sqrt{2V(x)}, \forall x\in(-1,+1), \quad \text{if} \quad \Delta V(x)>0, \forall x\in(a_1,a_2)$$

$$\lim_{\varepsilon\to 0} b^\varepsilon(x) = +\sqrt{2V(x)}, \forall x\in(-1,+1), \quad \text{if} \quad \Delta V(x)<0, \forall x\in(a_1,a_2).$$

ii) There exists a point $a_1^*>1$ such that if $a_1>a_1^*$ then

$$\lim_{\varepsilon\to 0} b^\varepsilon(x) = -\sqrt{2V(x)}, \forall x\in[1,0),$$

$$\lim_{\varepsilon\to 0} b^\varepsilon(x) = +\sqrt{2V(x)}, \forall x\in(0,1]$$

independently of the sign of ΔV.

Proof. i) We treat only the case $\Delta V>0$, the proof of the opposite one being identical. Let us first assume that there is no point y^ε such that $b^\varepsilon(y^\varepsilon)=0$, $\frac{d}{dx} b^\varepsilon(x)|_{x=y^\varepsilon}>0$; by Proposition 2.2 we conclude:

$$\lim_{\varepsilon\to 0} b^\varepsilon(x) = \pm\sqrt{2V(x)}, \forall x\in[-1,+1]. \tag{5.6}$$

On the other hand formula (5.3) with $\Delta V>0$ tells us that

$$b^\varepsilon(x)+b^\varepsilon(-x)<0, \ \forall x\in(0,+1)$$

and this implies with (5.6) that

$$\lim_{\varepsilon\to 0} b^\varepsilon(x) = -\sqrt{2V(x)}, \ \forall x\in(-1,+1).$$

Now if a jump point y^ε is present, the negativity of $b^\varepsilon(x)+b^\varepsilon(-x)$ implies $y^\varepsilon>0$, $\forall\varepsilon>0$; we will show that the boundedness of $b^\varepsilon(x)$ for $x\in[-1,+1]$ implies $\lim_{\varepsilon\to 0} y^\varepsilon=+1$. Suppose in fact that $1-y^\varepsilon>C$ for some $\varepsilon<\varepsilon_0$, $\forall\varepsilon_0>0$, with C an arbitrary positive constant. Then by Proposition 2.2

$$b^\varepsilon(x)>+\sqrt{2V(x)}-K\varepsilon^\gamma, \qquad 0<\gamma<2, \qquad \forall x\in(1-C,1)$$

for some $\varepsilon<\varepsilon_0$, $\forall\varepsilon_0$ sufficiently small; but this in turn implies:

$$\exp\left\{\int_x^{+1} (b^\varepsilon(x')-\bar{b}^\varepsilon(x'))\,dx'/\varepsilon^2\right\} > \exp\left\{\left(2\int_x^{+1} \sqrt{2V(x')}\,dx' - K_1\varepsilon\right)\Big/\varepsilon^2\right\}$$
$$\cdot\forall x\in(1-C,1),$$

where K_1 is a positive constant. Inserting this estimate in (5.3) we get:

$$|b^\varepsilon(x)+\bar{b}^\varepsilon(x)| > \exp\left\{\left(2\int_x^1 \sqrt{2V(x')}\,dx' - K_1\varepsilon\right)\Big/\varepsilon^2\right\}\int_1^{a_2} \Delta V(x') \tag{5.7}$$
$$\cdot\exp\left\{\int_1^{x'} (b^\varepsilon(x'')-\bar{b}^\varepsilon(x''))\,dx''/\varepsilon^2\right\}dx', \ \forall x\in(1-C,1),$$

for some $\varepsilon<\varepsilon_0$, $\forall\varepsilon_0$ sufficiently small. The integral on the right hand side of (5.7) can be estimated by:

$$\int_{+1}^{a_2} \Delta V(x')\exp\left\{\int_1^{x'} (b^\varepsilon(x'')-\bar{b}^\varepsilon(x''))dx''/\varepsilon^2\right\}dx' > \int_1^{1+\varepsilon^2} \Delta V(x') \cdot\exp\left\{-\max_{y\in(1,1+\varepsilon^2)} |b^\varepsilon(y)-\bar{b}^\varepsilon(y)|\right\}dx' > K_2 A\varepsilon^{2(n_0+1)}, \tag{5.8}$$

by the assumption on $\Delta V(x)$, K_2 being a positive constant.

Combining (5.8) and (5.7) we would get:

$$|b^\varepsilon(x)+\bar{b}^\varepsilon(x)| > \exp\left\{\left(2\int_x^1 \sqrt{2V(x')}\,dx' - K_1\varepsilon\right)\Big/\varepsilon^2\right\}K_2 A\varepsilon^{2(n_0+1)},\ \forall x\in(1-C,1)$$

for some $\varepsilon<\varepsilon_0$, $\forall\varepsilon_0$ sufficiently small; but this is forbidden by the uniform boundedness in ε of $b^\varepsilon(x)$ for $x\in[-1,+1]$.

In conclusion, if an y^ε exists, $\lim_{\varepsilon\to 0} y^\varepsilon = +1$ which in turn implies $\lim_{\varepsilon\to 0} b^\varepsilon(x) = -\sqrt{2V(x)}$, $\forall x\in[-1,+1]$.

ii) In order to prove ii) we observe that for $0\leqq x\leqq 1$

$$\int_x^{a_1} (b^\varepsilon(x')-\bar{b}^\varepsilon(x'))dx' < \int_0^1 2\sqrt{2V(x)}\,dx - 2\int_1^{a_1}\sqrt{2V(x)}\,dx + K_3\varepsilon,$$

for some constant $K_3>0$. Let a_1^* be the point such that:

$$\int_1^{a_1^*}\sqrt{2V(x)}\,dx = \int_0^1 \sqrt{2V(x)}\,dx,$$

then for any $a_1>a_1^*$, $\int_x^{a_1} b^\varepsilon(x)-\bar{b}^\varepsilon(x) < K_4 < 0$ for ε sufficiently small. From (5.3) we have thus that if $a_1>a_1^*$ then:

$$\lim_{\varepsilon\to 0} b^\varepsilon(x)+\bar{b}^\varepsilon(x)=0,\ \forall x\in(0,1),$$

and this implies:

$$\lim_{\varepsilon\to 0} b^\varepsilon(x) = +\sqrt{2V(x)},\ \forall x\in(0,1]$$

$$\lim_{\varepsilon\to 0} b^\varepsilon(x) = -\sqrt{2V(x)},\ \forall x\in[-1,0).$$

The picture of b^ε in the case $a_1=1$ and $a_1>a_1^*$ is summarized in Figs. 6 and 7.

Once the sign of $b^0(x)$ is determined, the computation of the splitting of the ground state level and the estimate of the concentration as $\varepsilon\to 0$ of the invariant measure $|\psi_0(x)|^2$ can be carried out exactly as in the previous cases. In the first situation, i.e. when $a_1=\pm 1$, the tunneling disappears in the sense that there are no values of the parameter a such that $0<a<\exp\{-h/2\varepsilon^2\}$ and $\det(q_a^\varepsilon - I)=0$ where I is the identity matrix.

