Commun. Math. Phys. 79, 25–32 (1981)

Communications in
Mathematical
Physics

A Note on the Stability of Phase Diagrams in Lattice Systems

A.C.D. van Enter
Institute for Theoretical Physics, University of Groningen, P.O. Box 800, Groningen, The Netherlands

Abstract. We construct a class of non-symmetry breaking pair interactions, which change the phase diagram of the *n.n.* Ising and classical XY model. Furthermore we improve earlier obtained constraints on the decrease of interactions, necessary to get analyticity properties of the pressure in manifolds of non-symmetry breaking interactions.

1. Introduction, Notation and Some Known Results

Heuristically, in thermodynamics one expects that the Gibbs phase rule holds, in the sense that there are manifolds in some interaction space on which
1. suitable thermodynamic functions are analytic
2. the number of possible phases remains the same [4].

In [1] it was shown that in spaces $\mathscr{B}_g$ defined by

$$\|\Phi\|_g = \sum_{0\in X\subset \mathbb{Z}^d} g(X)|\Phi(X)|$$

2. does not hold if $g(X)$ is only dependent on the number of points in X and
1. does not hold if $g(X)$ increases more slowly than $(diam(X))^{1/2}$.

The existence of manifolds on which more than one phase coexist has been shown by Peierls contour arguments. In finite dimensional interaction spaces, containing classical interactions of finite range, this has been done in [2]; for a more restricted class of interactions, but in the infinite dimensional subspace of pair interactions with $g(X) = diam(X)$, in [3].

In this paper we improve the results of [1] for the 2 dimensional Ising model. We construct a classical long range perturbation, which does not break the symmetry of the transition, but changes the phase diagram when added to the Ising interaction with any positive strength. This implies that the result in [3] in a sense is best possible. For analyticity we give the improved condition (which is necessary but probably not sufficient in view of our result on the stability of the phase diagram) that $g(X)$ has to increase at least as $diam(X)^{2/3}$.

Furthermore we consider the broken rotation symmetry of the classical XY model and show that in that case the stability of the phase transition is lower than in the case of a broken discrete symmetry.

0010-3616/81/0079/0025/$01.60

Remark. We have taken the Ising and the classical XY model as examples, but our results can be directly generalized to a much wider class of classical lattice systems incorporating all short range interactions, which give rise to a phase transition with a broken symmetry. The generalization to quantum interactions is also straightforward (if we start with the special case of a discrete spin classical interaction it is a direct consequence of the fact that neighbourhoods in classical interaction spaces lie inside the corresponding quantum neighbourhood [14]).

For general properties of lattice systems we refer to [4]. We will make repeatedly use of the variational principle

$$P(\Phi) = \sup_{\rho \in I} s(\rho) - \rho(A_\Phi) \qquad (*)$$

[11 Chap. 7.4; 4 Chap. 2] where the equality is reached for elements of I_Φ, the translation invariant equilibrium states for the interaction Φ. As in [1] I is the set of translation invariant states, P the pressure and s the entropy per lattice site.

Further we write G_Φ for the Gibbs states for Φ, i.e. the states for which the DLR equations hold [5] (compare [14]) if $\|\Phi\|_1 = \sum_{0 \in X} |\Phi(X)| < \infty$. $G_\Phi \supset I_\Phi$. Both I_Φ and G_Φ are Choquet simplices. Hence, if $\rho \in I_\Phi$, ρ has a unique decomposition in extremal invariant equilibrium states and also a (in general finer) unique decomposition in extremal DLR states [5, 11]. Extremal DLR states have short range correlations i.e. for any observable $A \in C(\{\Omega\}^{\mathbb{Z}^d})$ there is a finite $\Lambda \subset \mathbb{Z}^d$ such that

$$|\rho(AB) - \rho(A)\rho(B)| \leqq \|B\|$$

if $B \in C(\{\Omega\}^{\mathbb{Z}^d/\Lambda})$. [5. Th. 3.4; see also 20] (Ω is a compact one spin configuration space). If ρ is extremal DLR and invariant, this implies

$$\lim_{x \to \infty} \rho(A\tau_x B) = \rho(A)\rho(B),$$

while if we only know that ρ is extremal invariant, but not necessarily extremal DLR, this limit needs only to exist in the Cesaro sense (ρ is weakly clustering [6, 12]).

