Possible New Strange Attractors With Spiral Structure

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Abstract. We define a class of three-dimensional differential equations which might present strange attractors of a new kind. This is illustrated by numerical observations on an explicit example.

I. Introduction

These last years a lot of efforts have been devoted to the study of chaotic behaviours which may arise in some dynamical systems. The relevance of these works to understand the transition to turbulence in physical and chemical experiments seems now well established [1]. A mathematical theory of Axiom A "strange attractors" has been fairly achieved and provides a satisfying description of some systems which do have complicated asymptotic behaviour [2,3]. Nevertheless this theory does not allow us to understand the stochasticity generated in numerical investigations of diffeomorphisms and differential equations given by explicit algebraic expressions. Indeed most of these numerical studies deal with generalizations of the Henon mapping [4,5] or its suspensions [6]. The existence of chaotic behaviour can be proved in these cases [7–10] but, according to [11], the very existence of strange attractors is far from being settled since homoclinic tangencies [12] arise quite naturally with one-parameter families of such systems[1]. Perhaps a special status should be attributed to the Lorenz system [14] which could fall in the class of topological models defined by Guckenheimer [15] and which posses effective strange attractors [16,17].

In this paper we propose a construction of what might reveal to be a new kind of strange attractors. This construction is illustrated by numerical observations on an explicit example of differential equations describing a forced oscillator [18].

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1 Recently Misiurewicz [13] proved the existence of strange attractors for the homeomorphism $(x, y) \rightarrow (1 - a|x| + by, x)$, under some conditions on the parameters a *and* b

II. A Class of Differential Equations

Our main tool is the following theorem:

Theorem. *Consider the system*

$$\begin{cases} \dot{x} = \varrho x - \omega y + P(x, y, z) \\ \dot{y} = \omega x + \varrho y + Q(x, y, z) \\ \dot{z} = \lambda z + R(x, y, z) \end{cases} \tag{1}$$

where P, Q, R are C^r functions $(1 \leqq r \leqq \infty)$ vanishing together with their first derivatives at the origin $0 = (0, 0, 0)^2$. Let us assume that one of the orbits, denoted by Γ_0, leaves the origin and returns to it as $t \to +\infty$. Then if

$$\lambda > -\varrho > 0 \tag{2}$$

every neighbourhood of the orbit Γ_0 contains a denumerable set of unstable periodic solutions of saddle type.

Proof. This theorem was proved in [19] under the inessential assumption that P, Q, R are analytic functions in $\mathbb{R}^3$. The associated symbolic dynamics was investigated in any dimension by the same author in [20] where it is shown that a Birkhoff-Morse system (shift with infinitely many symbols) is contained in a neighbourhood of the homoclinic trajectory Γ_0.

We will however sketch a proof of the above theorem based on a geometrical construction (Fig. 1) which explains the structure of the attractor we shall get.

Following Hartman-Grobman theorem [21, 22][3], instead of (1) we can work with a vector field which is conjugate to (1) and linear in some neighbourhood V of the origin 0 (then the local invariant manifolds of 0 are linear in V).

Let Ω_0 be a point of the homoclinic curve Γ_0 close enough to the saddle focus 0, and Π_0 part of a plane orthogonal to the local stable manifold W_{loc}^s of 0 and containing Ω_0. We define P_0 as the Poincaré's first return map on Π_0. Then let $A_1 \in \Pi_0$ and $A_1' = P_0(A_1)$. We can construct B_1' and B_1 in Π_0 such that $A_1 B_1' A_1' B_1$ is a rectangle (we choose A_1 such that the distances from Ω_0 to A_1 and B_1' are equal). We set $A_2' = B_1'$ and $A_2 = P_0^{-1}(A_2')$. Then continuing this iterative procedure, we generate a sequence of rectangles $\{A_i B_i' A_i' B_i\}$ in Π_0.

Let Ω_1 be a point of the local unstable manifold W_{loc}^u of 0, close enough to 0, and Π_1 part of a plane orthogonal to W_{loc}^u and containing Ω_1. We define $T_0 : \Pi_0 \to \Pi_1$ as the map which associates to each point M of Π_0, the first intersection with Π_1 of the orbit issued from M. Then the family of rectangles $\{A_i B_i' A_i' B_i\}$ transforms under T_0 as represented on Fig. 1.

