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A Note on Coherent States

W. J. M. A. Hochstenbach

Institut für Theoretische Physik, Universität Göttingen, D-3400 Göttingen, Federal Republic of Germany

Abstract. Given a connected Lie group G with an Abelian invariant Lie subgroup and a continuous unitary representation of G on the Hilbert space $\mathscr{H}$, we investigate a relationship between the first cohomology group $H^1(G,\mathscr{H})$ and classes of sectors, determined by coherent states with a projectively G-covariant Weyl system. This result is applied to calculate $H^1(G,\mathscr{H})$, if the group G has in addition a compact subgroup with certain properties.

1. Introduction

In a discussion of the coherent states of the free electromagnetic field Roepstorff [1] stressed the use of cohomological methods. In Proposition 2 of [1], necessary and sufficient conditions were derived such that the automorphisms of the space-time translation group $\mathbb{R}^4$ in the Gel'fand-Naimark-Segal (GNS) representation $(\mathscr{K}_0, W, \Omega_0)$, corresponding to a given coherent state, are implemented by a strongly continuous unitary projective representation U of the translation group $\mathbb{R}^4$ on $\mathscr{K}_0$. Such a Weyl system may be called a projectively covariant Weyl system. In proving these necessary and sufficient statements, 1-cocycles on the translation group with values in the 1-photon Hilbert space $\mathscr{H}$, carrying a continuous unitary representation V of the translation group, seemed extremely useful.

For the Poincaré group, Basarab-Horwath et al. [2] recently have shown the existence of an injective mapping from classes of sectors, defined by coherent states admitting a GNS representation with the above mentioned properties for the Poincaré group $P_+^\uparrow$, into the first cohomology group $H^1(P_+^\uparrow,\mathscr{H})$. With an extension of these notions to a connected, n-dimensional Lie group G, we will give a condition for this injective mapping to be bijective. For a group G with a structure, which resembles that of the Poincaré group, the first cohomology group $H^1(G,\mathscr{H})$ can be calculated.

2. Sectors and Cohomology

Let $\mathscr{H}$ be a complex Hilbert space, L a dense complex-linear subspace of $\mathscr{H}$ and $L^\times$ the algebraic dual of L. The pair (L,σ) is a symplectic real-linear space with

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$\sigma(f,g)=\mathrm{Im}(f,g)_{\mathscr{H}}$ $\forall f, g\in L$. For an account of the definitions omitted below the reader is referred to [1, 2]. We define the Fock state E_0 on L by $E_0(f) =\exp(-\frac{1}{2}\|f\|^2)$ and a coherent state E_F on L by $E_F(f)=E_0(f)\exp(i\,\mathrm{Im}\,F(f))$ ($f\in L$, $F\in L^\times$). As the corresponding GNS representations we take $(\mathscr{K}_0, W_0, \Omega_0)$ and $(\mathscr{K}_0, W_F, \Omega_0)$ with $W_F(f)=W_0(f)\exp(i\,\mathrm{Im}\,F(f))\,\forall f\in L$; $\mathscr{K}_0$ is the Fock space over $\mathscr{H}$, $W_0(f)=\exp(i\overline{(a(f)+a(f)^*)})$ $f\in L$, $a(f)$ is the usual annihilation operator on $\mathscr{K}_0$, the bar means operator closure, Ω_0 is the vacuum vector in $\mathscr{K}_0$. The Weyl systems are defined with respect to the symplectic form σ on L.

Let G be a topological group and $s\to V(s)$ ($s\in G$) a strongly continuous unitary representation of G on $\mathscr{H}$ such that $V(s)L\subset L$ $\forall s\in G$. Given $F\in L^\times$ and $s\in G$, the map $W_F(f)\to W_F(V(s)f)$ defines an automorphism of the Weyl system $(\mathscr{K}_0, W_F)$. Let $L_G^\times$ be the set of all $F\in L^\times$ such that there exists a strongly continuous unitary projective representation U_F of the group G on $\mathscr{K}_0$, which implements the Weyl automorphisms $\forall s\in G$ (i.e. $W_F(V(s)f)=U_F(s)W_F(f)U_F(s)^{-1}$ $\forall f\in L$, $\forall s\in G$). In the complex-linear space $L^\times$ the operator $V(s)$ induces an operator $V(s)^\times$, defined by $(V(s)^\times F)(f)=F(V(s)f)$ $\forall F\in L^\times$ and $\forall f\in L$. A 1-cocycle on G with values in $\mathscr{H}$ with respect to the strongly continuous unitary representation V of G on $\mathscr{H}$ is a continuous mapping $\xi: G\to\mathscr{H}$ such that $\xi(s)+V(s)^*\xi(t)=\xi(ts)$ $\forall t, s\in G$. The complex-linear space of such 1-cocycles is denoted by $Z^1(G,\mathscr{H})$. A 1-coboundary on G with values in $\mathscr{H}$ is a mapping $\xi: G\to\mathscr{H}$ of the form $\xi(s)=V(s)^*\xi-\xi$ with $\xi\in\mathscr{H}$. The 1-coboundaries determine a complex-linear subspace $B^1(G,\mathscr{H})$ of $Z^1(G,\mathscr{H})$. The 1-cohomology group $H^1(G,\mathscr{H})=Z^1(G,\mathscr{H})/B^1(G,\mathscr{H})$ is a complex-linear space; an element of $H^1(G,\mathscr{H})$ will be denoted by $\hat{\xi}$ ($\xi\in Z^1(G,\mathscr{H})$).

