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Integral Decomposition of Unbounded Operator Families

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Abstract. We give a meaning to the direct integral decomposition of unbounded operators and Op*-algebras on a metrizable dense domain of a Hilbert space, by considering them as bounded operators between several other Hilbert spaces.

Introduction

The decomposition of representations and states of $*$-algebras into irreducible representations and extremal states has been considered by Borchers and Yngvason [1] and Hegerfeldt [2] in the important case of nuclear $*$-algebras. The method of [2] consists in restricting the state to a dense subalgebra which is the finite linear span of countably many elements. Since the algebraic dual of such a subalgebra has a proper, metrizable and weakly complete positive cone, Choquet decomposition theory [3] can be applied to it. The restriction of the state is thus decomposed into extremal states of the subalgebra and these are continuous states (because the initial algebra is nuclear) and so can be extended to the whole algebra.

In this paper we want to consider non nuclear $*$-algebras and if we try the method of [2] we are not able to extend the extremal states of the subalgebra to the whole algebra (unless our algebra is such that every state on it is continuous; sufficient conditions for that are given in [4] p. 228). For that reason, we prefer to adopt the point of view of Borchers and Yngvason and to decompose first certain families of unbounded operators on a Hilbert space, the so-called Op*-algebras [5].

As usual for unbounded operators we have to distinguish between two different notions of commutant, the strong and the weak. A state is extremal if and only if the corresponding GNS-representation has a trivial weak commutant [6]. In the first part of [1] Borchers and Yngvason developed an extension theory for $*$-invariant families of unbounded operators. They showed that any such family $\mathscr{A}$ always has an extension $\hat{\mathscr{A}}$ such that its strong commutant $\hat{\mathscr{A}}'_s$ contains an Abelian von Neumann algebra $\mathscr{M}$ which is at the same time maximal Abelian in the weak commutant $\hat{\mathscr{A}}'_w$ (in order to get the irreducibility of the decomposition which will be performed with respect to $\mathscr{M}$). This extension theory is valid for any Op*-algebra (the nuclearity assumption comes only in the second part of [1]) so we are going to use it as well in our framework.

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Beginning thus with an Op*-algebra $\mathscr{A}$ defined on a dense domain $\mathscr{D}$ of a separable Hilbert space $\mathscr{H}$ and with an Abelian von Neumann algebra $\mathscr{M} \subset \mathscr{A}'_s$ such that $\mathscr{M}' \cap \mathscr{A}'_w = \mathscr{M}$ we decompose $\mathscr{H}$ in a direct integral with respect to $\mathscr{M}$. In Sect. II, we associate to each $A \in \mathscr{A}$ a Hilbert space $\mathscr{H}_A$ which is the domain of its closure $\bar{A}$, provided with the graph-norm. To the whole algebra $\mathscr{A}$ is then associated in a natural way a lattice of Hilbert spaces. The intersection $\bar{\mathscr{D}}$ of these Hilbert spaces is the completion of $\mathscr{D}$ for the $\mathscr{A}$-graph topology and is the domain of the "closure" $\bar{\mathscr{A}} = \{\bar{A}\lceil_{\bar{\mathscr{D}}} | A \in \mathscr{A}\}$. Considering the elements of $\mathscr{M}$, we prove that they are bounded operators in $\mathscr{H}_A$, for every $A \in \mathscr{M}$ and that the restrictions of $\mathscr{M}$ to the different $\mathscr{H}_A$ give unitarily equivalent representations $\mathscr{M}_A$ of $\mathscr{M}$ in each $\mathscr{H}_A$. This fact allows us in Sect. III to decompose each of the $\mathscr{H}_A$'s individually into a direct integral with respect to the measure μ on the compact space Λ occurring in the decomposition of $\mathscr{H}$. In order to get a coherent decomposition of the different $\mathscr{H}_A$'s we have to introduce and additional hypothesis on $(\mathscr{A}, \mathscr{D})$: we assume that the $\mathscr{A}$-graph topology of $\mathscr{D}$ is actually given by a countable number of graph-norms, i.e. that $\mathscr{D}$ is metrizable. This assumption is not really very restrictive because in most concrete cases the algebra $\mathscr{A}$ is either countable itself or dominated by a countable subset as introduced in [6] (i.e. there exists a countable subset $\mathscr{B}_0 \subset \mathscr{A}$ such that for every $A \in \mathscr{A}$, there exists $B \in \mathscr{B}_0$ and a constant K such that $\forall f \in \mathscr{D}$ we have: $\|Af\| \leq K \|Bf\|$). With that hypothesis $\bar{\mathscr{D}} = \bigcap_{A \in \mathscr{B}_0} \mathscr{H}_A$ is a separable Frechet space. We then just consider the decomposition of these $\mathscr{H}_A$ with $A \in \mathscr{B}_0$ and we get for almost every $\lambda \in \Lambda$ a lattice $\{\mathscr{H}_A(\lambda) | A \in \mathscr{B}_0\}$ of Hilbert spaces embedded into each other in exactly the same way as the $\mathscr{H}_A$'s. (The restriction to countably many Hilbert spaces avoids problems with union of null sets and yields the lattice *for almost every* λ). We can then define $\mathscr{D}(\lambda) \equiv \bigcap_{A \in \mathscr{B}_0} \mathscr{H}_A(\lambda)$ and show that it is a dense domain in $\mathscr{H}(\lambda)$ for almost every $\lambda \in \Lambda$.