Definition. An interaction Ψ does not break the symmetry of the phase transition at Φ if $\rho(A_\Psi)$ is the same for all $\rho \in I_\Phi$.

2. Instability of the Ising Phase Diagram

For the 2-d ferromagnetic $n.n.$ Ising model with interaction Φ_{Is}, it is well known that, if the external field $H = 0$ and $T < T_c$, there exist two invariant extremal DLR states ρ^+ and ρ^- with magnetization $+m$ and $-m$ [7] (It is even true that $I_{\Phi_{Is}} = G_{\Phi_{Is}}$ [13]).

As in [1] we will use spinflip operators.

For any $\Omega \subset \mathbb{Z}^2$, R_Ω is the operator which flips the spins in Ω. For $N \in \mathbb{N}$ we divide $\mathbb{Z}^2$ in layers as follows:

$$\mathbb{Z}^2 = \bigcup_{i=-\infty}^{\infty} L_N^i \text{ where } L_N^i = \{x_1, x_2 \mid N^i \leqq x_1 < N(i+1)\}.$$

Furthermore we define: $\Omega_N = \bigcup_{i=-\infty}^{\infty} L_N^{2i+1}$ and $R_N = R_{\Omega_N}$. Now we introduce a class of perturbations which destroy the Ising transition, denoted by Φ_Z^a. Φ_Z^a is a non-symmetry breaking pair interaction with only antiferromagnetic terms, defined by:

$$\Phi_Z^a(x\tau_{j,0}x) = \sum_{i=0}^{\infty} \delta_{j,2^i} a^i \quad \tfrac{1}{2} < a < 1 \quad \forall x \in \mathbb{Z}^2$$

$$\Phi_Z^a(X) \quad = 0 \text{ otherwise.}$$

Note that

$$\| \Phi_Z^a \| = \tfrac{1}{2} \| \Phi_Z^a \|_1 = \sum_{i=0}^{\infty} a^i = \frac{1}{1-a} < \infty.$$

Now we can prove the following:

Theorem 1. *For $\Phi_{Is} + \lambda \Phi_Z^a$, there do not exist two invariant extremal DLR states with a magnetization different from 0 for any a with $\frac{1}{2} < a < 1$ and $\lambda > 0$.*

Proof. Suppose that there did exist $\omega_{a,\lambda}^+$ and $\omega_{a,\lambda}^-$, translation invariant extremal *DLR* states with a nonzero magnetization $\pm m_1$. Then (*) implies:

$$P(\Phi_{Is} + \lambda \Phi_Z^a) = s(\omega_{a,\lambda}^{\pm}) - \omega_{a,\lambda}^{\pm}(A_{\Phi_{Is} + \lambda \Phi_Z^a}) = \sup_{\rho \in I} s(\rho) - \rho(A_{\Phi_{Is} + \lambda \Phi_Z^a}). \tag{1}$$

We will now construct a state $\tilde{\omega}_{a,\lambda,N_0} \in I$ such that:

$$s(\tilde{\omega}_{a,\lambda,N_0}) - \tilde{\omega}_{a,\lambda,N_0}(A_{\Phi_{Is} + \lambda \Phi_Z^a}) > s(\omega_{a,\lambda}^{\pm}) - \omega_{a,\lambda}^{\pm}(A_{\Phi_{Is} + \lambda \Phi_Z^a}). \tag{2}$$

Clearly (2) contradicts (1) which implies that, contrary to the assumption $\omega_{a,\lambda}^{\pm}$ cannot be equilibrium states.