In the second case the situation of the symmetric case is recovered in the sense that for the value of a such that $a<\exp\{-h/2\varepsilon^2\}$ and $\det(q_a^\varepsilon-I)=0$ we obtain the

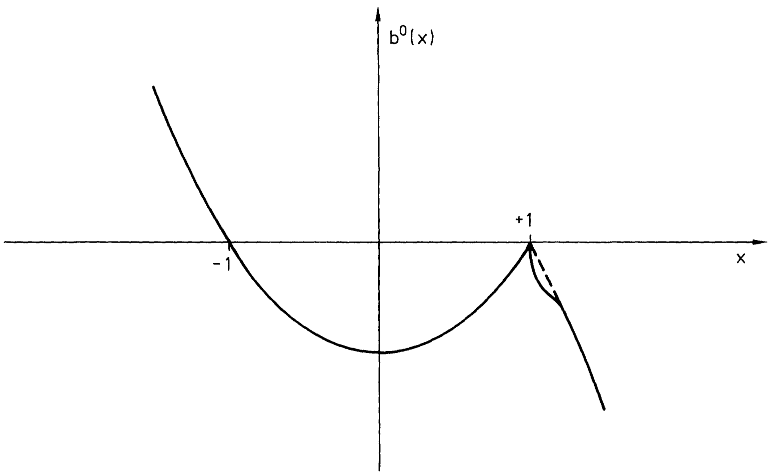

Fig. 6

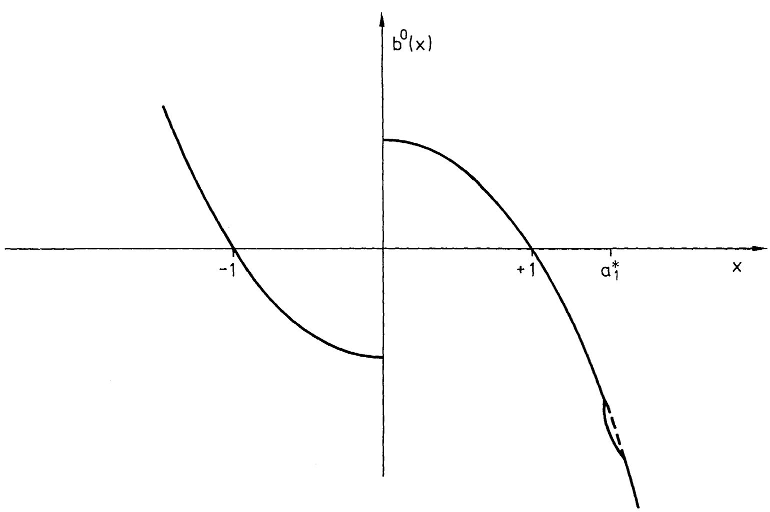

Fig. 7

estimate:

$$\exp\{(-V_{+1,-1}-h(\varepsilon,\delta))/2\varepsilon^2\}<a<\exp\{(-V_{+1,-1}+h(\varepsilon,\delta))/2\varepsilon^2\},$$

with $V_{+1,-1}=4\int\limits_{-1}^{0}\sqrt{2V(x)}\,dx$, which is identical to estimate (5.1) at least for the main logarithmic term.

6. Multiwell Potentials

Let us consider a C^∞ positive function $F(x)$ with support $(-\pi,\pi)$ such that:

1) $F(x)$ is symmetric, i.e. $F(x)=F(-x)$.

2) $F(\pi)=F(-\pi)=0$ and $\lim\limits_{x\to-\pi}\dfrac{F(x)}{(x+\pi)^2}=\omega^2>0$.

With such a function we construct a potential $V(x)$, which is periodic over the finite interval $[-N\pi, N\pi]$ with period 2π, and grows at infinity outside, as follows (Fig. 2)

(i) $V(x)=F(x-K\pi)$ for $x\in[(K-1)\pi,(K+1)\pi]$ with K even and $0\leqq K\leqq N-1$.

(ii) $V(x)$ for $x\geqq N\pi$ is a C^∞ positive function such that $\lim_{x\to N\pi} V(x)/(x-N\pi)^2 =\omega^2>0$ and there exists a real number $1<\alpha<\infty$ such that $\lim_{x\to+\infty} V(x)/x^\alpha$ exists and it is finite and positive.

(iii) For $x\in[N\pi, (N+1)\pi]$, $V(x)>F(x-(N+1)\pi)$, for $x>(N+1)\pi$, $V(x) > \max_{x\in[0,\pi]} F(x)$.

(iv) $V(x)=V(-x)$.

We analyze the quantities $b^\varepsilon(x)+\bar{b}^\varepsilon(x)$ and $b^\varepsilon(x)-\bar{b}^\varepsilon(x)$ as before but we now let the point with respect to which the inversion $x\to -x$ is taken, vary over the points $|x|=K\pi$, K even, $0\leqq K<N-1$. Since this technique is explained in detail in the next more general case, we omit the proof of the following result: $\lim_{\varepsilon\to 0} b^\varepsilon(x)=b^0(x)$ exists in the complement of the points x, $|x|=K\pi$, K even, $0\leqq K<N-1$ where $b^0(x)$ jumps from $-\sqrt{2V(x)}$ to $+\sqrt{2V(x)}$. Besides $b^0(x)=-\sqrt{2V(x)}$ for $(N-2)\pi \leqq x$, $b^0(x)=+\sqrt{2V(x)}$ for $x<-(N-2)\pi$. Once the sign of b^0 is determined we can easily conclude that the invariant measure as far as the main logarithmic term is concerned is equally distributed among the different wells except the ones at extreme left and right.

As for the computation of the splitting of the ground state energy, as before, we must estimate using (4.7) and (4.8) the values of $0<a<\exp\{-h/2\varepsilon^2\}$ such that $\det(q_a^\varepsilon-I)=0$ where q_a^ε is a $2(N+1)\times 2(N+1)$ matrix defined in the usual way, and I is the $2(N+1)\times 2(N+1)$ identity matrix.

We find that for $\varepsilon\to 0$ there are at least $N-2$ eigenvalues of L^ε, $\frac{E_i-E_0}{\varepsilon^2}$ with the same exponential term $\exp\{-V/2\varepsilon^2\}$, $V=4\int_0^\pi \sqrt{2V(x)}\,dx$ [3].

More interesting is the case in which one of the barriers is lowered slightly and the symmetry is thus destroyed (see e.g. Fig. 2).

