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The Boltzmann Equation with a Soft Potential

II. Nonlinear, Spatially-Periodic

Russel E. Caflisch*

Department of Mathematics, Stanford University, Stanford, CA 94305, USA

Abstract. The results of Part I are extended to include linear spatially periodic problems–solutions of the initial value are shown to exist and decay like $e^{-\lambda t^{\beta}}$. Then the full non-linear Boltzmann equation with a soft potential is solved for initial data close to equilibrium. The non-linearity is treated as a perturbation of the linear problem, and the equation is solved by iteration.

1. Introduction

The linear Boltzmann equation with a soft intermolecular potential was solved globally in time in Part I [1], if the initial density is a spatially homogeneous perturbation of a global Maxwellian. Moreover it was proven that this perturbation decays in $\mathscr{L}^2$ or sup norm like $e^{-\lambda t^{\beta}}$, with $\lambda>0$, $1>\beta>0$, if it is initially bounded by a Maxwellian. We will refer to formulas or results from Part I by preceeding their numbers with an "I" as in (I 1.7).

In this paper we find the same result even if the initial perturbation is *spatially dependent* in the cube with periodic boundary conditions. In addition we can solve the spatially periodic *nonlinear* problem globally in time if the initial perturbation is small enough, and we find that the solution decays to the Maxwellian equilibrium.

The linear, spatially-dependent Boltzmann equation is

$$\frac{\partial}{\partial t}f+\boldsymbol{\xi}\cdot\frac{\partial}{\partial \mathbf{x}}f+Lf=0, \tag{1.1}$$

$$f(t=0)=f_0\in\mathcal{N}, \tag{1.2}$$

where f_0 and $f=f(t,\mathbf{x},\boldsymbol{\xi})$ are periodic in $\mathbf{x}\in T^3=[0,2\pi]^3$, $t\geqq 0$, $\boldsymbol{\xi}\in\mathbb{R}^3$, and $\mathcal{N}=\left\{g(\mathbf{x},\boldsymbol{\xi}):\int\limits_{T^3}\int\limits_{\mathbb{R}^3}\psi(\boldsymbol{\xi})g(\mathbf{x},\boldsymbol{\xi})\,d\boldsymbol{\xi}\,d\mathbf{x}=0 \text{ for } \psi(\boldsymbol{\xi})=1,\ \xi_i, \text{ or } \xi^2\right\}$. The requirement that $f_0\in\mathcal{N}$ just means that we have chosen the right Maxwellian equilibrium to

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perturb about, so that it has the total mass, momentum, and energy. Our first result, Theorem 2.1, is that the solution of this problem decays like $e^{-\lambda t^\beta}$.

As in Part I we remove the null space of $L=\nu+K$ by adding on a finite rank operator. $N(L)$ is spanned by the functions $\psi_i(\boldsymbol{\xi})$ defined in (I2.14). We define the modified linear operator

$$\bar{L}=\nu+\bar{K}, \tag{1.3}$$

$$\bar{K}=K+P, \tag{1.4}$$

$$P=\sum_{i=0}^{4}(\psi_i,\cdot)\psi_i, \tag{1.5}$$

where now the inner product is the $\mathscr{L}^2(\mathbf{x},\boldsymbol{\xi})$ inner product. Since ψ_i are independent of x and $Pf_0=0$, the linear problem (1.1), (1.2) is not changed if we replace L by $\bar{L}$. Since the nonlinearity $\nu\Gamma$ of the Boltzmann equation is also perpendicular to ψ_i, this replacement of L by $\bar{L}$ does not affect the nonlinear problem either.

The relevant norms, which are defined in Sect. 2, are $\mathscr{L}^2$ norms of f and its spatial derivatives, then $\mathscr{L}^2$ or sup over $\boldsymbol{\xi}$. The derivatives are introduced in order to be able to use the Sobolev inequality when estimating the nonlinear terms. For consistency they are also included in the linear theory of Sects. 2 and 3 where they are not really needed. The estimates on K given in Part I all have analogues which are presented in Sect. 2.

Just as in Part I the velocities are cut off by defining the characteristic function

$$\chi_w(\xi)=\begin{cases}1 & \xi\leqq w,\\ 0 & \xi>w,\end{cases} \tag{1.6}$$

and introducing

$$B_w=\boldsymbol{\xi}\cdot\frac{\partial}{\partial\mathbf{x}}+\nu(\xi)+\chi_w\bar{K}, \tag{1.7}$$

as an operator on $\mathscr{L}^2(\xi\leqq w,\mathbf{x}\in T^3)$. The only new twist in the spatially dependent problem comes in the analysis of the semigroup e^{-tB_w}, given in Sect. 3. This employs spectral perturbation theory [4] and an argument given by Ukai [5]. The rest of the proof of Theorem 2.1 goes exactly as in Part I.

The nonlinear Boltzmann equation is

$$\frac{\partial}{\partial t}f+\boldsymbol{\xi}\cdot\frac{\partial}{\partial\mathbf{x}}f+Lf=\nu\Gamma(f,f), \tag{1.8}$$

$$f(t=0)=f_0\in\mathcal{N}, \tag{1.9}$$

where f and f_0 are periodic in $\mathbf{x}$. If f_0 is sufficiently small, this problem can be solved for all time and the solution $f(t)$ decays to 0, as stated in Theorem 4.1 in Sect. 4. The estimates on Γ in Sect. 5 state that if f is small, $\nu\Gamma(f,f)$ is even smaller. So this problem is just a perturbation of the linear problem, which also keeps its solution small. The solution is found by an iterative procedure described in Sect. 7, after the iteration equation is analyzed in Sect. 6.