Section IV is devoted to the decomposition of unbounded operators. In [7], Nussbaum gives a method of reduction for closed operators in Hilbert space and relates the decomposability of an unbounded operator to the decomposability of its characteristic matrix (which is constructed from the projection in $\mathscr{H} \times \mathscr{H}$ onto the graph of the operator). Here, our method will consist in the identification of each unbounded operator with a family of bounded operators between pairs of Hilbert spaces. Since each element $\bar{A} \in \bar{\mathscr{A}}$ is a continuous operator from $\bar{\mathscr{D}}$ into itself (with the $\mathscr{B}_0$-graph topology), $\bar{A}\lceil_{\bar{\mathscr{D}}}$ can be extended to bounded operators between various pairs of spaces $(\mathscr{H}_B, \mathscr{H}_C)$, $B, C \in \mathscr{B}_0$. We then use the well-known result of [8] for bounded operators between two direct integrals of Hilbert spaces: a bounded operator is decomposable iff it permutes with the diagonalizable operators. For a pair $(\mathscr{H}_B, \mathscr{H}_C)$ the diagonalizable operators are $\mathscr{M}_B$, $\mathscr{M}_C$ respectively, which are both restrictions of $\mathscr{M}$. Since $\bar{A}$ is bounded between the two spaces and permutes with $\mathscr{M}(\mathscr{M} \subset \mathscr{A}'_s)$, it follows that there exists measurable fields $\lambda \rightsquigarrow \bar{A}(\lambda) \in \mathscr{B}(\mathscr{H}_B(\lambda), \mathscr{H}_C(\lambda))$ between the corresponding pairs $(\mathscr{H}_B(\lambda), \mathscr{H}_C(\lambda))$. Since $B, C \in \mathscr{B}_0$ there is only a countable number of such pairs so that the various decompositions of $\bar{A}$ can be made coherent and it makes sense to consider $\bar{A}(\lambda)\lceil_{\mathscr{D}(\lambda)}$ which are continuous operators on $\mathscr{D}(\lambda)$. With this definition of the decomposition of an unbounded operator, it is easily shown that the algebraic relations (sum, product, adjoint) are preserved almost everywhere in the decomposition. So if we begin with a countable algebra $\mathscr{A}$ we decompose it in $(\mathscr{A}(\lambda), \mathscr{D}(\lambda))$ which are countable Op*-algebras almost everywhere. Because of the assumptions on $\mathscr{M}$ we have considered, the $(\mathscr{A}(\lambda), \mathscr{D}(\lambda))$ obtained in the decomposition are irreducible.

The extension of this decomposition to uncountable Op*-algebras defined on a metrizable $\mathscr{D}$, can be made but it involves a different kind of techniques (a judicious choice of topology on $\mathscr{A}$ will be necessary) and for that reason, it will be presented in a separate paper together with the application to the decomposition of representations and states of $*$-algebras.

I. Definitions and Elementary Properties

1.1 Let $\mathscr{D}$ be a prehilbert space and $\mathscr{H}$ its completion. Let $\mathscr{L}_+(\mathscr{D})$ denote the set of all linear operators A such that:
(a) The domain $D(A) = \mathscr{D}$ and $A\mathscr{D} \subseteq \mathscr{D}$
(b) The adjoint operator A^* is such that $D(A^*) \supseteq \mathscr{D}$ and $A^*\mathscr{D} \subseteq \mathscr{D}$

$\mathscr{L}_+(\mathscr{D})$ is a $*$-algebra with the involution $A \to A^+ = A^*\restriction_{\mathscr{D}}$

A $*$-subalgebra $\mathscr{A}$ of $\mathscr{L}_+(\mathscr{D})$ is called an Op*-algebra [5]. In this paper we shall always assume that $\mathscr{A}$ contains a unit element (the restriction to $\mathscr{D}$ of the identity operator on $\mathscr{H}$). Sometimes if confusion is possible, the Op*-algebra $\mathscr{A}$ will be denoted by $(\mathscr{A}, \mathscr{D})$.

If we equip $\mathscr{D}$ with the graph topology which is given by the set of semi-norms:

$$\{p_A(f) = \|Af\| \mid A \in \mathscr{A}\} \tag{1}$$

then all the elements of $\mathscr{A}$ are continuous from $\mathscr{D}$ into itself.

1.2. Since $\mathscr{A}$ consists of unbounded operators we have to distinguish between the weak and the strong commutant [1]. Let $B(\mathscr{H})$ denote the set of all bounded operators on $\mathscr{H}$.

The Weak Commutant

$$\mathscr{A}'_w \equiv \{C \in B(\mathscr{H}) \mid (f, CAg) = (A^+f, Cg),\ \forall A \in \mathscr{A},\quad \forall f, g \in \mathscr{D}\} \tag{2}$$

is a weakly closed linear subspace of $B(\mathscr{H})$, containing the identity and $*$-invariant but is not an algebra in general.

The Strong Commutant

$$\mathscr{A}'_s \equiv \{C \in B(\mathscr{H}) \mid C\mathscr{D} \subseteq \mathscr{D},\ CAf = ACf,\ \forall A \in \mathscr{A}, \forall f \in \mathscr{D}\} \tag{3}$$

is an algebra but is not $*$-invariant nor weakly closed in general.

If $\mathscr{A}$ consists of bounded operators only and $\mathscr{D} = \mathscr{H}$, the weak and the strong commutant coincide with the usual commutant. For unbounded operators $\mathscr{A}'_s \subseteq \mathscr{A}'_w$. To avoid confusion we shall sometimes denote these commutants by $(\mathscr{A}, \mathscr{D})'_s$ and $(\mathscr{A}, \mathscr{D})'_w$.

1.3. By definition, every element $A \in \mathscr{A}$ is a closable operator. We shall denote by $\bar{A}$ its closure as an operator in $\mathscr{H}$.

Consider the completion $\bar{\mathscr{D}}$ of $\mathscr{D}$ with respect to the graph topology (1). Because $\mathscr{A}$ possesses a unit element we get

$$\bar{\mathscr{D}} = \bigcap_{A \in \mathscr{A}} D(\bar{A}) \tag{4}$$

Remark. This is not true in general for any family of operators possessing a common domain. The proof ([9] Theorem 1) is based on the fact that since $1\lceil_{\mathscr{D}} \in \mathscr{A}$, for any two $A, B \in \mathscr{A}, \exists C = (1 + A^*A + B^*B)\lceil_{\mathscr{D}}$ in $\mathscr{A}$ such that $p_C(f) \geqq p_A(f), p_B(f), \forall f \in \mathscr{D}$ i.e. the system of semi-norms defining the graph topology is directed.