Lemma 1. *Given $F\in L^\times$, then $F\in L_G^\times$ if and only if there exists a $\xi\in Z^1(G,\mathscr{H})$ such that $F(f)-(V(s)^\times F)(f)=2(\xi(s),f)_{\mathscr{H}}$ $\forall s\in G$, $\forall f\in L$.*

The 1-*cocycle ξ is uniquely determined and $L_G^\times$ is a complex-linear subspace of $L^\times$.*

Proof. The proof is analogous to that of Proposition 1 in [1] and Theorem 2c in [2]. $L_G^\times$ is a complex-linear subspace of $L^\times$ because $Z^1(G,\mathscr{H})$ is a complex-linear space.

Let $\mathscr{H}^*$ be the topological dual of $\mathscr{H}$; $\mathscr{H}^*$ is a complex-linear subspace of $L_G^\times$. The elements of $L_G^\times/\mathscr{H}^*$ are in one-to-one correspondence with sectors containing a coherent state E_F with $F\in L_G^\times$. All states in such a sector trivially induce projectively covariant Weyl systems. It is obvious that these coherent sectors depend on the given group G and the representation $(\mathscr{H}, V)$ of G. Because for a given $F\in L_G^\times$ the 1-cocycle ξ in Lemma 1 is uniquely determined, one can consider the antilinear mapping $F\to\hat{\xi}\in H^1(G,\mathscr{H})$. This mapping induces an antilinear mapping $\hat{F}\to\hat{\xi}$ from $L_G^\times/\mathscr{H}^*$ into $H^1(G,\mathscr{H})$. Let $\mathrm{Inv}_G L$ be the set of all G-invariant complex-linear functionals on L; $\mathrm{Inv}_G L$ is a complex-linear subspace of $L_G^\times$ and so $\widehat{\mathrm{Inv}_G} L$ is a complex-linear subspace of $L_G^\times/\mathscr{H}^*$. The elements of $(L_G^\times/\mathscr{H}^*)/\widehat{\mathrm{Inv}_G} L=\{E_F^G\}$ are denoted by E_F^G; to an element E_F^G corresponds a class of coherent sectors. Again an antilinear mapping $E_F^G\to\hat{\xi}$ is induced. The element E_0^G defines even covariant Weyl systems.

Theorem 2. *The antilinear mapping* $\{E_F^G\} \to H^1(G, \mathscr{H})$ *is injective.*

Proof. Let $E_{F_1}^G$ and $E_{F_2}^G$ be elements of $(L_G^\times/\mathscr{H}_G^*)/\widehat{\mathrm{Inv}_G} L$ and suppose that $\hat{\xi}_1 = \hat{\xi}_2$. Then $\xi_1 - \xi_2 \in B^1(G, \mathscr{H})$, i.e. $\exists \xi \in \mathscr{H}$ such that $\xi_1(s) - \xi_2(s) = (\mathbb{1} - V(s)^*)\xi \ \forall s \in G$. The linear functional B, defined by $B(f) = F_1(f) - F_2(f) - 2(\xi, f)_{\mathscr{H}} \ \forall f \in L$, is G-invariant. This implies $\hat{F}_1 = \hat{F}_2 + \hat{B}$. Because $\hat{F}_1 \in E_{F_1}^G$ and $\hat{F}_2 + \hat{B} \in E_{F_2}^G$ it follows that $E_{F_1}^G = E_{F_2}^G$.

The next theorem relies on results of [3]; we collect a few definitions and single out some statements, whose proofs can be found loco citato.