References to previous work and more explanation of the Boltzmann equation are found in Part I. I am very grateful to Harold Grad, who suggested this problem, and to Percy Deift, George Papanicolaou, and Robert Turner for a number of helpful discussions. This work was performed at the Courant Institute and the Mathematics Research Center; I am happy to acknowledge their support.

2. The Linear Equation

We will use an $\mathscr{L}^2$ Sobolev norm over space alone, as well as norms over both $\mathbf{x}$ and $\boldsymbol{\xi}$, which are sup or $\mathscr{L}^2$ norm over $\boldsymbol{\xi}$ of the Sobolev norm over space. If the function is not spatially dependent these $(\mathbf{x}, \boldsymbol{\xi})$-norms are exactly those used in Part I and we will use the same notation.

Definition. Let $f = f(\mathbf{x}, \boldsymbol{\xi})$ be periodic in $\mathbf{x}$. Define

$$\|f(\boldsymbol{\xi}, \cdot)\|_{H_4(x)} = \sum_{s=1}^{4} \left(\int_{T^3} |\nabla^s f(\mathbf{x}, \boldsymbol{\xi})|^2 \, d\mathbf{x} \right)^{1/2}, \tag{2.1}$$

$$\|f\| = \left(\int_{\mathbb{R}^3} \|f(\boldsymbol{\xi}, \cdot)\|_{H_4(x)}^2 \, d\boldsymbol{\xi} \right)^{1/2}, \tag{2.2}$$

$$\|f\|_{\alpha, r} = \sup_{\xi} (1+\xi)^r e^{\alpha \xi^2} \|f(\boldsymbol{\xi}, \cdot)\|_{H_4(x)}, \tag{2.3}$$

$$\|f\|_{\alpha} = \|f\|_{\alpha, 0}, \tag{2.4}$$

$$\|f\|_{\infty} = \|f\|_{0,0}. \tag{2.5}$$

Denote $\mathscr{H}_\alpha = \{f(\mathbf{x}, \boldsymbol{\xi}) : \|f\|_\alpha < \infty$ and f periodic in $\mathbf{x}\}$. As in Part I, α will always refer to exponential decay and r to algebraic decay in ξ. If γ ever appears in the subscript of a norm it is in the algebraic decay part. The algebraic decay is used in the following proofs, but not in the statements of the theorems. The Sobolev inequality in T^3 states that

$$\|fg\|_{H_4(x)} \leqq c \|f\|_{H_4(x)} \|g\|_{H_4(x)}. \tag{2.6}$$

The main result for the linear problem is the following:

Theorem 2.1. *Let $0 < \alpha < \frac{1}{4}$, and let $f_0 \in \mathscr{N} \cap \mathscr{H}_\alpha$. Then there is a unique solution of the linear Boltzmann equation* (1.1) *and* (1.2) *in $\mathscr{H}_\alpha$. It decays in time like*

$$\|f(t)\| \leqq c \|f_0\|_\alpha e^{-\lambda t^\beta}, \tag{2.7}$$

$$\|f(t)\|_\infty \leqq c \|f_0\|_\alpha e^{-\lambda t^\beta}, \tag{2.8}$$

$$\|f(t)\|_\alpha \leqq c \|f_0\|_\alpha. \tag{2.9}$$

In which $\beta = \dfrac{2}{2+\gamma}$ and $\lambda = (1-2\varepsilon)\alpha^{1-\beta} \left(\dfrac{c_0}{\beta}\right)^\beta$, for any $\varepsilon > 0$. The constant c depends on ε.

The estimates on K are exactly as before. We first note that, since K is independent of $\mathbf{x}$,

$$\|Kf(\boldsymbol{\xi}, \cdot)\|_{H_4(x)} \leqq K(\|f(\cdot, \cdot)\|_{H_4(x)})(\boldsymbol{\xi}). \tag{2.10}$$

Using that inequality we easily show

Proposition 2.2.

$$\|Kf\|_{0,\gamma+3/2} \leqq c\|f\|, \tag{2.11}$$

$$\|Kf\|_{\alpha,r+\gamma+2} \leqq c\|f\|_{\alpha,r}, \tag{2.12}$$

$$\|Kf\| \leqq c\|f\|_{\infty}. \tag{2.13}$$

These estimates and Theorem 3.1 of the next section are used to prove Theorem 2.1 just as in Part I. In the proof we solve two types of equations:

$$\frac{\partial}{\partial t} g + B_w g = g_1, \quad \text{on} \quad \xi < w, \tag{2.14}$$

in which $B_w = \xi \cdot \frac{\partial}{\partial \mathbf{x}} + v + \chi_w \bar{K}$, and

$$\frac{\partial}{\partial t} h + \xi \cdot \frac{\partial}{\partial \mathbf{x}} h + v h = h_1. \tag{2.15}$$

We rewrite these as

$$g(\mathbf{x}, t, \xi) = e^{-tB_w} g_0(\mathbf{x}, \xi) + \int_0^t e^{-(t-s)B_w} g_1(\mathbf{x}, s, \xi)\, ds, \tag{2.16}$$

$$h(\mathbf{x}, t, \xi) = e^{-tv(\xi)} h_0(\mathbf{x} - t\xi, \xi) + \int_0^t e^{-(t-s)v(\xi)} h_1(\mathbf{x} - (t-s)\xi, s, \xi)\, ds. \tag{2.17}$$

Now take the $H_4(x)$ norm and use Theorem 3.1 to estimate

$$\begin{aligned} \|g\|_{H_4(x)}(t, \xi) \leqq e^{-\mu t v(w)} \|g_0\|_{H_4(x)}(\xi) \\ + \int_0^t e^{-\mu(t-s)v(w)} \|g_1\|_{H_4}(s, \xi)\, ds, \end{aligned} \tag{2.18}$$

$$\begin{aligned} \|h\|_{H_4(x)}(t, \xi) \leqq e^{-tv(\xi)} \|h_0\|_{H_4(x)}(\xi) \\ + \int_0^t e^{-(t-s)v(\xi)} \|h_1\|_{H_4}(s, \xi)\, ds. \end{aligned} \tag{2.19}$$

These are exactly like the equations treated in Sects. 9–12 of Part I.

