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Boundedness of Total Cross-Sections in Potential Scattering. II

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Abstract. If, in addition to the condition

$$\frac{1}{(4\pi)^2}\int d^3x d^3x' \frac{|V(x)|\,|V(x')|}{|x-x'|^2} < 1$$

in units where $2M/\hbar^2 = 1$, which guarantees that the total cross-section averaged over incident directions is finite, we have also

$$\frac{1}{(4\pi)}\int d^3x d^3x' \frac{|V(x)|\,|V(x')|}{|x-x'|}$$

finite, the total cross-section is finite for all energies and all directions of the incident beam.

We have recently shown, in a paper bearing the same title [1], that if the quantity

$$I = \frac{1}{(4\pi)^2}\int d^3x d^3x' \frac{|V(x)|\,|V(x')|}{|x-x'|^2} \tag{1}$$

is such that

$$I < 1 \tag{2}$$

in units where $2M/\hbar^2 = 1$, M being the mass of the particle scattered by the potential V, the total cross-section averaged over angles

$$\bar{\sigma}_T(k) = \int \frac{d\Omega_k}{4\pi}\,\sigma_T(\mathbf{k}) \tag{3}$$

is finite. More precisely

$$\bar{\sigma}_T(k) < \frac{4\pi}{k^2}\,\frac{I}{(1-\sqrt{I})^2}\,. \tag{4}$$

Here we want to show that if we supplement condition (1) by

$$J = \frac{1}{4\pi} \int d^3x d^3x' \frac{|V(x)|\,|V(x')|}{|x-x'|} \quad \text{finite} \tag{5}$$

the total cross-section is finite for any given direction of the incident beam.

First, like in [1] we notice that if condition (2) is fulfilled the Green function $G_k(x, x')$, entering in the scattering amplitude as

$$T(\mathbf{k}', \mathbf{k}) = T_{\mathrm{BORN}}(\mathbf{k}', \mathbf{k}) + \int e^{-ik' \cdot x} V(x) G_k(x, x') V(x') e^{ik \cdot x} d^3x d^3x' \tag{6}$$

admits a convergent series expansion

$$G_k(x, x') = \sum_{n=0}^{\infty} \frac{1}{(4\pi)^{n+1}} \frac{e^{ik|x-x_1|}}{|x-x_1|} V(x_1) \dots V(x_n) \frac{e^{ik|x^n - x'|}}{|x_n - x'|} \tag{7}$$

and we have to establish the convergence of the corresponding series for the total cross-section:

$$\sigma_T(\mathbf{k}) = \frac{4\pi}{k} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \int e^{-ik \cdot x_1} V(x_1) \frac{e^{-ik|x_1 - x_2|}}{4\pi|x_1 - x_2|} \dots V(x_n)$$
$$\cdot \frac{\sin k|x_n - y_1|}{4\pi|x_n - y_1|} V(y_1) \dots V(y_m) e^{ik \cdot y_m} \cdot d^3x_1 \dots d^3x_n d^3y_1 \dots d^3y_m . \tag{8}$$

This series is bounded by the series

$$\sigma_T(\mathbf{k}) \leqq \frac{4\pi}{k} \sum_{p=2}^{\infty} (p-1) \sigma_p \tag{9}$$

with

$$\sigma_p = \int |V(x_1)| \frac{1}{4\pi|x_1 - x_2|} |V(x_2)| \dots \frac{1}{4\pi|x_{p-1} - x_p|} |V(x_p)| \cdot d^3x_1 d^3x_2 \dots d^3x_p . \tag{10}$$

We now multiply and divide the integrand of (10) by $|x_1 - x_p|^{1/2}$, make use of the inequality

$$|x_1 - x_p|^{1/2} \leqq |x_1 - x_2|^{1/2} + |x_2 - x_3|^{1/2} + \dots |x_{p-1} - x_p|^{1/2}$$

and introduce the notation

$$K(x, y) = \frac{1}{4\pi} |V(x)|^{1/2} \frac{1}{|x-y|} |V(y)|^{1/2} , \tag{11}$$

$$H(x, y) = \frac{1}{(4\pi)^{1/2}} |V(x)|^{1/2} \frac{1}{|x-y|^{1/2}} |V(y)|^{1/2} . \tag{12}$$

In this way we get

$$\begin{aligned} \sigma_p \leqq & \int H(x, x_p) H(x_1, x_2) K(x_2, x_3) \dots K(x_{p-1}, x_p) d^3x_1 \dots d^3x_p \\ & + \int H(x_1, x_p) K(x_1, x_2) H(x_2, x_3) \dots K(x_{p-1}, x_p) d^3x_1 \dots d^3x_p \\ & + \dots \\ & + \int H(x_1, x_p) K(x_1, x_2) K(x_2, x_3) \dots H(x_{p-1}, x_p) . \end{aligned} \tag{13}$$

Using now the inequality

$$[\int A_1(x_1,x_2)A_2(x_2,x_3)\dots A_p(x_p,x_1)d^3x_1d^3x_2\dots d^3x_p]^2 \\ \leqq \int d^3x_1d^3x_2(A_1)^2 \int d^3x_2d^3x_3(A_2)^2 \dots \int d^3x_pd^3x_1(A_p)^2\,, \tag{14}$$

we get

$$\sigma_p<(p-1)JI^{\frac{p-2}{2}}$$

and

$$\sigma_T(\mathbf{k})<\frac{4\pi}{k}J\sum_{n=0}^{\infty}(n+1)^2I^{n/2}=\frac{4\pi}{k}J(1-\sqrt{I})^{-3}\,, \tag{15}$$

which is our main result.

Now the question is to know how good is condition (2). It is easy to see that it is fulfilled if

$$|V(x)|<|x|^{-5/2}(\log|x|)^{-1-\varepsilon}, \qquad \varepsilon>0,\ |x|>R\,. \tag{16}$$

This, however, is not terribly interesting because (16) is a spherically symmetric upper bound, and it is probably not possible to construct an example of a potential violating (16) only marginally and producing a divergent total cross-section. What is easy to show from (16) is that there are potentials producing total cross-sections finite for any direction of the incident beam but divergent forward scattering amplitudes. More precisely, it is rather clear from the derivation that under conditions (2) and (5) the difference between the forward amplitude and the Born approximation for the forward amplitude is finite. Therefore, if the Born approximation diverges in the forward direction the forward amplitude is infinite. So the potential

$$V(x)=|f(\Omega_x)|(|x|+a)^{-2.6}$$

where $f(\Omega_x)$ is an arbitrary, square integrable function of the direction of x, gives a finite total cross-section and an infinite forward amplitude, for all directions.

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Reference

1. Martin, A.: Commun. Math. Phys. **69**, 89 (1979)
   This paper contains references to previous work on the subject. However, the paper by T. A. Osborne and D. Bollé, J. Math. Phys. **20**, 1059 (1979) was unfortunately omitted therein

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Note added in proof. It has been pointed out to me that under weaker conditions, Amrein, Jauch and Sinha, in the book 'Scattering theory in quantum mechanics' show that $\sigma_T(\mathbf{k})$ is finite for almost all energies and angles.