Each $A \in \mathscr{A}$ extends to a continuous operator $\bar{A}\lceil_{\bar{\mathscr{D}}}$ from $\bar{\mathscr{D}}$ into itself and the set

$$\bar{\mathscr{A}} \equiv \{\bar{A}\lceil_{\bar{\mathscr{D}}} \mid A \in \mathscr{A}\}$$

is an Op*-algebra called the "closure" of $\mathscr{A}$. The involution in $\bar{\mathscr{A}}$ is given by $\bar{A}\lceil_{\bar{\mathscr{D}}} \rightsquigarrow A^*\lceil_{\bar{\mathscr{D}}} = \bar{A^+}\lceil_{\bar{\mathscr{D}}}$. When $\mathscr{D} = \bar{\mathscr{D}}, \mathscr{A} = \bar{\mathscr{A}}$ is called a "closed" Op*-algebra. $(\bar{\mathscr{A}}, \bar{\mathscr{D}})$ is the minimal closed extension of $(\mathscr{A}, \mathscr{D})$ in the sense of [1]. The weak and the strong commutant of $\bar{\mathscr{A}}$ have the following properties:

a) $\bar{\mathscr{A}}'_s$ is weakly closed [1]

b) $\bar{\mathscr{A}}'_w = \mathscr{A}'_w$ [1]

c) $\mathscr{A}'_s \subseteq \bar{\mathscr{A}}'_s$

Proof. c) Let $C \in \mathscr{A}'_s$ and $f \in \bar{\mathscr{D}}$, there exists a Cauchy net $\{f^\alpha\} \subseteq \mathscr{D}$ converging to f in the graph topology i.e. $f = \lim_\alpha f^\alpha$ and $\bar{A}f = \lim_\alpha Af^\alpha, \forall A \in \mathscr{A}$ (lim = strong limit in $\mathscr{H}$).

Because C is a bounded operator in $\mathscr{H}, \{Cf^\alpha\}$ is also a Cauchy net and $Cf = \lim_\alpha Cf^\alpha$. Moreover the set $\{ACf^\alpha\}$ which is equal to $\{CAf^\alpha\}$ since $C \in \mathscr{A}'_s$, is a Cauchy net as well, for every $A \in \mathscr{A}$.

Since the $\bar{A}$ are closed operators, it follows that $Cf \in D(\bar{A})$ and $\bar{A}Cf = \lim_\alpha ACf^\alpha$, $= \lim_\alpha CAf^\alpha = C \lim_\alpha Af^\alpha = C\bar{A}f, \forall A \in \mathscr{A}, \forall f \in \mathscr{D}$. Since $\bar{\mathscr{D}} = \bigcap_{A \in \mathscr{A}} D(\bar{A})$ it follows that $Cf \in \bar{\mathscr{D}}$.

II. The Natural Structure Associated to an Op*-algebra

2.1. The Hilbert Spaces $\mathscr{H}_A$

For each $A \in \mathscr{A}, \bar{A}$ is a closed operator in $\mathscr{H}$, so its domain $D(\bar{A})$ provided with the graph norm

$$\|f\|_A^2 = \|f\|^2 + \|\bar{A}f\|^2 \tag{5}$$

is a Hilbert space that we shall denote by $\mathscr{H}_A$. By the representation theorem [10] applied to the (closed symmetric) quadratic form $a(f,f) = (f,f) + (\bar{A}f, \bar{A}f)$ defined

$\forall f\in\mathscr{H}_A$ we have that $\mathscr{H}_A$ is also the domain of the self-adjoint operator $(1+A^*\bar{A})^{1/2}$ associated to that form. So the graph norm can be rewritten as:

$$\|f\|_A = \|(1+A^*\bar{A})^{1/2}f\| \tag{6}$$

By construction $\mathscr{D}$ and $\bar{\mathscr{D}}$ are cores for $\bar{A}$ and $(1+A^*\bar{A})^{1/2}$.

The inverse operator $(1+A^*\bar{A})^{-1/2}$ is a bounded self-adjoint operator in $\mathscr{H}$ and $(1+A^*\bar{A})^{\pm 1/2}$ are reciprocal unitary isomorphisms between $\mathscr{H}$ and $\mathscr{H}_A$. So the norm of every element $g\in\mathscr{H}$ can be written as:

$$\|g\| = \|(1+A^*\bar{A})^{-1/2}g\|_A \tag{7}$$

Repeating the same construction for every $B\in\mathscr{A}$ we get a family of Hilbert spaces $\{\mathscr{H}_B\}$, each of them continuously embedded in $\mathscr{H}$ and unitarily isomorphic to $\mathscr{H}$. That family possesses a lattice structure: given $\mathscr{H}_A$ and $\mathscr{H}_B$, there exists $C=(1+A^*A+B^*B)\lceil_{\mathscr{D}}$ in $\mathscr{A}$ such that $\mathscr{H}_C\subseteq\mathscr{H}_A\cap\mathscr{H}_B$. So any Op*-algebra $(\mathscr{A},\mathscr{D})$ is associated to the following structure:

$$\begin{array}{ccccc} & & \mathscr{H}_A & & \\ & & \cup\!\shortmid & \subseteq & \\ \mathscr{D}\subseteq\bar{\mathscr{D}}=\bigcap_{A\in\mathscr{A}}\mathscr{H}_A\subseteq & \cdots\cdots\cdots & \mathscr{H}_C & \cdots\cdots\cdots & \mathscr{H} \\ & & \subseteq & \cup\!\shortmid & \\ & & \mathscr{H}_B & & \end{array} \tag{8}$$

where each space is dense and continuously embedded in the next ones. Let us now study the behaviour of the elements of the commutants with respect to that structure.

2.2. Lemma 2.2.1. *Let $C\in\mathscr{A}'_s$. Then for every $A\in\mathscr{A}$, C is a bounded operator from $\mathscr{H}_A$ into itself.*

Proof. By definition $C\in\mathscr{A}'_s$ means $C\bar{\mathscr{D}}\subseteq\bar{\mathscr{D}}$ and $C\bar{A}f=\bar{A}Cf, \forall A\in\mathscr{A}, \forall f\in\bar{\mathscr{D}}$. Consider $f\in\bar{\mathscr{D}}$ and compute the norm of Cf in $\mathscr{H}_A$:

$$\begin{aligned}\|Cf\|_A^2 &= \|Cf\|^2+\|\bar{A}Cf\|^2=\|Cf\|^2+\|C\bar{A}f\|^2\\ &\leqq \|C\|^2(\|f\|^2+\|\bar{A}f\|^2)=\|C\|^2\|f\|_A^2\end{aligned}$$

which shows that C is bounded because $\bar{\mathscr{D}}$ is dense in $\mathscr{H}_A$.