We consider the case in which, for example in the interval $[a_1,a_2]$, $a_1\equiv(i-1)\pi$, $a_2\equiv(i+1)\pi$, i even, $0<i<N-1$, the function F is replaced by a symmetric C^∞ function $0\leqq f(x)<F(x)$ with support $(-\pi,\pi)$ such that $f(-\pi)=f(\pi)=0$, $\lim_{x\to\pi} f(x)/(x-\pi)^2=\omega^2$, there exists a real positive number n_0 and a constant A such that $\lim_{x\to-\pi}(F(x)-f(x)/(x+\pi)^{n_0}=A$. *We want now to prove that in this situation as* $\varepsilon\to 0$ *the ground state wave function is localized in neighborhoods of each of the points* a_1 *and* a_2.

Since our potential is obtained from the symmetric one by a perturbation localized between the points a_1, a_2, formula (2.15):

$$b^\varepsilon(x)+\bar{b}^\varepsilon(x)=-\int_x^\infty 2/\varepsilon^2\Delta V(x')\exp\left\{\int_x^{x'}(b^\varepsilon(x'')-\bar{b}^\varepsilon(x''))dx''/\varepsilon^2\right\}dx'$$

tells us that $b^\varepsilon(x)+b^\varepsilon(-x)=0$, $\forall x\geqq a_2$, i.e. $b^\varepsilon(x)$ is antisymmetric for $x\geqq a_2$.

3 We remark here that there is a theorem of Ventsel [17] which provides an estimate as $\varepsilon\to 0$ of these values of a for any $N\times N$ matrix q_a^ε

Let us take as an inversion point the point a_2. With this choice we are in the hypotheses of Proposition 2.3 and we conclude

$$\lim_{\varepsilon\to 0} b^\varepsilon(x) = -\sqrt{2V(x)}, \ \forall x\in(a_2, a_2+2\pi).$$

If we let the inversion point move over the points $K\pi$, $K>i+1$, we can use Proposition 2.3 each time and conclude:

$$\lim_{\varepsilon\to 0} b^\varepsilon(x) = -\sqrt{2V(x)}, \ \forall x>a_2.$$

By the antisymmetric property of $b^\varepsilon(x)$, for $x>a_2$ we get

$$\lim_{\varepsilon\to 0} b^\varepsilon(x) = +\sqrt{2V(x)}, \ \forall x<-a_2.$$

If we then take as the inversion point precisely the symmetry point of the original potential ($x=0$) and we observe that in this case $V(x)-V(-x)<0$ only for $x\in(a_1, a_2)$, we get by Proposition 2.3 that:

$$\lim_{\varepsilon\to 0} b^\varepsilon(x) = \sqrt{2V(x)} \quad \text{for} \quad x\in(-a_2, -a_1).$$

It remains then to determine the sign of $b^\varepsilon(x)$ in the region $[-a_1, a_2]$. To do this we put the inversion point in π. In this case it is no longer true that $V(x)-V(2\pi-x)=\Delta V(x)$ has a definite sign for $x>\pi$, in fact it is negative for $x\in(a_1, a_2)$ and positive between $N\pi$ and infinity.

However, we can show that $b^\varepsilon(x)+\bar{b}^\varepsilon(x)$, where $\bar{b}^\varepsilon(x)\equiv b^\varepsilon(2\pi-x)$ is positive for $x\in(a_1, a_2)$, provided ε is sufficiently small.

In fact

$$\begin{aligned} b^\varepsilon(x)+\bar{b}^\varepsilon(x) = {} & \frac{2}{\varepsilon^2}\exp\left\{\frac{1}{\varepsilon^2}\int_x^{a_2}(b^\varepsilon(x'')-\bar{b}^\varepsilon(x''))\,dx''\right\} \\ & \cdot\left[\int_x^{a_2}|\Delta V(x')|\exp\left\{-\frac{1}{\varepsilon^2}\int_{x'}^{a_2}(b^\varepsilon(x'')-\bar{b}^\varepsilon(x''))\,dx''\right\}\right. \\ & -\exp\left\{\int_{a_2}^{N\pi}\frac{1}{\varepsilon^2}(b^\varepsilon(x'')-\bar{b}^\varepsilon(x''))\,dx''\right\}\cdot\int_{N\pi}^{\infty}|\Delta V(x')| \\ & \left.\cdot\exp\left\{\frac{1}{\varepsilon^2}\int_{N\pi}^{x'}(b^\varepsilon(x'')-\bar{b}^\varepsilon(x''))\,dx''\right\}dx'\right], \end{aligned}$$

and the expression inside the square bracket is positive if ε is sufficiently small. We have in fact:

$$\begin{aligned} & \int_x^{a_2}|\Delta V(x')|\exp\left\{-\frac{1}{\varepsilon^2}\int_{x'}^{a_2}(b^\varepsilon(x'')-\bar{b}^\varepsilon(x''))\,dx''\right\}dx' \\ & \geqq \exp\left\{-\frac{1}{\varepsilon^2}\int_{a_1}^{a_2}|b^\varepsilon(x'')-\bar{b}^\varepsilon(x'')|\,dx''\right\}\cdot\int_x^{a_2}|\Delta V(x')|\,dx', \end{aligned} \tag{6.1}$$

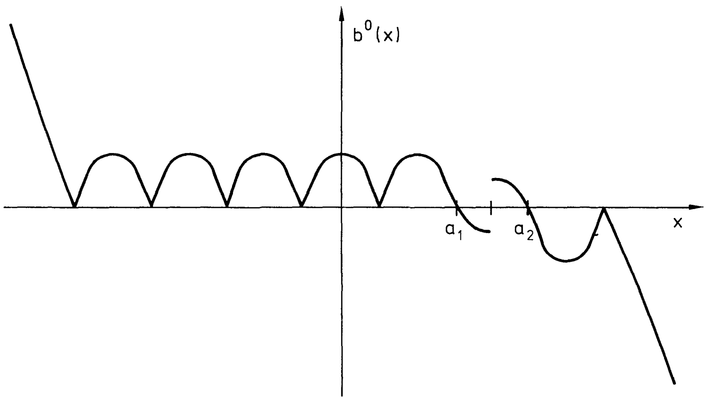

Fig. 8

$$\exp\left\{\frac{1}{\varepsilon^2}\int_{a_2}^{N\pi}(b^\varepsilon(x'')-\bar{b}^\varepsilon(x''))\,dx''\right\}\cdot\int_{N\pi}^{\infty}|\Delta V(x')|$$
$$\exp\left\{\frac{1}{\varepsilon^2}\int_{N\pi}^{x'}(b^\varepsilon(x'')-\bar{b}(x''))\,dx''\right\}dx'$$
$$\leqq\exp\left\{\frac{1}{\varepsilon^2}\int_{a_2}^{N\pi}(b^\varepsilon(x'')-\bar{b}^\varepsilon(x''))\,dx''\right\}\cdot\int_{N\pi}^{\infty}|\Delta V(x')|$$
$$\cdot\exp\left\{-\int_{N\pi}^{x'}\sqrt{2V(x'')}\,dx''\right\}dx',\tag{6.2}$$

and for ε sufficiently small (6.1) is greater than (6.2) if we observe that by the previous results $\int_{a_2}^{N\pi}\frac{1}{\varepsilon^2}(b^\varepsilon(x'')-\bar{b}^\varepsilon(x''))\,dx''$ is negative and greater in absolute value for ε sufficiently small than $\int_{a_1}^{a_2}\frac{1}{\varepsilon^2}(b^\varepsilon(x'')-\bar{b}^\varepsilon(x''))\,dx''$ because $V(x)<V(y)$ for $x\in(a_1,a_2)$, $y\notin(a_1,a_2)$.