Now let $T_1 : \Pi_1 \to \Pi_0$ be the map which associates to each point M of Π_1, the first intersection with Π_0 of the orbit originating from M. Then $T = T_1 \circ T_0$ maps Π_0 into itself. The above theorem results from the fact that if A_1 is close enough to W_{loc}^s then (2) and the regularity of the vector field imply that for each i, $T|_{A_i B_i' A_i' B_i}$ is a topological horseshoe [2, 7]. □

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2   Using Poincaré's terminology, the origin is then a saddle focus
3   Actually a C^r linearization $(1 \leqq r \leqq \infty)$ would be available under mild (non resonance) assumptions on eigenvalues [23]. This let T_1 regular enough. One then uses density of non resonant systems and stability of topological horseshoes to conclude
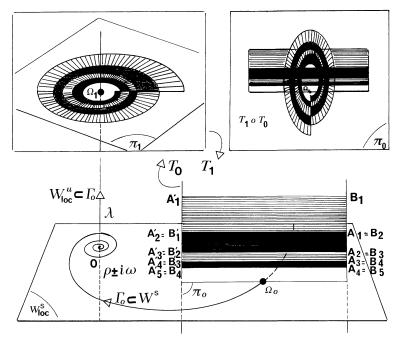
Fig. 1. Geometrical construction used for the proof of Shil'nikov's theorem. We have also represented how acts the map $T = T_1 \circ T_0$

At this point let us remark that T maps parts of each rectangle $A_i\, B'_i\, A'_i\, B_i$ on the opposite side of W^s_{loc}, which prevents defining a T-invariant region in Π_0. This leads us to specify the dynamics on both side of W^s_{loc}. We proceed in a simple way by imposing on (1) the symmetry with respect to the origin: $(x, y, z) \rightarrow (-x, -y, -z)$. Then if $\bar{A}'_1$ and $\bar{B}_1$ are symmetric to A'_1 and B_1 with respect to W^s_{loc} (W^s_{loc} is linear in V), we can find systems such that, as soon as A_1 is far enough from W^s_{loc}, the set theoretical complementary of W^s_{loc} in the open rectangle $\bar{B}_1\, \bar{A}'_1\, A'_1\, B_1$, is invariant under the Poincaré first-return map defined on it. (The rectangles $\{A_i\, B'_i\, A'_i\, B_i\}$ with i large enough still support topological horseshoes.)

For such systems it is obvious that $T_0 \left(\bigcup_{i=1}^{\infty} A_i\, B'_i\, A'_i\, B_i \right)$ is invariant under the Poincaré first-return map P_1 defined on Π_1. Using the foliation depicted in Fig. 1, we can conclude that any attractor of P_1 must be included in a spiral asymptotic to Ω_1.

III. An Example

Now we come to the numerical investigation of a specific example choosen in a class of differential equations introduced in [18].

$$\begin{cases} \dddot{x} + \beta \ddot{x} + \dot{x} = \eta \\ \\ \dot{\eta} = f_\mu(x) \end{cases} \quad \text{or} \quad \begin{cases} \dot{x} = y \\ \dot{y} = z \\ \dot{z} = -y - \beta z + f_\mu(x), \end{cases} \tag{3}$$
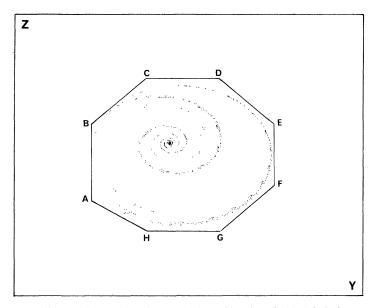
Fig. 2. For system (3) with $f_\mu(x)$ given by (5) and $a \simeq 0.2171604$ and $\mu \simeq 0.2061612$, the image under P_1 of a lattice of 1000 points taken in the polygon $ABCDEFGH$ with $A(0.000001, -0.162)$, $B(0.000001, 0.108)$, $C(0.141, 0.270)$, $D(0.329, 0.270)$, $E(0.470, 0.108)$, $F(0.470, -0.108)$, $G(0.329, -0.270)$, $H(0.141, -0.270)$

where $\beta > 0$ is the dissipation and f_μ a real function depending on the parameter μ. In a previous paper [24], we considered system (3) with the particular choice

$$f_\mu(x) = \begin{cases} 1 + ax & \text{if} \quad x \leqq 0 \\ 1 - \mu x & \text{if} \quad x \geqq 0 \end{cases} \tag{4}$$

and proved the existence for some a_0, μ_0 of a homoclinic orbit Γ_0 satisfying the conditions of Shil'nikov's theorem [19]. This allowed us to understand the structure of the chaotic behaviours numerically displayed by (3) for values of μ less than μ_0. Nevertheless let us emphasize that for such functions $f_\mu(x)$ with only one extremum, no attractor is observable simultaneously with Γ_0 [24]. In order to remedy this fact, we impose the above mentioned symmetry $(x, y, z) \rightarrow (-x, -y, -z)$ by choosing $f_\mu(x)$ as defined by

$$f_\mu(x) = \begin{cases} -\mu x - \mu - a & \text{if} \quad x \leqq -1 \\ ax & \text{if} \quad |x| \leqq 1 \\ -\mu x + \mu + a & \text{if} \quad x \geqq 1 \end{cases} \tag{5}$$

