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On a Normal Form of Symmetric Maps of [0, 1]

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Abstract. A class of continuous symmetric mappings of $[0,1]$ into itself is considered leaving invariant a measure absolutely continuous with respect to the Lebesgue measure.

We consider a continuous map f of the closed unit interval onto itself and try to put it into the normal form $N=\varphi^{-1}\circ f\circ\varphi$,

$$N(x)=\begin{cases}2x & \text{if} \quad 0\leqq x\leqq \frac{1}{2}\\ 2(1-x) & \text{if} \quad \frac{1}{2}\leqq x\leqq 1\end{cases} \tag{1}$$

by means of a homeomorphism φ of $[0,1]$. The statement is as follows:

Theorem. *Let* $f:[0,1]\to[0,1]$ *be continuous and satisfying*

$$f(0)=0,\, f(\tfrac{1}{2})=1\,, \tag{2}$$

$$f(x)=f(1-x), \qquad 0\leqq x\leqq 1\,. \tag{3}$$

Assume that

$$1<c\leqq\frac{f(x)-f(y)}{x-y}, \qquad 0\leqq y<x\leqq\tfrac{1}{2} \tag{4}$$

for a real constant c. Then there is a strictly increasing continuous map φ *of* $[0,1]$ *onto itself such that*

$$\varphi(Nx)=f(\varphi(x)), \qquad 0\leqq x\leqq 1 \tag{5}$$

with N as defined by (1). *Moreover, if* $c>2^{\sigma}$ *for some* σ *with* $0<\sigma<1$ *then* φ *is Hölder-continuous with exponent* σ.

Remark 1. Observe the condition (4) is essentially a smallness condition on the Lipschitz distance between the functions $x\to f(x)$ and $x\to 2x$, $0\leqq x\leqq\frac{1}{2}$.

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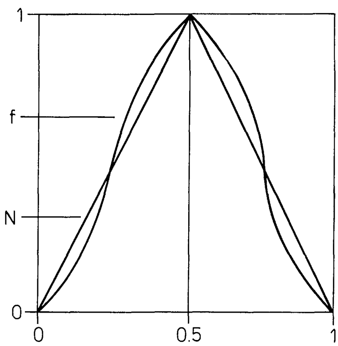

Fig. 1

Remark 2. Mappings $f:[0,1]\to[0,1]$ satisfying the conditions of the theorem leave invariant a measure absolutely continuous with respect to the Lebesgue measure, see [1]. For more general results see [2].

For the proof of the theorem we reduce the problem to a conjugacy problem on the circle. By definition of N in (1) the functional equation (5) is equivalent to

$$\begin{aligned}\varphi(2x) &= f(\varphi(x))\\ &= f(1-\varphi(1-x)), \qquad 0 \leqq x \leqq \tfrac{1}{2}\end{aligned} \tag{6}$$

in view of (3). The idea is to continue $f/[0,1/2]$ to a function $F:\mathbb{R}\to\mathbb{R}$ such that

$$\left.\begin{aligned}F(x+1) &= F(x)+2\\ F(-x) &= -F(x),\end{aligned}\right\} \quad x\in\mathbb{R}. \tag{7}$$

Then we look for a $\Phi:\mathbb{R}\to\mathbb{R}$ continuous and strictly increasing, such that the conditions

$$\left.\begin{aligned}\Phi(x+1) &= \Phi(x)+1\\ \Phi(-x) &= -\Phi(x),\end{aligned}\right\} \quad x\in\mathbb{R} \tag{8}$$

and

$$\Phi(2x) = F(\Phi(x)), \qquad x\in\mathbb{R} \tag{9}$$

are satisfied.