Thus $b^\varepsilon(x)+\bar{b}^\varepsilon(x)>0$, $\forall x\in(a_1,a_2)$ and ε sufficiently small; this implies that $b^\varepsilon(x)>0$ for $x\in(-a_1,-a_1+2\pi)$ for ε sufficiently small. If we repeat the same discussion with the inversion point varying over $x=K\pi$, $0<K\leqq i-1$ we get $b^\varepsilon(x)\geqq 0$ for $x<a_1$, i.e. by Proposition 2.2

$$\lim_{\varepsilon\to 0}b^\varepsilon(x)=+\sqrt{2V(x)}\quad\text{for}\quad x<a_1\,.$$

As a final step one can show that $b^0(x)$ has a jump for $y=i\pi$ by taking $i\pi$ as the inversion point and by noting that by formula (2.15), $b^\varepsilon(i\pi)$ is small like $\exp\left\{-\frac{c}{\varepsilon^2}\right\}$, if ε is sufficiently small, for some constant $c>0$. The shape of b^0 may look something like in Fig. 8.

By the identity $\psi_0(x)/\psi_0(a)=\exp\left\{\int_a^x\frac{b^\varepsilon(x')}{\varepsilon^2}dx'\right\}$ it is clear now that the invariant measure ψ_0^2 concentrates as $\varepsilon\to 0$ on each of the two wells on either sides of the lower barrier, and that, as far as the splitting of the ground state is concerned, this situation is similar to a double well potential, i.e. there is a unique exponentially small splitting $E_1 - E_0$ for which the following estimate holds:

$$\exp\left\{-\frac{1}{2\varepsilon^2}\left[4\int_{a_1}^{a_1+\pi}\sqrt{2V(x)}\,dx+h\right]\right\}<E_1-E_0$$

$$<\exp\left\{-\frac{1}{2\varepsilon^2}\left[4\int_{a_1}^{a_1+\pi}\sqrt{2V(x)}\,dx-h\right]\right\},$$

with h as the usual error term.

Appendix A

Let ψ_n, ψ_0 be the eigenfunctions of the Hamiltonian corresponding to the n^{th} energy level and to the ground state respectively. If the potential V satisfies the assumptions of Sect. 2 then ψ_n/ψ_0 grows at infinity at most as a power law.

Proof. Let $d(\varepsilon, n)$ be such that for $x \in (d, +\infty)$

1) $V(x) - E_n > 0$,
2) $V'(x)/\sqrt{2V(x)} > 2E_0/\varepsilon^2$,
3) $V(x) > Cx^2$ for some positive C.

Due to our hypotheses such a point clearly exists. From 1) and the Schrödinger equation it follows that in $(d, +\infty)$, $|\psi_n|$ decreases monotonically due to the fact that ψ'' has the same sign as ψ. We have therefore that $b_n^\varepsilon = \frac{\varepsilon^2}{2}\frac{d}{dx}(\ln \psi_n^2) < 0$ in $(d, +\infty)$. The ratio ψ_n/ψ_0 can be written

$$\psi_n(x)/\psi_0(x) = [\psi_n(d)/\psi_0(d)] \cdot \exp\left\{\frac{1}{\varepsilon^2}\int_d^x (b_n^\varepsilon(x') - b_0^\varepsilon(x'))\,dx'\right\}.$$

Exploiting as we did, in Sect. 2 the quadratic structure of the Riccati equation for b_n^ε and b_0^ε, we can express the difference $b_n^\varepsilon - b_0^\varepsilon$ in terms of their sum $b_n^\varepsilon + b_0^\varepsilon$. We have in fact:

$$b_n^\varepsilon(x) - b_0^\varepsilon(x) = \int_x^\infty \frac{2}{\varepsilon^2}(E_n - E_0)\exp\left\{\frac{1}{\varepsilon^2}\int_x^{x'} (b_n^\varepsilon(x'') + b_0^\varepsilon(x''))\,dx''\right\} dx', \quad \forall x \in (d, +\infty), \tag{A.1}$$

that is

$$\begin{aligned} b_n^\varepsilon(x) - b_0^\varepsilon(x) &\leqq \frac{2}{\varepsilon^2}(E_n - E_0)\int_x^\infty \exp\left\{\frac{1}{\varepsilon^2}\int_x^{x'} (b_0^\varepsilon(x''))\,dx''\right\} dx' \\ &\leqq \frac{1}{\varepsilon^2}(E_n - E_0)\int_x^\infty \exp\left\{-\frac{1}{\varepsilon^2}\inf_{x''>x}|b_0^\varepsilon(x'')|\,(x'-x)\right\} dx' \\ &= \frac{2(E_n - E_0)}{\inf\limits_{x''>x}|b_0^\varepsilon(x'')|}. \end{aligned} \tag{A.2}$$

Using now the estimate (2.10) on b_0^ε:

$$|b_0^\varepsilon(x)| \geqq \sqrt{2V(x)}, \quad \forall x \in (d, +\infty)$$

and the estimate 3), we conclude that:

$$b_n^\varepsilon(x) - b_0^\varepsilon(x) \leqq \frac{2(E_n - E_0)}{\sqrt{C}x},$$

that is

$$\frac{\psi_n(x)}{\psi_0(x)} < \frac{\psi_n(d)}{\psi_0(d)} \exp\left\{\frac{2(E_n - E_0)}{\varepsilon^2 \sqrt{C}} \ln\left(\frac{x}{d}\right)\right\} < C' \frac{\psi_n(d)}{\psi_0(d)} x^{C''}, \tag{A.3}$$

where C' and C'' are positive constants.

The same discussion holds for the study of the growth of

$$\left|\frac{\psi_n(x)}{\psi_0(x)}\right| \quad \text{for} \quad x \to -\infty .$$

Appendix B

This appendix is devoted to the proof of the theorem stated in Sect. 3. For the all necessary notations we refer to that section.

We recall for simplicity the statement of the theorem: let $a > 0$ be such that $\|M_x^\varepsilon e^{2a\tau_1}\| < \infty$[4]. Then there is a one-to-one correspondence between the a-eigenfunctions of L^ε and the eigenvectors of $(q_a^\varepsilon)_{ij}$ with eigenvalue 1.