Remark. The relation $C\bar{A}f=\bar{A}Cf$ can then be extended to every $f\in\mathscr{H}_A$.

We have seen before that the strong commutant is in general not $*$- invariant. Nevertheless if we consider subsets of the strong commutant which are $*$-invariant we can prove the following lemma.

Lemma 2.2.2. *Let $\mathscr{M}$ be a $*$-invariant subset of $\mathscr{A}'_s$. If $C\in\mathscr{M}$, C leaves $D(A^*)$ invariant and commutes with A^*, for every $A\in\mathscr{A}$.*

Proof. Since $\mathscr{M}$ is $*$-invariant, $C\in\mathscr{M}$ implies $C^*\in\mathscr{M}\subseteq\mathscr{A}'_s$. Thus by Lemma 2.2.1. C^* leaves each $\mathscr{H}_A$ invariant. Let $f\in D(A^*)$. For every $h\in\mathscr{H}_A$ we have $|(f,\bar{A}C^*h)|\leqq K\|C^*h\|\leqq K'\|h\|$ for some constant $K, K'=\|C^*\|K$. The left hand side of this inequality equals $|(f, C^*\bar{A}h)| = |(Cf, \bar{A}h)|$. So $|(Cf, \bar{A}h)| \leqq K' \|h\|$ for some constant K' and for every $h \in D(\bar{A})$ which means that $Cf \in D(A^*)$ and $(Cf, \bar{A}h) = (A^*Cf, h)$.

On the other hand we have $(f, \bar{A}C^*h) = (CA^*f, h)$ and comparing these two last equalities we get: $(A^*Cf, h) = (CA^*f, h)$ for every $h \in \mathscr{H}_A$. Since $\mathscr{H}_A$ is dense in $\mathscr{H}$ we have finally $CA^*f = A^*Cf$ for every $f \in D(A^*)$ which means that C commutes with A^*. This can be done for every $A \in \mathscr{A}$.

2.3. In the next section we are going to decompose the Hilbert space $\mathscr{H}$ in a direct integral (of Hilbert spaces). That decomposition will be associated to an Abelian von Neumann algebra $\mathscr{M} \subseteq \bar{\mathscr{A}}'_s$. It follows thus from Lemmas 2.2.1. and 2.2.2. that each $M \in \mathscr{M}$ is a bounded operator in every $\mathscr{H}_A$ and commutes with each A^*.

In particular, every $M \in \mathscr{M}$ commutes with the two self-adjoint operators $(1 + A^*\bar{A})^{\pm 1/2}$, for every $A \in \mathscr{A}$.

Denoting by $\mathscr{M}_A = \{M\restriction_{\mathscr{H}_A} | M \in \mathscr{M}\} \subseteq B(\mathscr{H}_A)$ we get a family $\{\mathscr{M}_A | A \in \mathscr{A}\}$ of unitarily equivalent representations of the von Neumann algebra $\mathscr{M}$ in the different $\mathscr{H}_A$. The equivalence is given by the unitary operators $\{(1 + A^*\bar{A})^{\pm 1/2}\}$.

Remark. At this point, we would like to mention that the kind of natural structure introduced in 2.1. is a "nested Hilbert space" if with the $\mathscr{H}_A$ we consider also their dual spaces with respect to the scalar product of $\mathscr{H}$. That type of structure has been studied in [11] and in a more general framework in [12]. In [13], several algebras of operators on nested Hilbert spaces are introduced. The connection with the present situation is the following: $\bar{\mathscr{A}}'_s$ belongs to the algebra called $\mathscr{A}$ in [13] and the above $\mathscr{M}$ belongs to the von Neumann algebra $\mathscr{C}$. Theorem 3.2. of [13] provides an alternative proof of the fact that the $\mathscr{M}_A$ are all unitarily equivalent representations of $\mathscr{M}$.

III. Integral Decomposition of the Different Hilbert Spaces

3.1. Decomposition of $\mathscr{H}$

From now on we shall assume that $\mathscr{H}$ is separable. Let $(\mathscr{A}, \mathscr{D})$ be an Op*-algebra in $\mathscr{H}$ and $(\bar{\mathscr{A}}, \bar{\mathscr{D}})$ its closure. Let us assume that we can find an Abelian von Neumann algebra $\mathscr{M} \subseteq \mathscr{A}'_s$, containing 1 and maximal in the sense that $\mathscr{M} = \mathscr{M}' \cap \mathscr{A}'_w$.

(This hypothesis is justified by the extension theory developed in [1]. Any $(\mathscr{A}, \mathscr{D})$ admits an extension $(\hat{\mathscr{A}}, \hat{\mathscr{D}})$ for which such a $\mathscr{M}$ exists.

Let us notice that because of the properties mentioned in 1.3b) and c), we have also $\mathscr{M} \subseteq \bar{\mathscr{A}}'_s$ and $\mathscr{M} = \mathscr{M}' \cap \bar{\mathscr{A}}'_w$.

Since $\mathscr{M}$ is an Abelian von Neumann algebra in a separable Hilbert space, there exists [8] a compact metrizable space Λ, a positive regular Borel measure μ on Λ, and a μ-measurable field $\lambda \rightsquigarrow \mathscr{H}(\lambda)$ of Hilbert spaces such that

$$\mathscr{H} \simeq \int_\Lambda \mathscr{H}(\lambda) d\mu(\lambda) \tag{9}$$

and such that $\mathscr{M}$ consists in the diagonalized operators in that decomposition ($\mathscr{M} \simeq L^\infty(\Lambda, \mu)$). The set of bounded operators in $\mathscr{H}$ which are decomposed in that direct integral consists exactly of the elements of $\mathscr{M}'$ [8].