3. Spectral Theory for the Cutoff Linear Operator

Consider the transport and collision operator

$$B = \xi \cdot \frac{\partial}{\partial \mathbf{x}} + v + \bar{K} \tag{3.1}$$

on $\mathscr{L}^2(\mathbf{x}, \xi)$. Recall that $\bar{K}$ is the modification of K defined in (1.4). We shall show that, after restriction to a bounded set of velocities, this operator generates a strictly contracting semi-group. Our main result is

Theorem 3.1. *Consider the operator* $B_w = \boldsymbol{\xi} \cdot \dfrac{\partial}{\partial \mathbf{x}} + \nu(\xi) + \chi_w \bar{K}$ *on* $\mathscr{L}^2\{\mathbf{x}, \boldsymbol{\xi} : \xi < w\}$.

i) $-B_w$ *is maximally dissipative.*

ii) *Let* $0 < \mu < 1$. *If* w *is sufficiently large,*

$$\|e^{-tB_w}\| \leqq e^{-t\mu\nu(w)}. \tag{3.2}$$

The theorem is proved by looking at the Fourier transform of B_w. The modification of K only affects the 0 Fourier variable, so that

$$B_{w,\mathbf{k}} = -i\mathbf{k} \cdot \boldsymbol{\xi} + \nu + K, \qquad \mathbf{k} \neq 0, \tag{3.3}$$

$$B_{w,0} = \nu + \bar{K}, \tag{3.4}$$

where $\mathbf{k}$ a vector with integer components. Each $B_{w,\mathbf{k}}$ is an operator on $\mathscr{L}^2(\xi < w)$ and satisfies

$$\operatorname{Re}(B_{w,\mathbf{k}} f, f) \geqq 0. \tag{3.5}$$

The following results are analogous to Theorem 7.1 and Proposition 7.2 in Part I. An important point is that the statements are independent of $\mathbf{k}$.

Proposition 3.2. *Let* $0 < \mu < 1$. *For* w *sufficiently large,* $B_{w,\mathbf{k}}$ *has spectrum whose real part is bigger than* $\mu\nu(w)$, *i.e.*

$$\sigma(B_{w,\mathbf{k}}) \subset \{\lambda : \operatorname{Re}\lambda > \mu\nu(w)\}. \tag{3.6}$$

Moreover the sufficient size of w *is independent of* $\mathbf{k}$.

Proposition 3.3. *Let* f *be an eigenfunction of* $B_{w,\mathbf{k}}$ *with eigenvalue* λ *such that* $\operatorname{Re}\lambda < \mu\nu(w)$. *Then* f *is rapidly decreasing in* ξ, *i.e.*

$$\sup_{\boldsymbol{\xi}} (1+\xi)^m |f(\boldsymbol{\xi})| \leqq c_m \int f(\boldsymbol{\xi})^2 \, d\boldsymbol{\xi}, \tag{3.7}$$

in which the constants c_m *are independent of* λ, w, f, $\mathbf{k}$.

The following lemma will be used in the proof of Proposition 3.2.

Lemma 3.4. *Let* $f \in \mathscr{L}^2$, $\theta \in R$, *and* $\mathbf{k} \in \mathbb{R}^3$ *with* $k = 1$. *Then*

$$\lim_{\varepsilon \to 0} \sup_{\theta, k=1} \int_A f^2 \, d\boldsymbol{\xi} = 0 \tag{3.8}$$

in which $A = \{\boldsymbol{\xi} : |\mathbf{k} \cdot \boldsymbol{\xi} + \theta| < \varepsilon\}$.

Proof of Proposition 3.3. Rewrite the eigen-equation as $\chi_w K f = \{-(\nu - \lambda) + i\mathbf{k} \cdot \boldsymbol{\xi}\} f$. Therefore $|K f(\boldsymbol{\xi})| \geqq (1-\mu)\, \nu(\xi)\, |f(\boldsymbol{\xi})|$. Then proceed as in Proposition I 7.2 using this inequality and the estimates (I 6.1) and (I 6.2).

Proof of Proposition 3.2. If $\mathbf{k} = 0$, the proposition is exactly Theorem I 7.1. So we consider only $\mathbf{k} \neq 0$

a) First we show that the values $\lambda \in \sigma(B_{w,\mathbf{k}})$ with $\operatorname{Re}\lambda < \mu\nu(w)$ are necessarily discrete eigenvalues with finite multiplicity. [In fact we could put here $\nu(w)$ instead of $\mu\nu(w)$.] The proof is exactly as in [2] using the methods of [4].