Proof. We first prove that if f_a^ε is an a-eigenfunction then:

$$f_a^\varepsilon(\bar{x}_i) = \sum_{j=1}^{M} M_{\bar{x}_i}^\varepsilon e^{a\tau_1} f_a^\varepsilon(\bar{x}_j) \chi(x_{\tau_1} = \bar{x}_j) \equiv M_{\bar{x}_i}^\varepsilon e^{a\tau_1} f_a^\varepsilon(x_{\tau_1}) \, \forall i =, \ldots, M .$$

It is sufficient to show that $f_a^\varepsilon(x) = M_x^\varepsilon(e^{a\tau_1} f_a^\varepsilon(x_{\tau_1}))$ for any x in an arbitrary compact set. Due to the fact that both $f_a^\varepsilon(x)$ and $M_x^\varepsilon(e^{a\tau_1} f_a^\varepsilon(x_{\tau_1}))$ are continuous in x any compact set it is enough to show:

$$\| f_a^\varepsilon(x) - M_x^\varepsilon(e^{a\tau_1} f_a^\varepsilon(x_{\tau_1})) \| = 0 .$$

The following inequality is straightforward for any $T \geqq 0$:

$$\begin{aligned} \| f_a^\varepsilon(x) - M_x^\varepsilon(e^{a\tau_1} f_a^\varepsilon(x_{\tau_1})) \| &\leqq \| f_a^\varepsilon(x) - M_x^\varepsilon(e^{a\tau_1} f_a^\varepsilon(x_{\tau_1})) \\ &+ M_x^\varepsilon(e^{a\tau_1} f_a^\varepsilon(x_{\tau_1}) \chi(\tau_1 \geqq T)) - M_x^\varepsilon(e^{aT} f_a^\varepsilon(x_T) \chi(\tau_1 \geqq T)) \| \\ &+ \| M_x^\varepsilon(e^{a\tau_1} f_a^\varepsilon(x_{\tau_1}) \chi(\tau_1 \geqq T)) \| + \| M_x^\varepsilon(e^{aT} f_a(x_T) \chi(\tau_1 \geqq T)) \| . \end{aligned} \tag{B.1}$$

The first term of the right hand side of (B.1) is zero; in fact it is equal to the norm of $f_a^\varepsilon(x) - M_x^\varepsilon(e^{a(\tau_1 \wedge T)} f_a^\varepsilon(x_{\tau_1 \wedge T}))$ where $a \wedge b = \min(a, b)$ and this function is shown to be zero on each compact set by means of the strong Markov property applied to the equality:

$$e^{aT} M_x^\varepsilon(f_a^\varepsilon(x_T)) = f_a^\varepsilon(x) .$$

4 The symbol $\| \cdot \|$ will denote in this Appendix the $L_2(\psi_0^2(x)dx)$ norm

By means of the dominated convergence theorem and using the assumption on $\| M_x^\varepsilon(e^{2a\tau_1})\|$ the second term of the r.h.s. of (B.1) goes to zero as $T\to\infty$. Finally we estimate the last term.

Using twice the Schwartz inequality we get:

$$\| M_x^\varepsilon(e^{aT} f_a^\varepsilon(x_T)\chi(\tau_1 \geqq T))\| \leqq \| M_x^\varepsilon(e^{2a\tau_1}\chi(\tau_1 \geqq T))\|^{1/2} \cdot \|(f_a^\varepsilon(x))^2\|^{1/2}$$

and this last quantity goes to zero as $T\to\infty$ because $\|(f_a^\varepsilon)^2\|^{1/2}<\infty$ [see Sect. 2d)].

We now prove the theorem in the opposite direction, that is, from an eigenvector of $(q_a^\varepsilon)_{\bar{x}_i\bar{x}_j}$ with eigenvalue 1 we construct a unique a-eigenfunction of L^ε.

Let us denote by $\{f(\bar{x}_i)\}_{i=1}^M$ an eigenvector of $(q_a^\varepsilon)_{\bar{x}_i\bar{x}_j}$ with eigenvalue 1. With this eigenvector we define two continuous functions: $f_1(x)$, $f_2(x)$ as follows:

$$f_1(x) = M_x^\varepsilon\left(e^{a\tau_E}\sum_{i=1}^{M} f(\bar{x}_j)\chi(x_{\tau_E}=\bar{x}_j)\right) = M_x^\varepsilon(e^{a\tau_E} f(x_{\tau_E}))$$

$$f_2(x) = M_x^\varepsilon(e^{a\tau_D} f_1(x_{\tau_D}))\,.$$

Clearly $f_1(\bar{x}_i)=f(\bar{x}_i)\,\forall i=1,\dots,M$ and $f_1(x)=f_2(x)$ on D. Because $\{f(\bar{x}_i)\}_{i=1}^M$ is an eigenvector with eigenvalue 1 of $(q_a^\varepsilon)_{\bar{x}_i\bar{x}_j}$, by the definition of τ_1, we get

$$f_1(\bar{x}_i) = f(\bar{x}_i) = f_2(\bar{x}_i)\,\forall i=1,\dots,M\,.$$

Lemma 1. *$h^{-1}(e^{ah}P^h f_i(x) - f_i(x))\to 0$, $i=1,2$ as $h\to 0$ uniformly on any closed set F disjoint from E for f_1 (D for f_2).*

Proof. By the strong Markov property we get

$$f_1(x) = M_x^\varepsilon(e^{a\tau_E} f_1(x_{\tau_E})) = M_x^\varepsilon(e^{a(\tau_E\wedge h)} f_1(x_{(\tau_E\wedge h)}))\,,$$

so that

$$\begin{aligned} h^{-1}(e^{ah}P^h f_1(x) - f_1(x)) &= h^{-1} M_x^\varepsilon\{\chi(\tau_E<h)\,[e^{ah} f_1(x_h) - e^{a\tau_E} f_1(x_{\tau_E})]\} \\ &= h^{-1} M_x^\varepsilon\{\chi(\tau_E<h) e^{a\tau_E}[e^{a(h-\tau_E)} f_1(x_h) - f_1(x_{\tau_E})]\} \\ &= h^{-1} M_x^\varepsilon\{\chi(\tau_E<h) e^{a\tau_E} M_{x_{\tau_E}}^\varepsilon[e^{a(h-\tau_E)} f_1(x_{(h-\tau_E)}) - f_1(x_{\tau_E})]\} \end{aligned}$$

again by the strong Markov property. This last term is less than:

$$e^{ah}\sup_{\bar{x}_i\in E}\sup_{t\leqq h} |M_{\bar{x}_i}^\varepsilon(e^{at} f_1(x_t)) - f_1(\bar{x}_i)|\,\frac{p_x^\varepsilon(\tau_E<h)}{h}\,.$$

From the continuity of the process x_t^ε with respect to t it follows that $\frac{p_x^\varepsilon(\tau_E<h)}{h}\to$const uniformly on each closed set F disjount from E. Besides $|M_y^\varepsilon(e^{at} f_1(x_t)) - f_1(y)|$ goes to zero as $t\to 0$ uniformly on compact sets. A proof of this assertion goes as follows:

$$\begin{aligned} |M_y^\varepsilon(e^{at} f_1(x_t)) - f_1(y)| \leqq{}& |M_y^\varepsilon(e^{at} f_1(x_t)\chi(|x_t-y|<M) - f_1(y)| \\ &+ |M_y^\varepsilon(e^{at} f_1(x_t)\chi(|x_t-y|\geqq M))| \end{aligned} \tag{B.2}$$

for any $M>0$. By the dominated convergence theorem, the first term in (B.2) goes to zero as $t\to 0$. The second term can be estimated by:

$$\begin{aligned}&M_y^\varepsilon(e^{at}f_1(x_t)\chi(|x_t-y|\geqq M))\\&\quad\leqq e^{at}(M_y^\varepsilon f_1^2(x_t))^{1/2}\cdot p_y^\varepsilon(|x_t-y|\geqq M)^{1/2}.\end{aligned}$$