3.2. Decomposition of $\mathscr{H}_A$, $\forall A \in \mathscr{A}$

Consider the natural structure associated to $(\mathscr{A}, \mathscr{D})$ introduced in Section II. In each $\mathscr{H}_A$ we get a representation $\mathscr{M}_A$ of the von Neumann algebra $\mathscr{M}$. Since all those representations are unitarily equivalent as we saw in 2.3, we may expect to be able to decompose each $\mathscr{H}_A$ in direct integral with respect to the same measure μ on the same Λ, such that each $\mathscr{M}_A$ is diagonalized in the decomposition. Due to the fact that the unitary operators between the different Hilbert spaces are functions of elements of $\bar{\mathscr{A}}$ and that, on the other hand, the von Neumann algebra we want to be diagonalized lies in $\bar{\mathscr{A}}'_s$, we can effectively decompose all the $\mathscr{H}_A$'s. This can be done explicitely as follows:

3.2.1. For each $A \in \mathscr{A}$, $(1 + A^*\bar{A})^{-1/2}$ is a bounded operator on $\mathscr{H}$ and commutes with all the elements of $\mathscr{M}$ (2.3) i.e. belongs to $\mathscr{M}'$ and is thus a decomposable operator. That means that there exists a measurable field $\lambda \rightsquigarrow (1 + A^*\bar{A})^{-1/2}(\lambda) \in B(\mathscr{H}(\lambda))$ essentially bounded such that:

$$(1 + A^*\bar{A})^{-1/2} = \int_\Lambda (1 + A^*\bar{A})^{-1/2}(\lambda)\, d\mu(\lambda) \tag{10}$$

and $\|(1 + A^*\bar{A})^{-1/2}\| = \text{ess}\cdot\sup \|(1 + A^*\bar{A})^{-1/2}(\lambda)\|$

Since $(1 + A^*\bar{A})^{-1/2}(\lambda) \in B(\mathscr{H}(\lambda))$ a.e. we may define for almost every $\lambda \in \Lambda$:

$$\begin{aligned}\mathscr{H}_A(\lambda) &\equiv \text{Range}[(1 + A^*\bar{A})^{-1/2}(\lambda)]\\ &= \{f(\lambda) \in \mathscr{H}(\lambda) \mid \exists g(\lambda) \in \mathscr{H}(\lambda) \text{ such that } f(\lambda) = (1 + A^*\bar{A})^{-1/2}(\lambda) g(\lambda)\}\end{aligned} \tag{11}$$

By that definition we get $\mathscr{H}_A(\lambda)$ as a subset of $\mathscr{H}(\lambda)$. (In the null set $\mathscr{N}_A$ where $(1 + A^*\bar{A})^{-1/2}(\lambda) \notin B(\mathscr{H}(\lambda))$ we put $\mathscr{H}_A(\lambda) \equiv 0$). $\mathscr{H}_A(\lambda)$ becomes a Hilbert space if we equip it with the scalar product $(f(\lambda) \mid f'(\lambda))_{A,\lambda} \equiv (g(\lambda) \mid g'(\lambda))$ where $f^{(\prime)}(\lambda) = (1 + A^*\bar{A})^{-1/2}(\lambda) g^{(\prime)}(\lambda)$ and $(1 + A^*\bar{A})^{-1/2}(\lambda)$ is a unitary isomorphism from $\mathscr{H}(\lambda)$ onto $\mathscr{H}_A(\lambda)$. Repeating the same construction for almost every $\lambda \in \Lambda$ we get a measurable field of Hilbert spaces $\lambda \rightsquigarrow \mathscr{H}_A(\lambda)$. Taking the direct integral of that family we reconstruct $\mathscr{H}_A$:

$$\begin{aligned}\int_\Lambda \mathscr{H}_A(\lambda)\, d\mu(\lambda) &= \int_\Lambda (1 + A^*\bar{A})^{-1/2}(\lambda) \mathscr{H}(\lambda)\, d\mu(\lambda)\\ &= (1 + A^*\bar{A})^{-1/2} \int_\Lambda \mathscr{H}(\lambda)\, d\mu(\lambda) = (1 + A^*\bar{A})^{-1/2} \mathscr{H} = \mathscr{H}_A\end{aligned}$$

3.2.2. Because the measurable field $\lambda \rightsquigarrow (1 + A^*\bar{A})^{-1/2}(\lambda)$ actually consists of isomorphisms (hence bijections) from $\mathscr{H}(\lambda)$ onto $\mathscr{H}_A(\lambda)$, the field $\lambda \rightsquigarrow [(1 + A^*\bar{A})^{-1/2}(\lambda)]^{-1}$ is also measurable and

$$(1 + A^*\bar{A})^{1/2} = \int_\Lambda [(1 + A^*\bar{A})^{-1/2}(\lambda)]^{-1}\, d\mu(\lambda) \tag{12}$$

So the unbounded operator (in $\mathscr{H}$) $(1 + A^*\bar{A})^{1/2}$ is also decomposed. Defining $(1 + A^*\bar{A})^{1/2}(\lambda) \equiv [(1 + A^*\bar{A})^{-1/2}(\lambda)]^{-1}$ we can reexpress the norm of the elements of $\mathscr{H}_A(\lambda)$ as:

$$\|f(\lambda)\|_{A,\lambda} = \|g(\lambda)\| = \|(1 + A^*\bar{A})^{1/2}(\lambda) f(\lambda)\| \tag{13}$$

where $f(\lambda) = (1 + A^*\bar{A})^{-1/2}(\lambda) g(\lambda)$ with $g(\lambda) \in \mathscr{H}(\lambda)$.

3.3.1. Assumption of Metrizability of $\mathscr{D}$

The decomposition of the Hilbert spaces $\mathscr{H}_A$'s by the method presented in this paragraph is a coherent decomposition. Indeed, a vector f belonging to, say, $\mathscr{H}_B \cap \mathscr{H}_C$, will have a unique representation $f = \int\limits_{\Lambda} f(\lambda) d\mu(\lambda)$ namely its decomposition as an element of $\mathscr{H}$. The difference will appear in the expressions of $\| f(\lambda) \|_{B,\lambda}$ and $\| f(\lambda) \|_{C,\lambda}$ given by formulae analogous to (13).