Define

$$\tilde{\omega}_{a,\lambda,N} = \frac{1}{2N} \sum_{i=1}^{2N} \omega_{a,\lambda}^{+} \circ R_N \tau_{i,0} \left(= \frac{1}{2N} \sum_{i=1}^{2N} \omega_{a,\lambda}^{-} \circ R_N \tau_{i,0} \right).$$

Since $\omega_{a,\lambda}^+$ as an extremal DLR state has short range correlations there exists $k \in \mathbb{N}$ such that:

$$\omega_{a,\lambda}^+(\sigma_{0,0}\sigma_{i,0}) \geqq \tfrac{1}{2} m_1^2 \quad \forall i \geqq 2^k. \tag{3}$$

Furthermore we know that $s(\omega_{a,\lambda}^+) = s(\tilde{\omega}_{a,\lambda,N}) \forall N$ ([1] Lemma 3). To prove (2) we need to show that

$$\omega_{a,\lambda}^+(A_{\Phi_{Is} + \lambda \Phi_Z^a}) - \tilde{\omega}_{a,\lambda,N}(A_{\Phi_{Is} + \lambda \Phi_Z^a}) > 0 \tag{4}$$

for some N. Take $N = 2^l$ for some l.

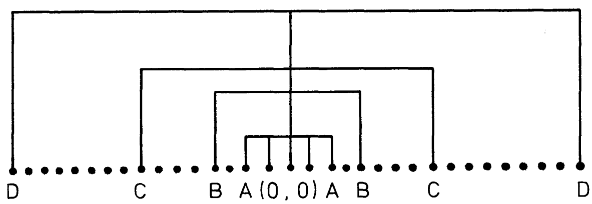
We will split $A_{\Phi_{Is}+\lambda\Phi_Z^a}$ in 4 parts consisting of

$$A = -\Phi_{Is}\tfrac{1}{2}(\sigma_{-1,0}\sigma_{0,0}+\sigma_{1,0}\sigma_{0,0}+\sigma_{0,1}\sigma_{0,0}+\sigma_{0,-1}\sigma_{0,0}) + \tfrac{1}{2}\lambda\sum_{i=0}^{k} a^i(\sigma_{0,0}\sigma_{2^i,0}+\sigma_{0,0}\sigma_{-2^i,0})$$

$$B = \tfrac{1}{2}\lambda\sum_{i=k+1}^{l-1} a^i(\sigma_{0,0}\sigma_{2^i,0}+\sigma_{0,0}\sigma_{-2^i,0})$$

$$C = \tfrac{1}{2}\lambda a^l(\sigma_{0,0}\sigma_{2^l,0}+\sigma_{0,0}\sigma_{-2^l,0})$$

$$D = \tfrac{1}{2}\lambda\sum_{i=l+1}^{\infty} a^i(\sigma_{0,0}\sigma_{2^i,0}+\sigma_{0,0}\sigma_{-2^i,0}).$$



Example with
k = 1 and l = 3

$$\omega^+_{a,\lambda}(A)-\tilde{\omega}_{a,\lambda,N}(A) > -\frac{2}{N}\|\Phi_{Is}\| - \frac{\lambda}{N}\sum_{i=1}^{k} 2^i a^i > -\frac{1}{N}\left(2\|\Phi_{Is}\| + \frac{\lambda(2a)^{k+1}}{2a-1}\right)$$
$$= \frac{-1}{2^l}\left(2\|\Phi_{Is}\| + \frac{\lambda(2a)^{k+1}}{2a-1}\right). \tag{5}$$

By (3) we have

$$\omega^+_{a,\lambda}(B)-\tilde{\omega}_{a,\lambda,N}(B) > \lambda\sum_{i=k+1}^{l-1}\tfrac{1}{2}m_1^2\frac{a^i}{2^{l-i}} > 0 \tag{6}$$

and

$$\omega^+_{a,\lambda}(C)-\tilde{\omega}_{a,\lambda,N}(C) > \lambda m_1^2 a^l. \tag{7}$$

Furthermore

$$\omega^+_{a,\lambda}(D)-\tilde{\omega}_{a,\lambda,N}(D) = 0. \tag{8}$$

Combining (5), (6), (7) and (8) we get (4) and hence (2) by choosing N large enough. □

Note that the same proof works whenever we take instead of a^l any positive function f for which

$$\lim_{l\to\infty} 2^l f(2^l) = \infty.$$

3. Constraints on Analyticity

In [1] it was shown that P is not analytic [18] on spaces $\mathscr{B}_g$ if g increases more slowly than $(diam\,(X))^{1/2}$ if $N(X)=2$.