First we show that for such a Φ the equalities (6) are fulfilled with $\varphi=\Phi/[0,1]$. Indeed from (8) we get

$$\Phi(x)+\Phi(1-x)=1, \qquad x\in\mathbb{R} \tag{10}$$

and therefore

$$\begin{aligned}\Phi(2x) &= F(\Phi(x))\\ &= F(1-\Phi(1-x)), \qquad x\in\mathbb{R}\end{aligned} \tag{11}$$

as a consequence of (9). Now (8) and (10) give

$$\Phi(0)=0, \qquad \Phi(\tfrac{1}{2})=\tfrac{1}{2}, \qquad \Phi(1)=1$$

so that the monotony of Φ leads to

$$0 \leqq \Phi(x) \leqq \tfrac{1}{2}, \quad 0 \leqq x \leqq \tfrac{1}{2},$$
$$\tfrac{1}{2} \leqq \Phi(x) \leqq 1, \quad \tfrac{1}{2} \leqq x \leqq 1.$$

Hence (11) with $\varphi = \Phi/[0,1]$ and $f = F/[0,1/2]$ yield (6).

In order to establish the existence of Φ we have to derive from (2), (3), (4) an $F: \mathbb{R} \to \mathbb{R}$ not only satisfying $F/[0,1/2] = f$ and (7) but also

$$1 < c \leqq \frac{F(x) - F(y)}{x - y}, \quad y < x. \tag{12}$$

We define $F: \mathbb{R} \to \mathbb{R}$ by

$$F(x) = f(x), \qquad 0 \leqq x \leqq \tfrac{1}{2},$$
$$F(x) = -f(x), \quad -\tfrac{1}{2} \leqq x \leqq 0,$$

and

$$F(x+k) = F(x) + 2k, \quad -\tfrac{1}{2} \leqq x \leqq \tfrac{1}{2}, \quad k \in \mathbb{Z}.$$

This definition is consistent in view of (2). Furthermore we obviously have $F/[0,1/2] = f$ and (7). Now (12) is valid for $x, y \in [-1/2, 0]$ by (4) and the definition of F. If $y \in [-1/2, 0]$ and $x \in [0, 1/2]$ we get

$$\begin{aligned} c(x-y) &= c(x-0) + c(0-y) \\ &\leqq F(x) - F(0) + F(0) - F(y) = F(x) - F(y), \end{aligned}$$

so that (12) is satisfied for $x, y \in [-1/2, 1/2]$.

If $x, y \in [-1/2+k, 1/2+k]$ for some $k \in \mathbb{Z}$ we find

$$\begin{aligned} c(x-y) &= c[(x-k)-(y-k)] \leqq F(x-k) - F(y-k) \\ &= F(x) - 2k - F(y) + 2k = F(x) - F(y) \end{aligned}$$

provided $y < x$. Finally let $y \in [-1/2+2, 1/2+2]$ and $x \in [-1/2+k, 1/2+k]$ with $k > 2$. Then we put $z_j = \tfrac{1}{2} + j$ and obtain

$$\begin{aligned} c(x-y) &= c(x - z_{k-1} + z_{k-1} - z_{k-2} + \ldots + z_2 - y \\ &\leqq F(x) - F(z_{k-1}) + F(z_{k-1}) - F(z_{k-2}) + \ldots + F(z_2) - F(y) \\ &= F(x) - F(y) \end{aligned}$$

so that (12) is completely proved.

Now the existence of a continuous strictly increasing function Φ with the desired properties (8) and (9) is guaranteed by the following

Lemma. *Let $F: \mathbb{R} \to \mathbb{R}$ be continuous, strictly increasing, and satisfying*

$$F(x+1) = F(x) + n, \quad x \in \mathbb{R}$$

for some integer $n \geqq 2$. Assume

$$1 < c \leqq \frac{F(x) - F(y)}{x - y}, \quad y < x$$

with some constant c. Then there is an unique $\Phi:\mathbb{R}\to\mathbb{R}$ *continuous, strictly increasing, and satisfying*

$$\Phi(x+1)=\Phi(x)+1\,,\qquad x\in\mathbb{R}$$

such that

$$\Phi(nx)=F(\Phi(x))\,.$$

Moreover, if for some σ, $0<\sigma<1$ *we require* $c>n^{\sigma}$, *then* Φ *is Hölder continuous with exponent* σ. *In addition, if* $F(-x)=-F(x)$, $x\in\mathbb{R}$ *then* $\Phi(-x)=-\Phi(x)$, $x\in\mathbb{R}$.

Remark. Since $F(x)=nx+\hat{F}(x)$, $\hat{F}(x+1)=\hat{F}(x)$, $x\in\mathbb{R}$ the above estimate is a Lipschitz smallness condition on the periodic part $\hat{F}$ of F.