However in order to be sure that there exists for almost every $\lambda \in \Lambda$ sufficiently many $\mathscr{H}_A(\lambda)$ different from zero we shall restrict ourselves to a countable number of $\mathscr{H}_A$'s. (Then the union of the null sets $\mathscr{N}_A$ for which $\mathscr{H}_A(\lambda) = 0$, will be a null set again). This leads us to impose an additional hypothesis on $\mathscr{D}$.

From now on, we shall consider dense subsets $\mathscr{D}$ of $\mathscr{H}$ for which the $\mathscr{A}$-graph-topology is a metrizable one, i.e. is given by a countable set of graph norms $\{ \| f \|_A^2 = \| f \|^2 + \| \bar{A} f \|^2 \mid A \in \mathscr{B}_0, \mathscr{B}_0 \subseteq \mathscr{A}, \mathscr{B}_0 \text{ countable} \}$. $\mathscr{B}_0$ can be considered as a $*$-subalgebra of $\mathscr{A}$ on the complex rational field by adding rational linear combinations, products and adjoints if necessary. In that way we have still that the system of norms $\| \ \|_A, A \in \mathscr{B}_0$ is directed. The $\mathscr{A}$-graph-topology and the $\mathscr{B}_0$-graph-topology are equivalent and $\bar{\mathscr{D}}$ is the completion of $\mathscr{D}$ with respect to those topologies.

$$\bar{\mathscr{D}} = \bigcap_{A \in \mathscr{A}} \mathscr{H}_A = \bigcap_{B_0 \in \mathscr{B}_0} \mathscr{H}_{B_0} \tag{14}$$

$\bar{\mathscr{D}}$ is a separable Frechet space (because $\mathscr{H}$ and $\mathscr{H}_A$ are separable).

3.3.2. We can now define for almost every $\lambda \in \Lambda$

(i.e. $\forall \lambda \notin \bigcup\limits_{A \in \mathscr{B}_0} \mathscr{N}_A$):

$$\mathscr{D}(\lambda) \equiv \bigcap_{A \in \mathscr{B}_0} \mathscr{H}_A(\lambda) \tag{15}$$

which is a Frechet space with the topology given by the set of norms (13) where A runs over $\mathscr{B}_0$.

Let us notice that the map $\mathscr{D} \to \mathscr{D}(\lambda) : f \rightsquigarrow f(\lambda)$ is not a continuous map in general, contrarily to what occurs in [1] where $\mathscr{D}$ is a nuclear space. Nevertheless, for almost every $\lambda \in \Lambda$ we have a structure similar to (8)

$$\begin{tikzcd}
 & & & \mathscr{H}_A(\lambda) & \\
\mathscr{D}(\lambda) = \bigcap\limits_{A \in \mathscr{B}_0} \mathscr{H}_A(\lambda) \subseteq \cdots\cdots\cdots \arrow[urrr, dashed, bend left] \arrow[drrr, dashed, bend right] & & \mathscr{H}_C(\lambda) \arrow[ur, phantom, sloped, "\subseteq"] \arrow[dr, phantom, sloped, "\supseteq"] & \cdots\cdots\cdots & \mathscr{H}(\lambda) \arrow[ul, phantom, sloped, "\supseteq"] \arrow[dl, phantom, sloped, "\subseteq"] \\
 & & & \mathscr{H}_B(\lambda) &
\end{tikzcd} \tag{16}$$

$(A, B, C \in \mathscr{B}_0)$

where each space is continuously embedded in the next ones and dense in the next ones in virtue of the following lemma.

Lemma 3.3.3. :$\mathscr{D}(\lambda)$ is dense in $\mathscr{H}(\lambda)$ a.e.

Proof. See the explicit decomposition of $\mathscr{H}$ in direct integral ([8] p. 208 theorem

1). Because $\mathscr{H}$ is separable we can begin with a sequence $(e_1, e_2 \dots)$ everywhere dense in it. The $\mathscr{H}(\lambda)$ are then constructed as the completion of the rational vector space X generated by the $\{e_i\}$ with respect to some sesquilinear form $h_{e_i,e_j}(\lambda)$ $i,j = 1,2\dots$ which comes from the Riesz–Markov theorem (after having taken the quotient by the kernel of that form).

Let $\{e_i(\lambda)\}$ be the set of images of $\{e_i\}$ by the canonical surjection $X \to \mathscr{H}(\lambda)$ for every λ. Then $\{e_i(\lambda)\}$ is total in $\mathscr{H}(\lambda)$ a.e. [8]. In our case, since $\mathscr{D}$ and $\bar{\mathscr{D}}$ are dense in $\mathscr{H}$, we may choose the set $\{e_i\}$ contained in $\bar{\mathscr{D}}$ or even in $\mathscr{D}$. Then $\{e_i(\lambda)\} \subseteq \mathscr{D}(\lambda)$ a.e. implies that $\mathscr{D}(\lambda)$ is dense in $\mathscr{H}(\lambda)$ a.e.

IV. Decomposition of Unbounded Operators

4.1. An unbounded operator is equivalent to a family of bounded operators

When we provide $\bar{\mathscr{D}}$ with the graph-topology given by the set of norms (5) $\{\|f\|_B \mid B \in \mathscr{A}\}$, every element $\bar{A}\restriction_{\bar{\mathscr{D}}} \in \mathscr{A}$ is a continuous operator from $\bar{\mathscr{D}}$ into itself. That means that $\forall B \in \mathscr{A}$, there exists $C \in \mathscr{A}$ such that

$$\|\bar{A}f\|_B \leqq K\|f\|_C, \quad \forall f \in \bar{\mathscr{D}}, \quad \text{for some constant } K \tag{17}$$

In fact, a suitable choice of C is $\bar{C} = 1 + A^*\bar{A} + (BA)^*\overline{BA}$. Since $\bar{\mathscr{D}}$ is dense in each one of the Hilbert spaces, (17) means that $\bar{A}$ can be extended to a bounded operator from $\mathscr{H}_C$ into $\mathscr{H}_B$. So given $A \in \mathscr{A}$, for every $\mathscr{H}_B$ there exists a $\mathscr{H}_C$ such that $\bar{A}$ is bounded from $\mathscr{H}_C$ in $\mathscr{H}_B$. We identify in that way the unbounded operator $\bar{A}$ with an infinite family of bounded operators between the various $\mathscr{H}_B$'s and we shall see that these bounded operators are in fact decomposable.

