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Convergent and Asymptotic Iteration Methods in General Relativity

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Abstract. We show that the fast motion iteration method in General Relativity gives an asymptotic approximation to exact solutions of the reduced Einstein equations. Rigorous estimates of the error commited at each step of the iteration are derived.

1. Introduction

In General Relativity powerful theorems are known, which guarantee existence and uniqueness of solutions to Einstein's equations [1–4] under very general conditions. In contrast to this, however, very little is known about the validity of the approximation methods, on which the comparison between theory and observations is based. One has in fact reasons to be rather sceptical about the usual procedures.

The main result of this paper is an exact estimate of the error one makes by solving the reduced Einstein equations in the usual fast motion iteration scheme.

We outline the basic arguments with the following model problem: Solve, for a given source $\varrho(x, y, z, t)$, and for a finite time interval $0 \leqq t \leqq T$, the *quasilinear* equation

$$\Box_\phi \phi := (\Box + \phi \partial_x^2)\phi = \varrho\,, \qquad \Box = -\partial_t^2 + \partial_x^2 + \partial_y^2 + \partial_z^2 \tag{1.1}$$

with zero initial data, i.e., $\phi = \partial_t \phi = 0$ for $t = 0$, in which case all properties of the solution should be determined by the source. In the scetch, which we shall give in this introduction, $\| \ \|$ will stand for norms in Banach spaces, which will be specified later in the text. Further "c" will stand for a constant depending only on T.

The existence theorems for the problem (1.1) are based on the iteration

$$\Box_{\phi_n} \phi_{n+1} = \varrho\,, \qquad \phi_0 = 0\,. \tag{1.2}$$

If ϱ is small enough this sequence has the property

$$\|\phi_{n+1} - \phi_n\| \leqq \lambda \|\phi_n - \phi_{n-1}\|\,, \qquad 0 < \lambda < 1\,, \tag{1.3}$$

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which shows that ϕ_n is a Cauchy sequence in a Banach space, hence it converges to a unique solution ϕ. Furthermore

$$\|\phi-\phi_n\| \leqq \frac{\lambda^n}{1-\lambda}\|\phi_1-\phi_0\| = \frac{\lambda^n}{1-\lambda}\|\phi_1\|. \tag{1.4}$$

The contraction constant λ as well as $\|\phi_1\|$ can both be bounded by a norm of the source ϱ, hence

$$\|\phi-\phi_n\| \leqq c\|\varrho\|^{n+1}. \tag{1.5}$$

Suppose we want to actually calculate the iterates ϕ_n. Then we face the problem of constructing the Greens functions of $\Box_{\phi_n}$, which is not possible analytically. The usual way around this difficulty is to rewrite (1.1) as

$$\Box\phi = \varrho - \phi\partial_x^2\phi \tag{1.6}$$

and to iterate as follows:

$$\Box\bar{\phi}_{n+1} = \varrho - \bar{\phi}_n\partial_x^2\bar{\phi}_n, \qquad \bar{\phi}_0 = 0. \tag{1.7}$$

Here we have to solve in each step just a flat space wave equation, whose source is determined by ϱ and the $\bar{\phi}_n$, calculated before. In general, however, this sequence $(\bar{\phi}_n)$ can not be expected to converge, as can be seen from domain of dependence arguments.

Unfortunately, we have now two equally useless sequences: $\{\phi_n\}$ converges, but cannot be calculated, whereas $\{\bar{\phi}_n\}$ can be calculated, but we don't know whether and how the $\bar{\phi}_n$'s are related to the exact solution ϕ.

In Einstein's theory of gravitation exactly the same difficulty occurs.

Nevertheless, it is possible to show, as we shall now outline, that the $\{\bar{\phi}_n\}$ sequence is in fact asymptotic to the exact solution ϕ as $\varrho\to 0$ (in a sense to be specified in the text) and the error commited at each step can be estimated.