If we observe that $f_1^2\in L^2(\psi_0^2dx)$ since

$$|f_1(x)|\leqq C_1M_x^\varepsilon(\exp\{a\tau_1\})\leqq C_1(M_x^\varepsilon(\exp\{2a\tau_1\}))^{1/2}$$

and $\|M_x^\varepsilon(\exp\{2a\tau_1\})\|<+\infty$ we can conclude that $M_y^\varepsilon f_1^2(x_t)$ is uniformly bounded for any $t\geqq 0$ and any y in a compact set. Since $p_y^\varepsilon(|x_t-y|\geqq M)\to 0$ for $t\to 0$, we conclude also that the second term of (B.2) goes to zero as $t\to 0$. Similarly Lemma 1 can be proved for f_2 by substituting D for E.

Lemma 2. *For all x in an arbitary compact:*

$$f_i(x)=M_x^\varepsilon(\exp\{a\tau_{E\cup D}\}\,f_i(x_{\tau_{E\cup D}})),\qquad i=1,2.$$

Proof. It is enough to show that

$$f_i(x)=M_x^\varepsilon(\exp\{a\tau_{\bar K}\}\,f_i(x_{\tau_{\bar K}})),$$

where $\bar K$ is the complement of an arbitrary closed set K disjoint from $E\cup D$, and x is in an arbitrary compact.

Indeed for arbitrary x there exists a sequence of bounded open sets $U_n\supseteqq E\cup D$, $U_n\supset U_{n+1}$ such that with p_x^ε-probability 1, $\tau_{U_n}\nearrow\tau_{E\cup D}$ as $n\to+\infty$. The continuity of the process implies almost surely that $x_{\tau_{U_n}}\to x_{\tau_{E\cup D}}$ so that $f_i(x_{\tau_{U_n}})\to f_i(x_{\tau_{E\cup D}})$. Thus almost surely

$$\exp\{a\tau_{U_n}\}\,f_i(x_{\tau_{U_n}})\to\exp\{a\tau_{E\cup D}\}\,f_i(x_{\tau_{E\cup D}}),$$

whereby the sequence is majorized by

$$\exp\{a\tau_{E\cup D}\}\sup_{x\in U_0}|f_i(x)|$$

so that a limit passage in the equality

$$f_i(x)=M_x^\varepsilon(\exp\{a\tau_{U_n}\}\,f_i(x_{\tau_{U_n}}))$$

yields

$$f_i(x)=M_x^\varepsilon(\exp\{a\tau_{E\cup D}\}\,f_i(x_{\tau_{E\cup D}})).$$

We now introduce the notation $\alpha(h)=\sup_{x\in K}h^{-1}|\exp(ah)P^hf_i(x)-f_i(x)|$; by definition $\alpha(h)\to 0$ as $h\to 0^+$. Put $\tau(h)=\min\{kh;\,x_{kh}\in\bar K\}$, k integer.

Applying the strong Markov property we get

$$\begin{aligned}&|M_x^\varepsilon\exp\{a[\tau(h)\wedge N]\}\,f_i(x_{\tau(h)\wedge N})-f_i(x)|\\&=\left|M_x^\varepsilon\sum_{k=0}^{h^{-1}[\tau(h)\wedge N]-1}\exp\{akh\}\,(\exp\{ah\}P^hf_i(x_{kh})-f_i(x_{kh}))\right|,\end{aligned}\tag{B.3}$$

in fact:

$$
\begin{aligned}
M_x^\varepsilon & \sum_{k=0}^{h^{-1}[\tau(h)\wedge N]-1} \exp\{a(k+1)h\} P^h f_i(x_{kh}) \\
&= \sum_{k=0}^{+\infty} \exp\{a(k+1)h\} M_x^\varepsilon\{\chi(\tau(h)\wedge N \geqq (k+1)h) M_{x_{kh}}^\varepsilon f_i(x_h)\} \\
&= \sum_{k=0}^{\infty} \exp\{a(k+1)h\} M_x^\varepsilon\{\chi(\tau(h)\wedge N \geqq (k+1)h)\, f_i(x_{(k+1)h})\} \\
&= M_x^\varepsilon\left\{\sum_{k=1}^{h^{-1}[\tau(h)\wedge N]} \exp\{akh\}\, f_i(x_{kh})\right\}.
\end{aligned}
\tag{B.4}
$$

From this (B.3) follows immediately.

The r.h.s. of (B.3) can be estimated as follows:

$$\left| M_x^\varepsilon \sum_{k=0}^{h^{-1}(\tau(h)\wedge N)-1} e^{akh}(e^{ah}P^h f_i(x_{kh}) - f_i(x_{kh}))\right| \leqq \alpha(h) M_x^\varepsilon \int_0^{\tau(h)\wedge N} e^{at}dt \leqq \alpha(h) N e^{aN}.$$

But as $n\to\infty$, the first hitting time $\tau(2^{-n})$ of $\bar{K}$ on a lattice with mesh 2^{-n} tends from above to τ_K, this means that $x_{\tau(2^{-n})}\to x_{\tau_{\bar{K}}}$ and $f_i(x_{\tau(2^{-n})})\to f_i(x_{\tau_{\bar{K}}})$. A limit passage in (B.3) for $h=2^{-n}\to 0$ yields

$$M_x^\varepsilon(e^{a(\tau_{\bar{K}}\wedge N)} f_i(x_{(\tau_{\bar{K}}\wedge N)}) = f_i(x).$$

Exactly as in the first part of the proof one gets

$$\|M_x^\varepsilon(e^{a\tau_{\bar{K}}} f_i(x_{\tau_{\bar{K}}})) - f_i(x)\| \to 0 \quad \text{as} \quad N\to\infty,$$

i.e. $M_{\bar{x}}^\varepsilon(e^{a\tau_K} f_i(x_{\tau_{\bar{K}}})) = f_i(x)$ in any compact set. Lemma 2 is proved.

Because $f_1(x)=f_2(x)$ for $x\in E\cup D$, by Lemma 2 it follows

$$f_2(x) = M_x^\varepsilon(e^{a\tau_{E\cup D}} f_2(x_{\tau_{E\cup D}})) = M_x^\varepsilon(e^{a\tau_{E\cup D}} f_1(x_{\tau_{E\cup D}})) = f_1(x),$$

i.e. f_1 and f_2 coincide for x in an arbitrary compact.