The Fredholm set of $(-i\mathbf{k} \cdot \boldsymbol{\xi} + \nu)$ is $\{\lambda : \lambda \neq -i\mathbf{k} \cdot \boldsymbol{\xi} + \nu(\xi)\}$. Since $\chi_w K$ is compact, then this is also the Fredholm set of $B_{w,\mathbf{k}}$. Therefore the set

$S=\{\lambda : \operatorname{Re}\lambda < \nu(w)\}$ is contained in a connected component of the Fredholm set of $B_{w,\mathbf{k}}$. This set S contains negative values of λ which are in the resolvent set of $B_{w,\mathbf{k}}$ because of (3.5), so that $\operatorname{nul}(B_{w,\mathbf{k}}-\lambda)=\operatorname{def}(B_{w,\mathbf{k}}-\lambda)=0$. Since the nullity and deficiency are constant in connected components of the Fredholm set, except at isolated points, $\operatorname{nul}(B_{w,\mathbf{k}}-\lambda)=\operatorname{def}(B_{w,\mathbf{k}}-\lambda)=0$ in S except at isolated points. These points are isolated eigenvalues of finite multiplicity. Every other point of S is in the resolvent set.

b) Now suppose the theorem is not true, so that there are sequences w_n, λ_n, $\mathbf{k}_n$ with $\operatorname{Re}\lambda_n \in \sigma(B_{w_n,\mathbf{k}_n})$, $\lambda_n < \mu\nu(w_n)$ and $\mathbf{k}_n \neq 0$. According to (a) each λ_n is an eigenvalue for $B_{w_n,\mathbf{k}_n}$ with eigenfunction f_n, i.e.

$$B_{w_n,\mathbf{k}_n} f_n = \lambda_n f_n \quad \text{and} \quad \|f_n\| = 1\,. \tag{3.9}$$

Write $\lambda_n = \varphi_n + i\theta_n$. Then just as in the proof of Theorem I7.1, $\varphi_n \to 0$ and $K f_n \to g$, after restricting to a subsequence, with the result that $\lim\limits_{n\to\infty} (-\nu(\xi) + i\mathbf{k}_n \cdot \boldsymbol{\xi} + i\theta_n) f_n = g$. As before we can divide by the factor on the right to obtain

$$f \equiv \lim_{n\to\infty} f_n = \lim_{n\to\infty} \frac{1}{-\nu(\xi) + i\mathbf{k}_n \cdot \boldsymbol{\xi} + i\theta_n}\, g\,. \tag{3.10}$$

Denote the function inside the last limit in (3.10) as g_n.

Next we show that $\overline{\lim}\, k_n \neq \infty$. Suppose to the contrary it was ∞ and restrict to a subsequence with $\lim k_n = \infty$. Choose ε as in Lemma 3.4, such that

$$\sup_{\varrho, k=1} \int_A f^2\, d\xi < \varepsilon\,, \tag{3.11}$$

in which $A = \{\boldsymbol{\xi} : |\mathbf{k} \cdot \boldsymbol{\xi} + \varrho| < \sqrt{\varepsilon}\}$. Choose n large enough that $\dfrac{1}{k_n} < \varepsilon$ and $\|f - g_n\|^2 < \varepsilon$. We will obtain a contradiction by integrating f^2 over the two sets $A_n = \{\boldsymbol{\xi} : |\mathbf{k}_n \cdot \boldsymbol{\xi} + \theta_n| < \sqrt{k_n}\}$ and $A_n^c = \mathbb{R}^3 - A_n$. Denote $\hat{\mathbf{k}}_n = \mathbf{k}_n / k_n$. Then $A_n = \{\boldsymbol{\xi} : |\hat{\mathbf{k}}_n \cdot \boldsymbol{\xi} + \theta_n / k_n| < 1/\sqrt{k_n}\}$. Since $1/\sqrt{k_n} < \sqrt{\varepsilon}$,

$$\int_{A_n} f^2\, d\xi \leqq \varepsilon\,. \tag{3.12}$$

In A_n^c, $g_n^2 < g^2 / k_n$ and

$$\begin{aligned} \int_{A_n} f^2\, d\xi &\leqq \int_{A_n^2} g_n^2\, d\xi + \varepsilon \\ &\leqq \varepsilon \|g\|^2 + \varepsilon\,. \end{aligned} \tag{3.13}$$

Adding (3.12) and (3.13) together results in

$$\|f\|^2 \leqq 2\varepsilon + \varepsilon \|g\|^2\,. \tag{3.14}$$

By choosing ε small enough we get a contradiction since $\|f\| = 1$, which shows that $\overline{\lim}\, k_n < \infty$.

Similarly θ_n must stay bounded, and we get $\mathbf{k}_n \to \mathbf{k}$ and $\theta_n \to \theta$ after restricting to a subsequence. Since $\mathbf{k}_n$ is on the integral lattice, $\mathbf{k}_n = \mathbf{k}$ for n large enough and so $\mathbf{k} \neq 0$. Take the limit $n \to \infty$ in the eigen-equation (3.9) again and find that

$$-i\mathbf{k} \cdot \boldsymbol{\xi} f + v f + K f = i\theta f. \tag{3.15}$$

Integrate this against f; the real part is $(vf + Kf, f) = 0$. Since $L = v + K$ is a positive semi-definite self-adjoint operator, then $f \in N(L)$, which means that

$$f(\boldsymbol{\xi}) = \alpha_0 + \boldsymbol{\alpha} \cdot \boldsymbol{\xi} + \alpha_4 \xi^2. \tag{3.16}$$

Since $(v+K)f = 0$, then $-(\mathbf{k} \cdot \boldsymbol{\xi}) f = \theta f$, which implies that $k = \theta = 0$. But this is a contradiction, since $\mathbf{k} \neq 0$. This concludes the proof of Proposition 3.2.