Since we have assumed that the $\mathscr{A}$-graph topology is equivalent to the $\mathscr{B}_0$-graph topology each element $A \in \mathscr{A}$ is also continuous from $\bar{\mathscr{D}}$ into itself considered this time with the $\mathscr{B}_0$-graph topology. That means that $\forall B \in \mathscr{B}_0$ there exists $C \in \mathscr{B}_0$ such that (17) holds. So $\bar{A}$ is in fact identified with a countable family of bounded operators between pairs of $\mathscr{H}_B$'s, $B \in \mathscr{B}_0$.

(Let us notice that to consider unbounded operators as a family of bounded ones is exactly the point of view of [11]. The operators we meet here are exactly well-defined operators in nested Hilbert spaces—see remark at the end of Sect. II).

4.2. As we mentioned in 3.1, when $\mathscr{H}$ is decomposed in a direct integral diagonalizing $\mathscr{M}$, the decomposable operators are exactly the elements of $\mathscr{M}'$. Here we want to apply a result slightly more general ([8] p. 164 theorem 1) which concerns bounded operators between two different Hilbert spaces $\mathscr{H}$ and $\mathscr{H}'$ both decomposed in a direct integral with respect to the same measure on the same compact space. A function $m(\lambda) \in L^\infty(\Lambda, \mu)$ determines a diagonalizable operator $M \in B(\mathscr{H})$ and another one $M' \in B(\mathscr{H}')$.

Then Dixmier's result says that any bounded operator $A : \mathscr{H} \to \mathscr{H}'$ such that $M'A = AM$ for every $m(\lambda) \in L^\infty(\Lambda, \mu)$, is decomposable. In our case, we have decomposed all the spaces $\mathscr{H}_B$ with respect to the same measure μ on the same compact space Λ. A function $m \in L^\infty(\Lambda, \mu)$ corresponds to a diagonal operator $M_B \in \mathscr{M}_B \subseteq B(\mathscr{H}_B)$ in each $\mathscr{H}_B$. Because all the $\mathscr{M}_B$'s are unitarily equivalent representations of $\mathscr{M}$, if $M_B \in \mathscr{M}_B$ and $M_C \in \mathscr{M}_C$ correspond to the same $m \in L^\infty(\Lambda, \mu)$, they are in fact the restrictions to $\mathscr{H}_B$ and $\mathscr{H}_C$ respectively of the same $M \in \mathscr{M}$.

Take now $A \in \mathscr{A}$: for every $\mathscr{H}_B$, there exists some $\mathscr{H}_C$ such that $\bar{A}$ is bounded from $\mathscr{H}_C$ into $\mathscr{H}_B$. Moreover $\bar{A} M_C f = M_B \bar{A} f$ for every $f \in \mathscr{H}_C$ because $\mathscr{M} \subseteq \mathscr{A}'_s$.

By Dixmier's result, $\bar{A}$ is thus decomposable i.e. there exists a measurable, essentially bounded field $\lambda \rightsquigarrow \bar{A}(\lambda) \in B(\mathscr{H}_C(\lambda), \mathscr{H}_B(\lambda))$ such that

$$\bar{A} = \int \bar{A}(\lambda) d\mu(\lambda) \tag{18}$$

and $\| \bar{A} \|_{BC} = \text{ess}\cdot\sup \| \bar{A}(\lambda) \|_{BC}$.

Doing the same thing for any pair $(\mathscr{H}_C, \mathscr{H}_B)$, $B, C \in \mathscr{B}_0$ between which $\bar{A}$ is bounded, we get a countable set of measurable fields $\lambda \rightsquigarrow \bar{A}(\lambda)_{(B,C)}$. As the various bounded operators representing $\bar{A}$ are coherent (i.e. if $\mathscr{H}_D \subseteq \mathscr{H}_C$ then $\bar{A}_{BC} \restriction_{\mathscr{H}_D} = \bar{A}_{BD}$) we have almost everywhere $\bar{A}(\lambda)_{BC} 1(\lambda)_{CD} = \bar{A}(\lambda)_{BD}$ (where $1(\lambda)_{CD}$ is the operator occurring in the decomposition of the inclusion of $\mathscr{H}_D(\lambda)$ into $\mathscr{H}_C(\lambda)$). Since there is only a countable number of such relations, we get a coherent decomposition of $\bar{A}$. The notation (18) is unambiguous and it makes sense to consider almost everywhere the restriction of $\bar{A}(\lambda)$ to $\mathscr{D}(\lambda)$.

4.3. Lemma 4.3.1. *For every $A \in \mathscr{A}$ and* a.e. $\lambda \in \Lambda$, *$\bar{A}(\lambda)$ leaves $\mathscr{D}(\lambda)$ invariant.*

Proof. Let $f(\lambda) \in \mathscr{D}(\lambda) = \bigcap_{C \in \mathscr{B}_0} \mathscr{H}_C(\lambda)$ and $\bar{A} = \int \bar{A}(\lambda) d\mu(\lambda)$

For every $B \in \mathscr{B}_0$, $\exists C \in \mathscr{B}_0$ such that

$$\| \bar{A}(\lambda) f(\lambda) \|_{B,\lambda} \leqq K \| f(\lambda) \|_{C,\lambda} \text{ for some constant } K.$$

Since $f(\lambda) \in \mathscr{D}(\lambda)$, the r.h.s. of this inequality is finite for every $C \in \mathscr{B}_0$ so the l.h.s. is finite for every $B \in \mathscr{B}_0$ which means $\bar{A}(\lambda) f(\lambda) \in \mathscr{D}(\lambda)$. ■