Let G be a connected n-dimensional Lie group; choose $h \in \mathscr{S}(\mathbb{R})$ such that its Fourier transform $\tilde{h}(p) = \int_{-\infty}^{\infty} e^{ipt} h(t)\,dt$ has the following properties: $\tilde{h}(0) = 1$, $\forall p \in \mathbb{R} \backslash \{0\}$ $0 \leqq \tilde{h}(p) < 1$ and $\tilde{h}''(0) \neq 0$. Let $\{X_1, \ldots, X_n\}$ be a basis for the Lie algebra of G and define the strong operator integral $K(X_k) = \mathbb{1} - \int_{-\infty}^{\infty} V(\exp tX_k) h(t)\,dt$ $(k = 1, \ldots, n)$. These properties imply that the operator $K = \sum_{k=1}^{n} K(X_k)$ is a bounded positive operator on $\mathscr{H}$ with spectral projections $E(\Delta)$ (Δ a Borel set in $\mathbb{R}$). The operator $k := K \restriction (\mathbb{1} - E(0))\,\mathscr{H}$ is a bounded injective positive operator on $(\mathbb{1} - E(0))\mathscr{H}$. On the domain, respectively the range of the operator $k^{1/2}$ the following norms are introduced:

$\forall f \in D(k^{1/2}) = (\mathbb{1} - E(0))\mathscr{H}$ $\quad \|f\|_+ := \|k^{1/2} f\|_{\mathscr{H}}$, respectively $\forall f \in D(k^{-1/2})$ $\|f\|_- := \|k^{-1/2} f\|_{\mathscr{H}}$. We denote the corresponding normed linear spaces by $D_+(G)$, respectively $D_-(G)$ and the completion of $D_+(G)$ in its norm by $\overline{D_+(G)}$.

Lemma 3. *Under the above mentioned conditions the following statements hold.*

1. $D_0 := E(0)\mathscr{H} = \operatorname{Ker} K = \{f \in \mathscr{H} \mid V(s) f = f \ \forall s \in G\}$.

2. *As topological linear spaces, the spaces* $D_+(G)$ *and* $D_-(G)$ *do not depend on the choice of the function* h *and of the basis* $\{X_1, \ldots, X_n\}$ *in the Lie algebra.*

3. k *is an isometry from* $D_+(G)$ *into* $D_-(G)$; *its bounded extension to* $\overline{D_+(G)}$, *also denoted by* k, *is an isometry from* $\overline{D_+(G)}$ *onto* $D_-(G)$.

4. *The mapping* $f, g \to (f, g)_{\mathscr{H}}$ *is a bounded bilinear functional on* $D_+(G) \times D_-(G)$, *and can be extended to a non-degenerate bounded bilinear functional* $(\cdot, \cdot)$ *on* $\overline{D_+(G)} \times D_-(G)$ *having the property* $(f, g) = (kf, g)_-$ $\forall f \in \overline{D_+(G)}$ *and* $\forall g \in D_-(G)$.

5. $\forall s \in G$ $V(s) - \mathbb{1}$ *is a bounded operator from* $\mathscr{H}$ *into* $D_-(G)$.

If A is a bounded operator from $\mathscr{H}$ into $D_-(G)$, then the adjoint A^+ may be defined as a bounded operator from $\overline{D_+(G)}$ into $\mathscr{H}$; if in addition the operator A^*, restricted to $D_+(G)$, is a bounded operator from $D_+(G)$ into $D_+(G)$, then $A^+ = \overline{A^*}$, where the bar means operator closure in the Hilbert space $\overline{D_+(G)}$.

Motivated by the structure of the Poincaré group, we assume in addition that there exists an invariant Abelian connected Lie subgroup H of G; for the representation V of G we demand that the representation $V \restriction H$ on $\mathscr{H}$ does not contain the identity as subrepresentation. The statements of the lemma can be improved [3]:

6. $\{0\}=D_0(H)\supset D_0(G)$ *and* $\mathscr{H}=D_+(H)=\overline{D_-(H)}^{\mathscr{H}}$ *for the underlying sets of the complex-linear spaces* $\mathscr{H}$, $D_+(H)$ *and* $D_-(H)$. *For all* $s\in G$ $V(s)$ *(suitably restricted) is a bounded operator from* $D_-(H)$ *into* $D_-(H)$ *and from* $D_+(H)$ *into* $D_+(H)$ *and* $\mathbb{1}-V(s)$ *is a bounded operator from* $\mathscr{H}$ *into* $D_-(H)$.