Proof of Theorem 3.1. i) Since $\bar{B}_w$ is densely defined on $\mathscr{L}^2(\mathbf{x}; \boldsymbol{\xi} : \xi < w)$ and

$$\mathrm{Re}(\bar{B}_w f, f) = \mathrm{Re}((v + \chi_w \bar{K}) f, f) \geqq 0, \tag{3.17}$$

then $\bar{B}_w$ is maximally dissipative.

ii) This proof is exactly that of Theorem 1.1 in [5], except that we have removed the null space by changing the operator K to $\bar{K}$. Denote $A_w = \boldsymbol{\xi} \cdot \dfrac{\partial}{\partial \mathbf{x}} + v(\xi)$, $K_w = \chi_w K$, and $B_w = A_w + K_w$, operators on $\mathscr{L}_w = \mathscr{L}^2\{(\mathbf{x}, \boldsymbol{\xi}) : \xi < w\}$. We outline the proof in the following steps

a) $K_w(\lambda - A_w)^{-1}$ *is compact on* $\mathscr{L}^2(\boldsymbol{\xi}, \mathbf{x})$, *for* $\mathrm{Re}\lambda < \mu v(w)$.

b) $\sigma(B_w) \subset \{\lambda : \mathrm{Re}\lambda < \mu v(w)\}$ *for w sufficiently large.*

From a), K_w is A_w-compact so that $\sigma_e(B_w) = \sigma_e(A_w) = \{\lambda : \mathrm{Re}\lambda \geqq v(w)\}$ [4]. In $\{\mathrm{Re}\lambda < v(w)\}$, A_w is Fredholm and so is B_w. Moreover if $\mathrm{Re}\lambda < 0$, then λ is in the resolvent set $\varrho(B_w)$. Therefore $\{\mathrm{Re}\lambda < v(w)\} \subset \varrho(B_w)$, except for a discrete set of points which are eigenvalues of B_w. But Proposition 3.2 shows that B_w has no eigenvalues to the left of $\mathrm{Re}\lambda = \mu v(w)$ for w large enough.

c) $\limsup\limits_{|\lambda| \to \infty\, \mathrm{Re}\lambda < \mu v(w)} \|K_w(\lambda - A_w)\| \to 0.$

d) *Denote* $\tilde{Z}(\lambda) = (\lambda - A_w)^{-1}(I - K_w(\lambda - A_w)^{-1})^{-1} K_w(\lambda - A_w)^{-1}$, *so that* $(\lambda - B_w)^{-1} = (\lambda - A_w)^{-1} + \tilde{Z}(\lambda)$. *Denote*

$$Z_\beta(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\gamma t} \tilde{Z}(\beta + i\gamma)\, d\gamma. \tag{3.18}$$

If $\beta \leqq \mu v(w)$, $Z_\beta(t)$ *converges absolutely in the weak topology and* $\|Z_\beta(t)\| \leqq c$ *where c is independent of t and λ.*

e) $e^{-tB_w} = e^{-tA_w} + e^{-\beta t} Z_\beta(t)$.

Choose $\beta = \mu v(w)$. Since $\|e^{-tA_w}\| \leqq e^{-tv(w)}$, the result (ii) in Theorem 3.1 follows.

4. The Nonlinear Equation

Theorem 4.1. *Let* $0 < \alpha < \frac{1}{4}$. *There is a positive constant* δ, *such that if* $\|f_0\|_\alpha < \delta$, *then the nonlinear Boltzmann equation* (1.8) *and* (1.9) *has a unique solution in* $\mathscr{H}_\alpha$, *which satisfies*

$$\|f(t)\| \leqq c\|f_0\|_\alpha e^{-\lambda t^\beta}, \tag{4.1}$$

$$\|f(t)\|_\infty \leqq c\|f_0\|_\alpha e^{-\lambda t^\beta}, \tag{4.2}$$

$$\|f(t)\|_\alpha \leqq c\|f_0\|_\alpha \tag{4.3}$$

in which $\beta = \dfrac{2}{2+\gamma}$ *and* $\lambda = (1-2\varepsilon)\left(\dfrac{\alpha}{2}\right)^{1-\beta}\left(\dfrac{c_0}{\beta}\right)^\beta$ *for any* $\varepsilon > 0$. *The constant c depends on* ε.

This β and λ are chosen just as in the linear problem, but they correspond to γ and $\dfrac{\alpha}{2}$ rather than α.

5. Estimates on Γ

The nonlinearity $\Gamma(f,g)$ was analyzed by Grad in the Appendix of [3]. We decompose Γ as (this is slightly different from [3])

$$\Gamma(f,g) = \Gamma_1(f,g) + \Gamma_2(f,g), \tag{5.1}$$

$$v\Gamma_1(f,g) = \tfrac{1}{2}\int (fg_1 + g_1 f)\omega_1^{1/2}\, d\mathbf{\Omega}, \tag{5.2}$$

$$v\Gamma_2(f,g) = \tfrac{1}{2}\int (f'g'_1 + f'_1 g')\omega_1^{1/2}\, d\mathbf{\Omega}, \tag{5.3}$$

$$d\mathbf{\Omega} = B(\theta, \mathbf{V})\, d\theta\, d\boldsymbol{\xi}_1, \tag{5.4}$$

in which $f'_1 = f(\boldsymbol{\xi}'_1)$ as given by (2.4) in Part I, etc. The following estimates are analogous to those proved by Grad.