Corollary 4.3.2. *For every $A \in \mathscr{A}$, $\bar{A}(\lambda) \restriction_{\mathscr{D}(\lambda)}$ is a continuous map from $\mathscr{D}(\lambda)$ into itself for the graph topology given by the set of norms* (13) *where A runs over $\mathscr{B}_0$.*

Corollary 4.3.3. *For every $A \in \mathscr{A}$, and every $f \in \bar{\mathscr{D}}$, $\bar{A} f = \int \bar{A}(\lambda) f(\lambda) d\mu(\lambda)$ in the sense that for every $B \in \mathscr{B}_0$ there exists $C \in \mathscr{B}_0$ such that*

$$\| \bar{A} f \|_B^2 = \int \| \bar{A}(\lambda) f(\lambda) \|_B^2 d\mu(\lambda) \leqq K \int \| f(\lambda) \|_C^2 d\mu(\lambda) = K \| f \|_C^2 \text{ with } K = \| \bar{A} \|_{BC}. \blacksquare$$

As a summary, we say that A is "decomposable" if $A = \int A(\lambda) d\mu(\lambda)$ where $A(\lambda)$ is a.e. the restriction to $\mathscr{D}(\lambda)$ of a countable coherent family $\{A(\lambda)_{BC}\}$, whose elements come from the decomposition of the countable coherent family of bounded operators $\{A_{BC}\}$ whose restrictions to $\mathscr{D}$ gives A. So we just showed that every $A \in \mathscr{A}$ is decomposable. (We shall not write the bar on $A(\lambda)$ any more).

Lemma 4.3.4. *Let $A, A' \in \mathscr{A}$. Then $A + A'$, AA' and A^+ are decomposable and we have almost everywhere $(A + A')(\lambda) = A(\lambda) + A'(\lambda)$, $(AA')(\lambda) = A(\lambda) A'(\lambda)$ and $A^+(\lambda) = (A(\lambda))^+ \equiv A(\lambda)^* \restriction_{\mathscr{D}(\lambda)}$.*

Proof. The only thing which is not obvious is the last statement about the adjoint. Let $f, g \in \mathscr{D}$, $A \in \mathscr{A}$ and $M \in \mathscr{M}$ corresponding to $m \in L^\infty(\Lambda, \mu)$. Since $M \in \mathscr{A}'_w$ we have $(f, MAg) = (A^+ f, Mg)$ i.e.

$$\int m(\lambda)[(f(\lambda), A(\lambda)g(\lambda)) - (A^+(\lambda)f(\lambda), g(\lambda))]d\mu(\lambda) = 0$$

Since this is true for every $m \in L^\infty(\Lambda, \mu)$ it follows that:

$$(f(\lambda), A(\lambda)g(\lambda)) = (A^+(\lambda)f(\lambda), g(\lambda)) \quad \text{a.e.} \tag{19}$$

and thus $|(f(\lambda), A(\lambda)g(\lambda))| \leqq \|A^+(\lambda)f(\lambda)\| \, \|g(\lambda)\|$ for every $g(\lambda) \in \mathscr{D}(\lambda)$, hence $f(\lambda) \in D(A(\lambda)^*)$. Since this is true for every $f(\lambda) \in \mathscr{D}(\lambda)$ it follows that $\mathscr{D}(\lambda) \subseteq D(A(\lambda)^*)$ i.e. the Hilbertian adjoint of $A(\lambda)$ is densely defined. If we put now $A(\lambda)^+ \equiv A(\lambda)^* \restriction_{\mathscr{D}(\lambda)}$, (19) shows that $A(\lambda)^+ = A^+(\lambda)$ almost everywhere.

4.3.5. An immediate consequence of this lemma is that any countable Op*-algebra $\mathscr{A}$ and its closure $\bar{\mathscr{A}}$ are decomposed in sets $\mathscr{A}(\lambda)$ which are themselves countable Op*-algebras almost everywhere. The Op*-algebras $(\mathscr{A}(\lambda), \mathscr{D}(\lambda))$ obtained in this way are irreducible a.e., i.e. $(\mathscr{A}(\lambda), \mathscr{D}(\lambda))'_w$ is trivial a.e. The proof of the irreducibility follows essentially from the hypotheses $\mathscr{M} \subseteq \mathscr{A}'_s$ and $\mathscr{M}' \cap \mathscr{A}'_w = \mathscr{M}$ and is a repetition of [1] lemma 3.4. and Theorem 3.3(V). adapted to our case. Since by [1] again we know that any Op*-algebra $(\mathscr{A}, \mathscr{D})$ admits an extension $(\hat{\mathscr{A}}, \hat{\mathscr{D}})$ for which such a $\mathscr{M}$ exists, we have proved the following theorem:

Theorem. Let $(\mathscr{A}, \mathscr{D})$ be a countable Op*-algebra in a separable Hilbert space $\mathscr{H}$. There exists a Hilbert space $\hat{\mathscr{H}}$ containing $\mathscr{H}$ as closed subspace and a direct integral decomposition $\hat{\mathscr{H}} = \int_\Lambda \mathscr{H}(\lambda)d\mu(\lambda)$ where μ is a regular Borel measure on a compact space Λ.

For almost every $\lambda \in \Lambda$, there exists a countable Op*-algebra $\mathscr{A}(\lambda)$ on a dense domain $\mathscr{D}(\lambda)$ of $\mathscr{H}(\lambda)$ such that $\forall f \in \mathscr{D}, \forall A \in \mathscr{A}$

$$Af = \int_\Lambda A(\lambda)f(\lambda)d\mu(\lambda)$$

where $A(\lambda) \in \mathscr{A}(\lambda)$ and $f(\lambda) \in \mathscr{D}(\lambda)$. The decomposition is irreducible, i.e. $(\mathscr{A}(\lambda), \mathscr{D}(\lambda))'_w$ is trivial a.e.

The extention of this decomposition to uncountable, separable Op*-algebras on a metrizable, dense domain is a non trivial step which can be done by considering some particular topologies on $\mathscr{A}$. This is more technical and will be the matter of a forthcoming paper.

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