This result can be improved somewhat as the following Theorem shows.

Theorem 2: *At Φ_{Is} P cannot be analytic on a space $\mathscr{B}_g$ if*

$$\lim_{\substack{N(X)=2\\ diam(X)\to\infty}} \frac{g(X)}{diam\,(X)^{2/3}} = 0.$$

Proof. Define Ψ_N to be the antiferromagnetic pair interaction between points at distance N:

$$\Psi_N(x\tau_{j,0}x) = \delta_{j,N}$$
$$\Psi_N(X) = 0 \text{ otherwise.}$$

If P is analytic in $\mathscr{B}_g$ and $\|\varepsilon\Psi_N\|_g$ small

$$P(\Phi_{Is}+\varepsilon\Psi_N) = P(\Phi_{Is}) - \varepsilon\rho^+(A_{\Psi_N}) + \varepsilon^2 P''_{Is}(\Psi_N, \Psi_N) + \varepsilon^3 O(\|\Psi_N\|_g^3) \tag{9}$$
$$P(\Phi_{Is}-\varepsilon\Psi_N) = P(\Phi_{Is}) + \varepsilon\rho^+(A_{\Psi_N}) + \varepsilon^2 P''_{Is}(\Psi_N, \Psi_N) + \varepsilon^3 O(\|\Psi_N\|_g^3) \tag{10}$$

where P''_{Is} is the bilinear form on $\mathscr{B}_g \times \mathscr{B}_g$ which is the second derivative of P at Φ_{Is}.
The difference of the right hand sides of (9) and (10) is

$$-2m^2\varepsilon + \varepsilon^3 O(\|\Psi_N)\|_g^3).$$

The difference of the left hand sides of (9) and (10) is of order $\frac{1}{N}$. This can be seen by the following argument:

Let $\omega_N \in I_{\Phi+\varepsilon\Psi_N}$ and $\tilde{\omega}_N = \frac{1}{2N}\sum_{i=1}^{2N} \omega_N \circ R_N\tau_{i,0}$.

From (*) we have

$$\begin{aligned} P(\Phi_{Is}-\varepsilon\Psi_N) - P(\Phi_{Is}+\varepsilon\Psi_N) &\geqq s(\tilde{\omega}_N) - s(\omega_N) - \tilde{\omega}_N(A_{\Phi_{Is}}) + \omega_N(A_{\Phi_{Is}}) \\ &\quad + \varepsilon\tilde{\omega}_N(A_{\Psi_N}) + \varepsilon\omega_N(A_{\Psi_N}) \geqq -\frac{2}{N}\|\Phi_{Is}\| \end{aligned} \tag{11}$$

because

$$\tilde{\omega}_N(A_{\Psi_N}) = -\omega_N(A_{\Psi_N}) \text{ and } s(\tilde{\omega}_N) = s(\omega_N).$$

In the same way we can prove:

$$P(\Phi_{Is}+\varepsilon\Psi_N) - P(\Phi_{Is}-\varepsilon\Psi_N) \geqq -\frac{2}{N}\|\Phi_{Is}\|. \tag{12}$$

Hence

$$|P(\Phi_{Is}+\varepsilon\Psi_N) - P(\Phi_{Is}-\varepsilon\Psi_N)| \leqq \frac{2}{N}\|\Phi_{Is}\|.$$

Thus

$$|-2m^2|\varepsilon + \varepsilon^3 O(\|\Psi_N)\|_g^3 \leqq \frac{2}{N}\|\Phi_{Is}\|. \tag{13}$$

By taking $\varepsilon = O\left(\frac{1}{N}\right)$ we see that

$$\| \Psi_N \|_g^3 = O(N^2).$$

Remark. Note that this should be true for any infinite subsequence N_k of $\mathbb{N}$.

4. Comments on the Ising Case

Ginibre et al. [3] proved that in applying the Peierls argument one is allowed to add small perturbations in the space of pair interactions satisfying:

$$\| \Phi \|_{iGGR} = \sum_{0 \in X} diam_i(X) |\Phi(X)| < \infty \quad \forall i$$

where $diam_i(X)$ is the maximal distance in the i'th direction between points of X.