Covering the whole line by two closed sets one of which does not intersect E and the other D, by Lemma 1 we find that $f_1=f_2=f_i$ satisfies

$$h^{-1}\|e^{ah}P^h f_i(x) - f_i(x)\| \to 0, \quad \text{as} \quad h\to 0,$$

i.e. f_i is an a-eigenfunction of $-L^\varepsilon$.

Appendix C

We prove here that $\|M_x^\varepsilon e^{2a\tau_1}\|_{L_2(\psi^2dx)} < \infty$ provided $2a < e^{-\frac{h(\varepsilon,\delta)}{2\varepsilon^2}}$ where $h(\varepsilon,\delta)$ is arbitrarily small, for ε and δ sufficiently small (see Sect. 4). We will use in the sequel all the notations of that section. By definition of $L_2(\psi_0^2dx)$ norm we have:

$$\|M_x^\varepsilon e^{2a\tau_1}\|^2 = \int_{\mathbb{R}\setminus\bigcup_{i=1}^N D_i} \psi_0^2(x)(M_x^\varepsilon e^{2a\tau_1})^2 dx + \int_{\bigcup_{i=1}^N D_i} \psi_0^2(x)(M_x^\varepsilon e^{2a\tau_1})^2 dx, \tag{C.1}$$

where D_i are the closed neighborhoods of the points x_i introduced in Sect. 4. The second term of the r.h.s. of (C.1) is finite if we use the estimate (4.8) with "a"

$<\exp\left\{-\frac{h}{2\varepsilon^2}\right\}$. By definition of expectation the first term of (C.1) is finite if and only if

$$\int_{\mathbb{R}\setminus\bigcup_{i=1}^{N} D_i} \psi_0^2(x)\left[2a\int_0^\infty e^{2at}p_x^\varepsilon(\tau_1>t)\,dt\right]^2 dx<\infty\,. \tag{C.2}$$

In order to estimate the probability $p_x^\varepsilon(\tau_1>t)$ it is convenient to work with the Wiener measure $\mathbb{W}_x$.

The relative density η is given by:

$$\eta(\mathbb{W}_t)=\frac{1}{\psi_0(x)}\psi_0(\mathbb{W}_t)\exp\left\{-\frac{1}{\varepsilon^2}\int_0^t(V(\mathbb{W}_s)-E_0)\,ds\right\}.$$

We have thus

$$p_x^\varepsilon(\tau_1>t)=E_{\mathbb{W}_x}\left[\frac{\psi_0(\mathbb{W}_t)}{\psi_0(x)}\exp\left\{-\frac{1}{\varepsilon^2}\int_0^t(V(\mathbb{W}_s)-E_0)\,ds\right\}\chi(\tau_1>t)\right] \tag{C.3}$$

By the Schwartz inequality we get:

$$\begin{aligned} p_x^\varepsilon(\tau_1>t)\leqq(\psi_0(x))^{-1}&\left[E_{\mathbb{W}_x}\left(\psi_0^2(\mathbb{W}_t)\exp\left\{-\frac{1}{\varepsilon^2}\int_0^t(V(\mathbb{W}_s)-E_0)\,ds\right\}\right)\right]^{1/2}\\ &\cdot\left[E_{\mathbb{W}_x}\left(\exp\left\{-\frac{1}{\varepsilon^2}\int_0^t(V(\mathbb{W}_s)-E_0)\,ds\right\}\chi(\tau_1>t)\right)\right]^{1/2}. \end{aligned} \tag{C.4}$$

The second factor in (C.4) is estimated by:

$$\begin{aligned} &\left[E_{\mathbb{W}_x}\left(\exp\left\{-\frac{1}{\varepsilon^2}\int_0^t(V(\mathbb{W}_s)-E_0)\,ds\right\}\cdot\chi(\tau_1>t)\right)\right]^{1/2}\\ &\quad\leqq\exp\left\{-\frac{1}{2\varepsilon^2}\left(\inf_{y\notin\bigcup_{i=1}^{N}D_i}(V(y)-E_0)t\right)\right\}, \end{aligned}$$

where $\inf_{y\notin\cup D_i}(V(y)-E_0)>0$ if ε is sufficiently small. The first factor is less than:

$$\left[\sup_{y\in\mathbb{R}}\psi_0(y)\right]^{1/2}\cdot[\psi_0(x)]^{-1/2}$$

$\left[\text{we have used the fact that } \psi_0(x)=E_{\mathbb{W}}\left(\psi_0(\mathbb{W}_t)e^{-\frac{1}{\varepsilon^2}\int_0^t(V(\mathbb{W}_s)-E_0)ds}\right)\right]$.

In conclusion:

$$p_x^\varepsilon(\tau_1>t)\leqq C[\psi_0(x)]^{-1/2}\cdot\exp\left\{-\frac{1}{\varepsilon^2}\inf_{y\notin\bigcup_{i=1}^{N}D_i}(V(y)-E_0)t\right\}. \tag{C.5}$$

Now combining (C.2) and estimate (C.5) and the fact that $[\psi_0(x)]^{1/2}$ is still in $L_2(dx)$ we have that $\|M_x^\varepsilon l^{2a\tau_1}\|$ is finite provided

$$2a<e^{-\frac{h(\varepsilon,\delta)}{2\varepsilon^2}}\wedge\inf_{y\notin\cup_i D_i}\left(\frac{V(y)-E_0}{2\varepsilon^2}\right)=e^{-\frac{h}{2\varepsilon^2}}$$

for ε sufficiently small.

Appendix D

We verify the right hand side of (4.4). It is sufficient to compute $p_{c_{i2}}(a_{j1}, a_{i2})$ using (4.3) (see Fig. 3 for reference)

$$p_{c_{i2}}(a_{j1}, a_{i2}) = \left[\int_{a_{i2}}^{c_{i2}} \phi^\varepsilon(x)dx\right] \cdot \left[\int_{a_{i2}}^{a_{j1}} \phi^\varepsilon(x)dx\right]^{-1}$$
$$\leqq \frac{\delta}{2} \max_{x\in[a_{i2}, c_{i2}]} \phi^\varepsilon(x) \cdot \left[\int_{a_{i2}}^{a_{j1}} \phi^\varepsilon(x)dx\right]^{-1}.$$

From the definition of $\phi^\varepsilon(x)$

$$\max_{x\in[a_{i2}, c_{i2}]} \phi^\varepsilon(x) \leqq \exp\left\{\frac{c(\delta,\varepsilon)}{2\varepsilon^2}\right\},$$

where $c(\delta,\varepsilon) = 2 \max_{x\in[a_{i2}, c_{i2}]} |b^\varepsilon(x)|\delta$ is small for δ, ε small.

$$\int_{a_{i2}}^{a_{j1}} \phi^\varepsilon(x)dx \geqq \int_{y_i^0}^{y_i^0+\Delta(\varepsilon)} \phi^\varepsilon(x)dx,$$

where y_i^0 is the position of the jump of $b^0(x)$ between x_i and x_j. $\Delta(\varepsilon)$ is a small interval which decreases with ε

$$\int_{y_i^0}^{y_i^0+\Delta(\varepsilon)} \phi^\varepsilon(x)dx \geqq \Delta(\varepsilon)\exp\left\{-\frac{2}{\varepsilon^2}\int_{a_{i2}}^{y_i^0} b^\varepsilon(x)dx\right\}\exp\left\{-\frac{2}{\varepsilon^2}\Delta(\varepsilon)\max_{x\in[y_i^0, y_i^0+\Delta(\varepsilon)]} b^\varepsilon(x)\right\}$$
$$\geqq \exp\left\{\frac{V_{ij}-\bar{c}(\varepsilon,\delta)}{2\varepsilon^2}\right\},$$

where $\bar{c}(\varepsilon,\delta)$ includes also the errors coming from the substitution of b^ε with $-\sqrt{2V}$.