Proposition 5.1.

$$\|v\Gamma_1(f,g)\|_{\alpha,r} \leqq c(\|f\|_{\alpha,r-\gamma}\|g\| + \|f\|\,\|g\|_{\alpha,r-\gamma}), \tag{5.5}$$

$$\|v\Gamma_2(f,g)\|_{\alpha,r} \leqq c\|f\|_{\alpha,r-k-\dot{\gamma}}\|g\|_{\alpha,r-1-\gamma}. \tag{5.6}$$

Proof. a) By the symmetry in Γ_1 it suffices to consider $v\Gamma_{11}(f,g) = \frac{1}{2}\int fg_1\omega_1^{1/2}\, d\mathbf{\Omega}$. First take the $H_4(x)$ norm and use the Sobolev inequality (2.6). Since the integral does not involve ξ, we can factor the f term out to get

$$\|v\Gamma_{11}(f,g)\|_{H_4(x)} \leqq c\|f\|_{H_4(x)} \tfrac{1}{2}\int \|g_1\|_{H_4(x)}\omega_1^{1/2} B(\theta, V)\, d\theta\, d\boldsymbol{\xi}_1. \tag{5.7}$$

Replace the first factor using

$$\|f\|_{H_4(x)}(\boldsymbol{\xi}) \leqq (1+\xi)^{-r} e^{-\alpha\xi^2}\|f\|_{\alpha,r}. \tag{5.8}$$

Then use the definition (I1.6) of ω and the bound (I2.21) on B and apply the Schwartz inequality to the integral over $\boldsymbol{\xi}_1$ to obtain

$$\begin{aligned}\int \|g_1\|_{H_4(x)}\omega_1^{1/2}\, d\mathbf{\Omega} &\leqq c\|g\| \int_{\mathbb{R}^3} e^{-1/2\xi_1^2}|\boldsymbol{\xi} - \boldsymbol{\xi}_1|^{-2\gamma}\, d\boldsymbol{\xi}_1^{1/2} \\ &\leqq c\|g\|(1+\xi)^{-\gamma}.\end{aligned} \tag{5.9}$$

Combining (5.8) and (5.9) results in

$$\|\nu\Gamma_{11}\|_{\alpha,r+\gamma} \leqq c\|f\|_{\alpha,r}\|g\|, \tag{5.10}$$

from which (5.5) follows.

b) Again we only estimate

$$\begin{aligned}\nu\Gamma_{21} &= \tfrac{1}{2}\int f_1' g' \omega_1^{1/2}\, d\boldsymbol{\Omega}\\ &= \tfrac{1}{2}\int_{\mathbb{R}^3}\int_{\mathbf{w}\perp\mathbf{v}} f(\boldsymbol{\xi}+\mathbf{v})\, g(\boldsymbol{\xi}+\mathbf{w})\, \omega^{1/2}(\boldsymbol{\xi}+\mathbf{v}+\mathbf{w})\\ &\quad\cdot \frac{Q(\mathbf{v},\mathbf{w})}{v^2}\, d\mathbf{w}\, d\mathbf{v},\end{aligned} \tag{5.11}$$

in which $\mathbf{w}$ and $\mathbf{v}$ are defined by (I2.10) and (I2.11). We continue exactly as Grad did. Resolve $\boldsymbol{\xi}$ into components $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$ parallel and perpendicular to $\mathbf{v}$ respectively, so that

$$\omega^{1/2}(\boldsymbol{\xi}+\mathbf{v}+\mathbf{w}) = \omega^{1/2}(\mathbf{v}+\boldsymbol{\xi}_1)\omega^{1/2}(\mathbf{w}+\boldsymbol{\xi}_2),$$

and, using also the Sobolev inequality,

$$\begin{aligned}&\|f(\boldsymbol{\xi}+\mathbf{v})g(\boldsymbol{\xi}+\mathbf{w})\|_{H_4(x)}\\ &\quad\leqq c(1+|\boldsymbol{\xi}+\mathbf{v}|)^{-r}(1+|\boldsymbol{\xi}+\mathbf{w}|)^{-r}\exp\{-\alpha(|\boldsymbol{\xi}+\mathbf{v}|^2+|\boldsymbol{\xi}+\mathbf{w}|^2)\}\\ &\qquad\cdot\|f\|_{\alpha,r}\|g\|_{\alpha,r}\\ &\quad\leqq c(1+\xi)^{-r+1}(1+\xi_1)^{-1}(1+\xi_2)^{-1}e^{-\alpha\xi^2}\|f\|_{\alpha,r}\|g\|_{\alpha,r}.\end{aligned} \tag{5.13}$$

After applying the $H_4(x)$ norm to $\nu\Gamma_{21}$ we can use (5.13) in estimating (5.11) to find

$$\begin{aligned}\|\nu\Gamma_{21}\|_{H_4(x)} &\leqq c(1+\xi)^{-r+1}e^{-\alpha\xi^2}\|f\|_{\alpha,r}\|g\|_{\alpha,r}\\ &\quad\cdot\int_{\mathbb{R}^3}\int_{\mathbf{w}\perp\mathbf{v}}(1+\xi_1)^{-1}(1+\xi_2)^{-1}\omega^{1/2}(\mathbf{v}+\boldsymbol{\xi}_1)\omega^{1/2}(\mathbf{w}+\boldsymbol{\xi}_2)\frac{Q(\mathbf{v},\mathbf{w})}{v^2}\, d\mathbf{w}\, d\mathbf{v}.\end{aligned} \tag{5.14}$$

Denote the integral on the right by I. According to Proposition 5.2 from Part I,

$$\frac{1}{v}\int_{\mathbf{w}\perp\mathbf{v}}\omega^{1/2}(\mathbf{w}+\boldsymbol{\xi}_2)Q(\mathbf{v},\mathbf{w})\, d\mathbf{v} \leqq c(1+\xi_2+v)^{-(\gamma+1)}, \tag{5.15}$$

so that

$$I \leqq c\int_{\mathbb{R}^3}(1+\xi_1)^{-1}(1+\xi_2)^{-1}(1+\xi_2+v)^{-(\gamma+1)}\frac{1}{v}\omega^{1/2}(\mathbf{v}+\boldsymbol{\xi}_1)\, d\mathbf{v}. \tag{5.16}$$

It is easy to see that

$$(1+\xi_2+v)^{-(\gamma+1)}\omega^{1/4}(\mathbf{v}+\boldsymbol{\xi}_1) \leqq c(1+\xi)^{-(\gamma+1)}. \tag{5.17}$$

Combine this with the estimate

which comes (almost exactly) from the Appendix of [3], to obtain

$$I \leqq c(1+\xi)^{-(\gamma+2)}. \tag{5.19}$$

Using this in (5.14), we find

$$\| v\Gamma_{21} \|_{H_4(x)} \leqq c(1+\xi)^{-(r+\gamma+1)} e^{-\alpha\xi^2} \| f \|_{\alpha,r} \| g \|_{\alpha,r}. \tag{5.20}$$

The result (5.6) follows after replacing r with $r+\gamma+1$, dividing, and taking sup over ξ.