Theorem 1 shows that this cannot be much improved in the sense that any condition of the form

$$\sum_{0 \in X} g(diam_i(X)) |\Phi(X)| < \infty$$

on the perturbation is too weak if

$$\lim_{l \to \infty} \frac{g(2^l)}{2^l} = 0.$$

(This implies that the GGR result is the best possible if g should be monotonic).

Analyticity results of Kunz and Souillard have been announced [8] in some space $\mathscr{B}_g$ with $\sum_{0 \in X} \frac{1}{g(X)} < \infty$ with the corresponding strong cluster-properties [9].

There are spaces $\mathscr{B}_g$ however, for which this condition holds which contain a Φ_Z^a of our Theorem 1 or in which the technique of our Theorem 2 can be applied [10], so without some other conditions this condition cannot be sufficient.

5. Instability of the Phase Diagram of the Classical XY Model

In the 3-*d* *n.n.* classical XY model it was proved in [15], that for $H = 0$ and T sufficiently low there is spontaneous magnetization in the sense that for some state $\rho \in I_{\Phi_{XY}}$

$$\lim_{n \to \infty} \rho(\vec{s}_{0,0,0} \cdot \vec{s}_{n,0,0}) = \lim_{n \to \infty} \rho(\cos(\Theta_{0,0,0} - \Theta_{n,0,0})) = m^2 > 0. \tag{14}$$

For this case we can prove

Theorem 3. *For $\Phi_{XY} + \lambda \Phi_Z^a$ there do not exist equilibrium states with spontaneous magnetization in the sense of* (14) *for any a with $\frac{1}{4} < a < 1$ and $\lambda > 0$.*

Here Φ_Z^a is the (rotation invariant) classical XY interaction defined analogous to the Φ_z^a in Theorem 1.

Proof. The proof goes essentially in the same way as in Theorem 1. However, instead of $\tilde{\omega}_{a,\lambda,N}$ we now use $\tilde{\tilde{\omega}}_{a,\lambda,N}$ defined as follows:

Let $\tilde{R}_N^{\pm}$ be defined by:

$$\tilde{R}_N^{\pm}\,\Theta_{i,j,k} = \Theta_{i,j,k} \pm \frac{i\pi}{N} \quad \forall\{i,j,k\}\in\mathbb{Z}^3$$
$$\Theta_{i,j,k}\in(S^1)_{i,j,k}.$$

Then

$$\tilde{\tilde{\omega}}_{a,\lambda,N} = \frac{1}{4N}\left(\sum_{i=1}^{2N}\omega_{a,\lambda}\circ\tilde{R}_N^{+}\tau_{i,0,0} + \omega_{a,\lambda}\circ\tilde{R}_N^{-}\tau_{i,0,0}\right)$$

(5) becomes in this case (using $\frac{1}{2}\cos\,(x+y)+\frac{1}{2}\cos(x-y)=\cos x\cos y$ and $1-\cos x<x^2$):

$$\begin{aligned}
&\omega_{a,\lambda}(A)-\tilde{\tilde{\omega}}_{a,\lambda,N}(A)\\
&\quad=\omega_{a,\lambda}\left(-\cos(\Theta_{0,0,0}-\Theta_{1,0,0})+\tfrac{1}{2}\cos\left(\Theta_{0,0,0}-\Theta_{1,0,0}+\frac{\pi}{N}\right)\right.\\
&\qquad\left.+\tfrac{1}{2}\cos\left(\Theta_{0,0,0}-\Theta_{1,0,0}-\frac{\pi}{N}\right)\right)+\lambda\omega_{a,\lambda}\left(\sum_{i=0}^{k}a^i\left(\cos(\Theta_{0,0,0}-\Theta_{2^i,0,0})\right.\right.\\
&\qquad\left.\left.-\tfrac{1}{2}\cos\left(\Theta_{0,0,0}-\Theta_{2^i,0,0}+\frac{2^i\pi}{N}\right)-\tfrac{1}{2}\cos\left(\Theta_{0,0,0}-\Theta_{2^i,0,0}-\frac{2^i\pi}{N}\right)\right)\right)\\
&\quad\geqq -C\frac{1}{N^2}=-C\frac{1}{4^l}
\end{aligned}\tag{15}$$

for C some positive constant independent of N. The rest of the proof is the same as for Theorem 1. □

As for analyticity, with the same techniques as in Sect. 3 it follows that $g(X)$ has to increase at least like $(diam(X))^{4/3}$ in this case for P to be analytic on a manifold in a rotation invariant Banach subspace $\mathscr{B}_g$.