In conclusion we have proved

$$p_{c_{i2}}(a_{j1}, a_{i2}) \leqq \exp\left\{\frac{1}{2\varepsilon^2}(-V_{ij}+h(\varepsilon,\delta))\right\}$$

with

$$h(\varepsilon,\delta) = c(\varepsilon,\delta) + \bar{c}(\varepsilon,\delta).$$

The left hand side of (4.4) can be obtained along similar lines.

Appendix E. Estimate of the Probability $p_x^\varepsilon(\tau_D < t)$

Let D be the interval $[c_{i1}, c_{i2}]$ around the i^{th} zero of $V(x)$, x_i, of length δ. It is known (see [15]) that $p_x^\varepsilon(\tau_D < t)$ where τ_D is the exit time from D for the Markov process starting at x, is the solution of the first boundary value problem:

$$\frac{\partial u^\varepsilon}{\partial t}(t,x) = b^\varepsilon(x)\frac{\partial u^\varepsilon}{\partial x}(t,x) + \frac{\varepsilon^2}{2}\frac{\partial^2 u}{\partial x^2}(t,x),$$
$$u^\varepsilon(0,x) = 0, \quad u^\varepsilon(t,\partial D) = 1. \tag{E.1}$$

The behaviour of the solution u^ε for small ε has been studied by Levinson [18] and Ventzel and Freidlin [19]. In particular in [19] the following result is proved:

$$\lim_{\varepsilon\to 0} 2\varepsilon^2 \ln u^\varepsilon(t_0, x) = -I_0(t_0, x), \tag{E.2}$$

where $I_0(t_0,x)$ is the infimum of $\int_0^{t_0} |b^0(\varphi_s)-\dot{\varphi}_s|^2 ds$ taken over all the absolutely continuous functions φ_s starting at x and reaching the boundary of D within the time t_0, provided $b^\varepsilon(x)\to b^0(x)$ as ε goes to zero, uniformly in D. In our one dimensional case it is not difficult to study the quantity $I_0(t_0,x)$. First of all we observe that

$$\int_0^{t_0} |b^0(\varphi_s)-\dot{\varphi}_s|^2 ds = 0 \Leftrightarrow b^0(\varphi_s)=\dot{\varphi}_s, \qquad \varphi_0 = x. \tag{E.3}$$

By choosing now the time t_0 in such a way that the solution of (E.3) does not exit from $D=[c_{i1}, c_{i2}]$ within the time t_0 we get $I_0(t_0,x)>0$. We remark that if the starting point x is equal to one of the two points $x_i+\frac{\delta}{4}$, $x_i-\frac{\delta}{4}$ and if δ is such that the potential $V(x)$ is quadratic in $[c_{i1}, c_{i2}]$ with an error proportional to δ^3, which is true by our assumptions, then the time t_0 is essentially independent of δ. Besides it is easy to show that $I_0(t_0,x)$ is as small as we like for δ sufficiently small; it is sufficient to evaluate the integral $\int_0^{t_0} |b^0(\varphi_s)-\dot{\varphi}_s|^2 ds$ for the trial function $\varphi_s = x + (c_{i2}-x)\frac{s}{t_0}$. We get:

$$I_0(t_0,x) < \max_{x\in D} |b^0(x)|^2 t_0 + \frac{8\delta^2}{t_0} \leqq 4 \max_{x\in D} V(x) + \frac{8\delta^2}{t_0} = O(\delta^2).$$

In conclusion from (E.2) we have:

$$e^{-\frac{h''(\varepsilon,\delta,t_0)}{2\varepsilon^2}} < e^{\frac{-I_0-h}{2\varepsilon^2}} \leqq p_x^\varepsilon(\tau_D < t_0) \leqq e^{\frac{-I_0+h}{2\varepsilon^2}} < e^{-\frac{h'(\varepsilon,\delta,t_0)}{2\varepsilon^2}},$$

where $h'(\varepsilon,\delta,t_0)$, $h''(\varepsilon,\delta,t_0)$ can be made arbitrarily small for δ and ε sufficiently small.

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References

1. Thompson, C.J., Kac, M.: Studies Appl. Math. **48**, 257 (1969)
2. Harrell, E.M.: Commun. Math. Phys. **60**, 73 (1978)
3. Harrell, E.M.: Commun. Math. Phys. **75**, 239 (1980)
4. Jona-Lasinio, G.: In: Quantum fields – algebras processes, p. 79. L. Streit (ed). Wien: Springer 1980

5. Nelson, E.: Phys. Rev. **150**, 1079 (1966), and Dynamical theories of brownian motion. Princeton: Princeton University Press 1967
6. Kac, M.: Probability and related topics in physical sciences, Chap. IV. New York, London: Interscience 1959
7. Guerra, F., Ruggiero, P.: Phys. Rev. Lett. **31**, 1022 (1973)
8. Albeverio, S., Hoegh-Krohn, R., Streit, L.: J. Math. Phys. **18**, 907 (1977)
9. Fukushima, M.: Dirichlet forms and Markov processes. Tokyo: Kodansha Ltd. Company 1980
10. Reed, M., Simon, B.: Methods of modern mathematical physics, Vol. IV. New York, London: Academic Press 1978
11. Ventzel, A.D.: Theory of Prob. and Appl. **18**, 1 (1973)
12. Reed, M., Simon, B.: Methods of modern mathematical physics, Vol. II, New York, London: Academic Press 1978
13. Carmona, R.: J. Functional Analysis **33**, 259 (1979)
14. Olver, F.W.J.: Proc. Camb. Phyl. Soc. **57**, 790 (1961)
15. Dynkin, E.B.: Markov processes. Berlin, Heidelberg, New York: Springer 1965
16. Gihman, I.I., Skorohod, A.V.: Stochastic differential equations, Chap. 3. Berlin, Heidelberg, New York: Springer 1972
17. Ventzel, A.D.: Sov. Math. Dokl. **13**, 65 (1972)
18. Levinson, N.: Ann. Math. **5**, 428 (1950)
19. Ventzel, A.D., Freidlin, M.I.: Usp. Math. Nauk. **25**, 3 (1970); [English translation Russ. Math. Surv. **25**, 1 (1970)]

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