Proof. Let $G:\mathbb{R}\to\mathbb{R}$ be the inverse of F; G is strictly increasing, continuous, satisfies

$$G(x+n)=G(x)+1\,,\qquad x\in\mathbb{R}$$

and

$$\frac{G(x)-G(y)}{x-y}\leqq\frac{1}{c}<1,\qquad x\neq y\,. \tag{13}$$

In order to solve

$$\Phi(x)=G(\Phi(nx))\,,\qquad x\in\mathbb{R}$$

we introduce the complete metric space

$$M:=\{\alpha:\mathbb{R}\to\mathbb{R},\ \alpha \text{ continuous, increasing, and satisfying } \alpha(x+1)=\alpha(x)+1,\ x\in\mathbb{R}\}$$

with the metric

$$|\alpha-\beta|:=\max_{0\leqq x\leqq 1}|\alpha(x)-\beta(x)|\,.$$

The required function Φ satisfies the equation

$$\Phi=T(\Phi)\,,$$

T being defined by

$$T(\alpha)(x)=G(\alpha(nx))\,,\qquad x\in\mathbb{R}\,. \tag{14}$$

Clearly $T(M)\subseteq M$, and T is a contraction in view of (13). It remains to show that the unique solution $\Phi\in M$ of (14) is strictly increasing. Assume $x<y$ and $\Phi(x)=\Phi(y)$ then $\Phi(nx)=\Phi(ny)$ and hence $\Phi(n^k x)=\Phi(n^k y)$ for all integers $k\geqq 1$. Since $n\geqq 2$ we can pick k so large that $n^k y>1+n^k x$, hence as $\Phi\in M$ we arrive at

$$\Phi(n^k y)\geqq\Phi(1+n^k x)=\Phi(n^k x)+1$$

which gives a contradiction. Therefore, if $x<y$ then $\Phi(x)<\Phi(y)$. Clearly $F(-x)=-F(x)$ implies $\Phi(-x)=-\Phi(x)$, since the set of odd functions in M is left invariant under T.

As far as the Hölder-continuity of Φ is concerned, assume $c > n^\sigma$, $0 < \sigma < 1$. If $\alpha \in M$, then

$$\alpha(x) = x + \hat{\alpha}(x), \qquad \hat{\alpha}(x+1) = \hat{\alpha}(x), \qquad x \in \mathbb{R}.$$

We shall show that $T(H_\sigma) \subseteqq H_\sigma$ where H_σ is the closed subset of M defined by

$$H_\sigma := \{\alpha \in M \mid H_\sigma(\hat{\alpha}) \leqq A\},$$

where

$$H_\sigma(\hat{\alpha}) := \sup_{\substack{x \neq y \\ 0 \leqq x, y \leqq 1}} \left| \frac{\hat{\alpha}(x) - \hat{\alpha}(y)}{|x-y|^\sigma} \right|$$

and $A > 0$ has still to be determined.

Let $\alpha \in H_\sigma$ and $\beta = T(\alpha)$ then for $x > y$ we have

$$\beta(x) - \beta(y) \leqq \frac{1}{c}(\alpha(nx) - \alpha(ny)),$$

and therefore if $0 \leqq x, y \leqq 1$

$$\left| \frac{\hat{\beta}(x) - \hat{\beta}(y)}{|x-y|^\sigma} \right| \leqq \left| \frac{n}{c} - 1 \right| + \frac{n^\sigma}{c} H_\sigma(\hat{\alpha}).$$

Hence, since $\frac{n^\sigma}{c} < 1$ we find $T(H_\sigma) \subseteqq H_\sigma$ if $A > 0$ is chosen sufficiently large, so we obtain $\Phi(x) = x + \hat{\Phi}(x)$, $x \in \mathbb{R}$ with $H_\sigma(\hat{\Phi}) \leqq A$ as was to be proved.

Asking why one loses smoothness in the above simple lemma one meets the equation

$$\varphi(2x) - 2\varphi(x) = a(x), \qquad a'(0) = 0,$$

to be solved for a periodic function φ. In general this equation does not admit differentiable solutions even if a is analytic.

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