6. The Inhomogeneous Iteration Equation

Consider the equation

$$\frac{\partial}{\partial t} f + \xi \cdot \frac{\partial}{\partial \mathbf{x}} f + Lf = v\Gamma(h_1, h_2), \tag{6.1}$$

$$f(t=0) = f_0 \in \mathcal{N} \cap \mathcal{H} \tag{6.2}$$

which is an inhomogeneous version of the iteration equations that will be solved in the next section. Pick λ and β as in Theorem 4.1, i.e. corresponding to $\alpha/2$. For f_0, h_1, h_2 we require

$$\| f_0 \|_\alpha \leqq b_0, \tag{6.3}$$

$$\sup_t \{ \| h_i(t) \|_\alpha, \, e^{\lambda t^\beta} \| h_i(t) \|, \, e^{\lambda t^\beta} \| h_i(t) \|_\infty \} \leqq b_i, \qquad i=1,2, \tag{6.4}$$

in which the sup is taken over time as well as over the three components.

Proposition 6.1. *The solution f of* (6.1) *and* (6.2) *satisfies*

$$\max \{ \| f(t) \|_\alpha, \, e^{\lambda t^\beta} \| f(t) \|, \, e^{\lambda t^\beta} \| f(t) \|_\infty \} \\ \leqq c(b_0 + b_1 b_2). \tag{6.5}$$

We will employ two useful inequalities. The first is a special case of an interpolation theorem for the α, r-norms.

Lemma 6.2.

$$\| f \|_{\alpha/2} \leqq 2 \| f \|_\alpha^{1/2} \| f \|_\infty^{1/2}. \tag{6.6}$$

Proof. For any $\xi_0 > 0$,

$$\begin{aligned} \| f \|_{\alpha/2} &\leqq e^{\alpha/2\,\xi_0^2} \sup_{\xi<\xi_0} |f(\xi)| + e^{-\alpha/2\,\xi_0^2} \sup_{\xi>\xi_0} e^{\alpha\xi^2} |f(\xi)| \\ &\leqq e^{\alpha/2\,\xi_0^2} \| f \|_\infty + e^{-\alpha/2\,\xi_0^2} \| f \|_\alpha \\ &\leqq 2 \sqrt{\| f \|_\infty / \| f \|_\alpha}. \end{aligned} \tag{6.7}$$

by choosing $e^{\alpha/2\,\xi_0^2} = \sqrt{\| f \|_\alpha / \| f \|_\infty}$.

Lemma 6.3. *For* $0<\beta<1$,

$$\int_0^t \exp\{-\lambda(t-s)^\beta-\lambda s^\beta\}\,ds \leqq c(1+t)^{-1}e^{-\lambda t}, \tag{6.8}$$

where c *depends on* β.

Proof. Just use the estimate $(t-s)^\beta-(t^\beta-s^\beta)>c\{(t/2)^2-(s-t/2)^2\}$ in the integral.

Proof of Proposition 6.1. a) First we infer from Lemma 6.2 and (6.4) that

$$\|h_i(t)\|_{\alpha/2} \leqq cb_i e^{1/2\,\lambda t^\beta}. \tag{6.9}$$

According to Proposition 6.1 and (6.4),

$$\|\nu\Gamma_1(h_1,h_2)(t)\|_\alpha \leqq cb_1b_2e^{-\lambda t^\beta}, \tag{6.10}$$

$$\|\nu\Gamma_2(h_1,h_2)(t)\|_{\alpha,\gamma+1} \leqq cb_1b_2, \tag{6.11}$$

$$\|\nu\Gamma(h_1,h_2)(t)\|_{\alpha/2} \leqq cb_1b_2e^{-\lambda t^\beta}. \tag{6.12}$$

Note that the $\frac{\alpha}{2}$-norm decays, while the α-norm does not. This decay is the reason for using the $\alpha/2$ and will be needed in the next estimate.

b) Using the estimates (2.7) and (2.8) for the linear problem and then (6.12) and Lemma 6.3, we find that (recall that λ corresponds to $\alpha/2$)

$$\begin{aligned}\max\{\|f(t)\|,\|f(t)\|_\infty\} &\leqq ce^{-\lambda t^\beta}\|f_0\|_{\alpha/2}+c\int_0^t e^{-\lambda(t-s)^\beta}\|\nu\Gamma(h_1,h_2)(s)\|_{\alpha/2}\,ds\\ &\leqq ce^{-\lambda t^\beta}b_0+c\int_0^t e^{-\lambda(t-s)^\beta}e^{-\lambda s^\beta}\,ds\,b_1b_2\\ &\leqq ce^{-\lambda t}(b_0+b_1b_2).\end{aligned} \tag{6.13}$$

c) To estimate $\|f(t)\|_\alpha$ we go back and redo the linear estimate. As in (2.19) we estimate

$$\begin{aligned}\|f(t,\xi)\|_{H_4(x)} &\leqq e^{-t\nu(\xi)}\|f_0\|_{H_4(x)}\\ &\quad+\int_0^t e^{-(t-s)\nu(\xi)}(\|Kf\|_{H_4(x)}(s,\xi)+\|\nu\Gamma\|_{H_4(x)}(s,\xi))\,ds.\end{aligned} \tag{6.14}$$