Two main reasons lead us to consider a piecewise linear function[4,5]:

   i) Using the method developed in [24], we can prove the existence of values of the parameters such that there exist two homoclinic orbits Γ_0 and Γ'_0 symmetric with respect to the saddle focus $(0,0,0)$, all conditions of Shil'nikov's theorem being fulfilled.

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4   Analogous results have also been obtained with analytical functions
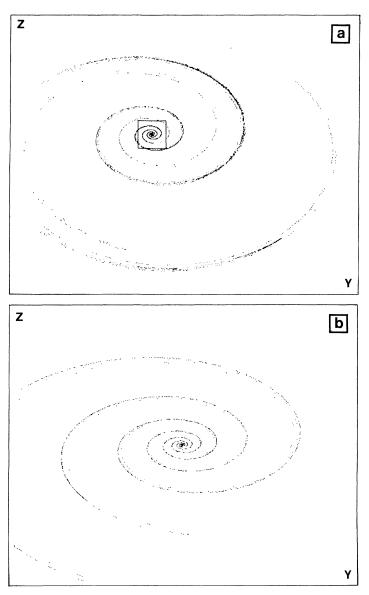5   The above theorem is readily adapted to this case

Fig. 3. a The spiral like attractor numerically observed for P_1; **b** Magnification of the boxed-in region in **a**; same model as in Fig. 2

ii) The solutions of (3) can be computed with a very accurate numerical scheme: the solutions are known analytically in each of the strata $x \leqq -1$, $|x| \leqq 1$ and $x \geqq 1$, and we only need to compute with Newton's method the return times on the planes

$$x = -1 \quad \text{and} \quad x = +1.$$

Our numerical procedure makes easy the computation of the Poincaré first return map on the plane $x = 1$, which we consider as including Π_1. Figure 2 gives
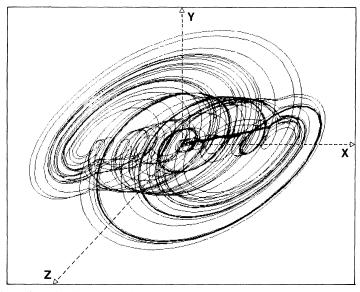
Fig. 4. A three-dimensional representation of the attractor; same model as in Fig. 2
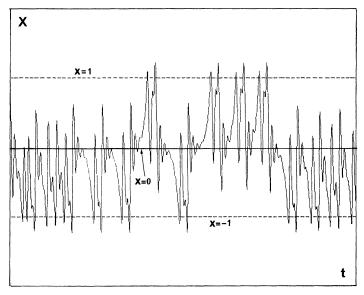

Fig. 5. A typical sample of the time evolution of the x-coordinate; same model as in Fig. 2

evidence that this Poincaré map P_1 admits an invariant polygonal surface for values of the parameters such that Γ_0 and Γ_0' exist ($a_0 \simeq 0.2171604$, $\mu_0 \simeq 0.2061612$). The corresponding attractor is represented in Fig. 3a, where the spiral structure previously predicted is clearly observed (this spiral structure can be guessed on a 3-dimensional plot of the attractor of the corresponding flow: See Figs. 4 and 5). The magnification of the boxed-in region (Fig. 3b) strongly suggests

that the attractor is scale-invariant and comes in each neighbourhood of Ω_1 which has no image under P_1. If it turns out to be true, the attractor cannot be a periodic orbit, and we conjecture that we are faced with a strange attractor of a new kind.

IV. Conclusion

We must emphasize that when we increase the parameter μ from 0 to the value μ_0 where Γ_0 and Γ_0' exist, the dynamics associated with system (3) becomes more and more complicated. The initial evolution (when observed on an arbitrary Poincaré first-return map) is similar to what is obtained when strengthening the non-linearities in the Henon mapping [4]. Further evolution corresponds to a succession of topological horseshoes with more and more branches i.e. semi-conjugate to full shifts on a alphabet with more and more symbols.

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