7. $\forall\xi\in Z^1(G,\mathscr{H})$ $\exists\eta\in\overline{D_+(H)}$ *such that* $\xi(s)=(\mathbb{1}-V(s))^+\eta$ $\forall s\in G$. η *is uniquely defined if and only if the representation* $s\to\overline{V(s)}$ *does not contain the identity as subrepresentation.*

With the foregoing Lemma we can now formulate a sufficient condition for the injection in Theorem 2 to be a bijection. If $H^1(G,\mathscr{H})=\{0\}$, this injection is automatically a bijection and Theorem 4 is superfluous.

Theorem 4. *Let* G *be a connected, n-dimensional Lie group,* $(\mathscr{H},V)$ *a strongly continuous unitary representation of* G, H *an invariant Abelian connected Lie subgroup of* G *such that* $V\restriction H$ *on* $\mathscr{H}$ *does not contain the identity as subrepresentation and* L *a dense complex-linear subset of* $\mathscr{H}$, *invariant under* G.

If $L\subset D_-(H)$, *then the injective mapping in Theorem 2 is surjective.*

Proof. For $\xi\in Z^1(G,\mathscr{H})$ there exists by Lemma 3.7 an $\eta\in\overline{D_+(H)}$ such that $\xi(s)=(\mathbb{1}-V(s))^+\eta$ $\forall s\in G$. Define $F(f)=2(\eta,f)$ $\forall f\in L$. Then $F\in L^\times$ and $F(f)-(V(s)^\times F)(f)=2(\xi(s),f)_{\mathscr{H}}$ $\forall f\in L$. Hence $F\in L_G^\times$ by Lemma 1.

3. Calculation of a Cohomology Group

The coherent states seem to be suitable to establish the cohomology group of a Lie group G, whose structure is analogous to that of the Poincaré group.

Theorem 5. *Let* $G,H,(\mathscr{H},V)$ *be as in Theorem 4. Let* R *be a compact subgroup of* G *such that* $\overline{V}\restriction R$ *on* $\overline{D_+(H)}$ *does not contain the identity as a subrepresentation. Then* $H^1(G,\mathscr{H})=\{0\}$.

(As to the possibility of defining the operators $\overline{V(s)}$ $\forall s\in G$ *see Lemma 3.6.)*

Proof. The complex-linear subset $L=D_-(H)$ is dense in $\mathscr{H}$ and is invariant under the group G (Lemma 3.6). Theorem 4 proves the existence of a bijection from $\{E_F^G\}$ onto $H^1(G,\mathscr{H})$. The subgroup R of G gives a linear mapping $E_F^G\to E_F^R$ from $\{E_F^G\}$ into $\{E_F^R\}$; this mapping is well defined because $R\subset G$. We prove that this mapping is faithful. Let $E_{F_1}^G$ and $E_{F_2}^G$ be two elements of $\{E_F^G\}$ such that $E_{F_1}^R=E_{F_2}^R$. Choosing for $F_i(i=1,2)$ the representatives $F_i(f)=2(\eta_i,f)$ $(i=1,2)$ with $\eta_i\in\overline{D_+(H)}$, this implies the existence of an R-invariant complex-linear functional b on L and of an $\eta\in\mathscr{H}$ such that $F_1(f)-F_2(f)=b(f)+2(\eta,f)_{\mathscr{H}}$ $\forall f\in L$. Hence $(\eta_1-\eta_2-\eta,(\mathbb{1}-V(s))f)=0$ $\forall f\in L$, $\forall s\in R$ and so one gets $\eta_1-\eta_2=\eta$, i.e. $E_{F_1}^G=E_{F_2}^G$. Theorem 2 gives an appropriate injective antilinear mapping from $\{E_F^R\}$ into $H^1(R,\mathscr{H})$. We now have an injective linear mapping from $H^1(G,\mathscr{H})$ into $H^1(R,\mathscr{H})$. Because R is a compact group, one has $H^1(R,\mathscr{H})=\{0\}$ and therefore $H^1(G,\mathscr{H})=\{0\}$.

This theorem can be applied to calculate the cohomology group $H^1(G,\mathscr{H})$, where G is the Poincaré group $P_+^\uparrow$, $\mathscr{H}$ the carrier space of the representation $[0,1]\oplus[0,-1]$ of G, H the subgroup of space-time translations and R the subgroup of rotations. The condition on the representation of R is fulfilled (Lemma 2.1 in [4]). Using Theorem 5 one gets $H^1(G,\mathscr{H})=\{0\}$. This implies that the interesting sectors are those sectors containing a coherent state $E_{F_\sigma}(\sigma \in \Sigma)$ with F_σ a G-invariant complex-linear functional on L and $F_{\sigma_1} - F_{\sigma_2}$ an unbounded complex-linear functional on L $\forall \sigma_1, \sigma_2 \in \Sigma$ with $\sigma_1 \neq \sigma_2$.

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