Using the argument in Sect. 12 of Part I, we find that

$$\begin{aligned}&\sup_\xi\{e^{\alpha\xi^2}e^{-(t-s)_\nu(\xi)}\|Kf\|_{H_4(x)}(s,\xi)\}\\ &\quad\leqq c(1+t-s)^{-1-\gamma/2}\{e^{\alpha\xi_0^2}\|f(s)\|+(1+\xi_0)^{-3/2}\|f(s)\|_\alpha\}\end{aligned} \tag{6.15}$$

for any ξ_0. Choose ξ_0 large enough and use (6.13) to obtain

$$\int_0^t \sup_\xi\{e^{\alpha\xi^2}e^{-(t-s)\nu(\xi)}\|Kf\|_{H_4(x)}(s,\xi)\}\,ds$$

The last term in (6.14) is split into two parts using $\Gamma=\Gamma_1+\Gamma_2$ [cf. (5.1)]. The reason for going back to the linear equation was to estimate the term containing Γ_2:

$$\begin{aligned}&\sup_{\xi} e^{\alpha\xi^2}e^{-(t-s)v(\xi)}\|v\Gamma_2\|_{H_4(x)}(s,\xi)\\&\quad\leqq\sup\{(1+\xi)^{-\gamma-1/2}e^{-(t-s)v(\xi)}\}\cdot\|v\Gamma_2\|_{\alpha,\gamma+1/2}\\&\quad\leqq c(1+t-s)^{-1-\gamma/2}b_1b_2,\end{aligned}\tag{6.17}$$

where we used Lemma I12.1 and (6.11) in the last step. Since this is integrable over time.

$$\int_0^t\sup_{\xi}\{e^{\alpha\xi^2}e^{-(t-s)v(\xi)}\|v\Gamma_2\|_{H_4(x)}(s,\xi)\}\,ds\leqq cb_1b_2.\tag{6.18}$$

The term containing Γ_1 is easily estimated

$$\begin{aligned}&\int_0^t\sup_{\xi}\{e^{\alpha\xi^2}e^{-(t-s)v(\xi)}\|v\Gamma_1\|_{H_4(x)}(s,\xi)\}\,ds\\&\quad\leqq\int_0^t\|v\Gamma_1\|_{\alpha}\,ds\leqq cb_1b_2,\end{aligned}\tag{6.19}$$

because of (6.10).

The three terms estimated in (6.16), (6.18), and (6.19) plus the initial term in (6.14) are just what appear on the right side of (6.14) after multiplying by $e^{\alpha\xi^2}$ and taking sup over ξ. The result is that

$$\|f(t)\|_{\alpha}\leqq c(b_0+b_1b_2)+\tfrac{1}{2}\sup_{0\leqq s\leqq t}\|f(s)\|_{\alpha},\tag{6.20}$$

from which it follows that

$$\|f(t)\|_{\alpha}\leqq c(b_0+b_1b_2).\tag{6.21}$$

This concludes the proof of the Proposition.

7. Proof of Theorem 4.1

The nonlinear Boltzmann equation (1.8) and (1.9) is solved by an iteration starting with

$$f_1(t)=e^{-\lambda t^{\beta}}f_0\tag{7.1}$$

and proceeding by

$$\frac{\partial}{\partial t}f_{n+1}+\xi\cdot\frac{\partial}{\partial\mathbf{x}}f_{n+1}+Lf_{n+1}=v\Gamma(f_n,f_n),\quad f_{n+1}(t=0)=f_0.\tag{7.2}$$

First we show the boundedness and decay of f_{n+1}. Denote $\|f_0\|_{\alpha}=b_0$ and suppose that

$$\max\{\|f_n\|_{\alpha},e^{\lambda t^{\beta}}\|f_n\|,e^{\lambda t^{\beta}}\|f_n\|_{\infty}\}\leqq b.\tag{7.3}$$

We need in addition that $b \geqq b_0$ in order to get the induction started. According to Proposition (6.1), the estimate (7.3) will also be true for f_{n+1} if $b \geqq c(b_0 + b^2)$. This can be fulfilled as long as b_0 is small enough, and we can even make b as small as desired.

Next we estimate the difference $h_{n+1} = f_{n+1} - f_n$. For h_{n+1} we have the equation

$$\frac{\partial}{\partial t} h_{n+1} + \xi \cdot \frac{\partial}{\partial \mathbf{x}} h_{n+1} + L h_{n+1} = \nu \Gamma(h_n, f_n + f_{n-1}), \quad h_{n+1}(t=0) = 0. \tag{7.4}$$

Denote $|||h||| = \sup_t \{ \|h(t)\|_\alpha, e^{\lambda t^\beta} \|h(t)\|, e^{\lambda t^\beta} \|h(t)\|_\infty \}$. Then $|||h_2||| \leqq 2b$ from (7.3), and using Proposition 6.1 again, $|||h_{n+1}(t)||| \leqq 2cb |||h_n|||$. After choosing $b < \frac{1}{2c}$, we find that $\sum_{n=2}^{\infty} |||h_{n+1}(t)||| < \infty$, and it follows that

$$f_n \to f \tag{7.5}$$

in the norm $||| \cdot |||$. Moreover f solves Eqs. (1.8) and (1.9). This concludes the proof of Theorem 4.1.

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