6. Final Comments

1. In the proof of [15] it is not allowed to have general non *n.n.* interactions, so it is not known if our Theorem 3 gives a "worst possible" result in some sense.
2. Theorems 1 and 3 can also be proven if we take instead of Φ_Z^a isotropic pair interactions with power decrease $\frac{1}{r^\alpha}$ with $d<\alpha<d+1$ in the d dimensional Ising case ($d\geqq 2$) and $d<\alpha<d+2$ in the d dimensional classical XY case ($d\geqq 3$).
3. The above results seem to support Griffiths' and Pearce's [16, 17] claim that renormalization group transformations in the thermodynamical limit may not exist as analytic transformations on interaction spaces, because the phase transition and hence also the critical behaviour is less stable than is generally assumed [16, 19].

Acknowledgements. I profited from discussions with H. A. M. Daniëls, M. Winnink, N. M. Hugenholtz and M. Aizenman. I am grateful to N. M. Hugenholtz and M. Winnink for advising me on the manuscript. This paper is part of the research program of the "Foundation for fundamental research on Matter" (F. O. M.) which is financially supported by the "Netherlands Organization for pure research" (Z. W. O.).

References

1. Daniëls, H. A. M., Enter, A. C. D. van: Commun. Math. Phys. **71**, 65–76 (1980)
2. Pirogov, S. A., Sinai, Ya. G. : Theor. Math. Phys. USSR **25**, 358–369 (1975), **26**, 61–76 (1976)
3. Ginibre, J., Grossmann, A., Ruelle, D. : Commun. Math. Phys. **3**, 187–193 (1966)
4. Israel, R. B. : Convexity in the theory of lattice gases. Princeton, NJ : Princeton University Press 1979
5. Lanford, O., Ruelle, D. : Commun. Math. Phys. **13**, 194–215 (1969)
6. Kastler, D., Robinson, D. W. : Commun. Math. Phys. **3**, 151–180 (1966)
7. Messager, A., Miracle-Sole, S. : Commun. Math. Phys. **40**, 187–196 (1975)
8. Iagolnitzer, D., Souillard, B. : Contribution to the Esztergom 1979. Conference on Random fields (to appear)
9. Iagolnitzer, D., Souillard, B.: Commun. Math. Phys. **60**, 131–152 (1978)
10. Simon, B. : Private Communication
11. Ruelle, D. : Statistical mechanics, New York, Amsterdam: Benjamin (1969)
12. Ruelle, D. : Commun. Math. Phys. **3**, 133–150 (1966)
13. Aizenman, M. : Phys. Rev. Lett. **43**, 407–409 (1979)
14. Brascamp, H. J. : Commun. Math. Phys. **18**, 82–96 (1970)
15. Fröhlich, J., Simon B., Spencer T. : Commun. Math. Phys. **50**, (1976)
16. Griffiths, R. B., Pearce, P. A. : Phys. Rev. Lett. **41**, (1978); J. Stat. Phys. **20**, 444 (1979)
17. Israel, R. B. : Proceedings Esztergom Conference on Random Fields (to appear)
18. Hille, E., Phillips, R. : Functional analysis and semigroups. Chap. III, Sect. 17. Baltimore: Waverly Press (1959)
19. Enter, A. C. D. van: (to be published)
20. Ruelle, D. : Symmetry breakdown in statistical mechanics. In: Cargèse Lectures in Physics, Vol. 4, Kastler, D. (ed.) pp. 169–194. New York, London, Paris: Gordon and Breach 1970

Communicated by E. Lieb

Received June 13, 1980