Combine the two iterations (1.5) and (1.7). One gets by eliminating ϱ

$$\Box_{\bar{\phi}_n}(\phi_{n+1}-\bar{\phi}_{n+1}) = (\phi_n-\bar{\phi}_n)\partial_x^2\phi_{n+1} + \bar{\phi}_n\partial_x^2(\bar{\phi}_{n+1}-\bar{\phi}_n) \tag{1.8}$$

provided one subtracts $\Box_{\bar{\phi}_n}\phi_{n+1}$ on both sides. This is a *linear* wave equation for $\phi_{n+1}-\bar{\phi}_{n+1}$ with wave operator $\Box_{\bar{\phi}_n}$ and a source depending only on quantities of the $\bar{\phi}_n$ iteration, $\partial_x^2\phi_{n+1}$ and the difference $(\phi_n-\bar{\phi}_n)$. We have in this case an energy inequality which boundes the unknown $\phi_{n+1}-\bar{\phi}_{n+1}$ in terms of the source. Hence one can estimate (provided that we can brake up the norm of the products into the product of the norms which we shall show in the text)

$$\|\phi_{n+1}-\bar{\phi}_{n+1}\| \leqq c\{\|\phi_n-\bar{\phi}_n\|\,\|\partial_x^2\phi_{n+1}\| + \|\bar{\phi}_n\|\,\|\partial_x^2(\bar{\phi}_{n+1}-\bar{\phi}_n)\|\}. \tag{1.9}$$

Since all three terms $\|\partial_x^2\phi_{n+1}\|$, $\|\bar{\phi}_n\|$ and $\|\partial_x^2(\bar{\phi}_{n+1}-\bar{\phi}_n)\|$ can be estimated by $\|\varrho\|$ one finds a recursion relation of the form:

$$\|\phi_{n+1}-\bar{\phi}_{n+1}\| \leqq c\|\varrho\|\,\|\phi_n-\bar{\phi}_n\| + c\|\varrho\|^{n+2} \tag{1.10}$$

which can easily be solved (since the first member $\phi_1-\bar{\phi}_1=0$), provided the norms of $(\phi_{n+1}-\bar{\phi}_{n+1})$ and $(\phi_n-\bar{\phi}_n)$ are taken in the same Banach space (a fact which we shall show in the text). Thus

$$\|\phi_n-\bar{\phi}_n\| \leqq c\|\varrho\|^{n+1}. \tag{1.11}$$

Combining this with (1.5) the result is

$$\|\phi-\bar{\phi}_n\| \leqq c\|\varrho\|^{n+1}. \tag{1.12}$$

It will turn out that the norm of ϱ will contain derivatives of ϱ up to order $n+1$, i.e. the right hand side, in contrast to (1.5), is not in fact proportional to a power of a fixed number. Equation (1.12) shows that the sequence $\{\bar{\phi}_n\}$ is in fact asymptotic to the exact solution ϕ.

Essentially the same estimate will be true for the reduced Einstein equations, i.e., Einstein's equation in a harmonic coordinate system

$$g^{\alpha\beta}\hat{\partial}_\alpha\hat{\partial}_\beta g^{\mu\nu}+P^{\mu\nu}(g^{\kappa\lambda},\hat{\partial}_\varrho g^{\kappa\lambda})=\varrho^{\mu\nu}. \tag{1.13}$$

The difference between the exact solution $g^{\mu\nu}$ and an approximate solution $\bar{g}_n^{\mu\nu}$, calculated by an iteration analogous to (1.7) will be bounded by a power of a norm of the source,

$$\|g-\bar{g}_n\|\leqq c\|\varrho\|^{n+1}. \tag{1.14}$$

The method by which we estimate the difference of the true solution and the iterates of a non-converging iteration procedure can be applied to other problems of classical field theories.

The plan of the paper is the following:

In Sect. 2 the tools from functional analysis is outlined.

The analysis of the convergent and asymptotic iterations is given in Sects. 3 and 4 respectively, and their comparison and final theorem appears in Sect. 5. Concluding remarks and comments follow in Sect. 6.

2. Function Spaces and Multiplication Properties

We define the following function spaces on $I\times\mathbb{R}^n$, where I is the interval $[0, T]$ [9].

Definition 1. $\tilde{E}_s$ is the space of tensor fields h on $I\times\mathbb{R}^n$ such that

i) the restriction h_t of h as well as the restriction $(D^\alpha h)_t$ of its derivatives of any order $|\alpha|\leqq s$, to each $\Sigma_t=\mathbb{R}^n\times\{t\}$ is almost everywhere defined and square integrable.

We set

$$\|h\|_s^{\Sigma_t}=\left[\int_{\Sigma_t}\sum_{|\gamma|\leqq s}|(D^\gamma h)_t|^2dx\right]^{1/2}. \tag{2.1}$$

ii) The mapping $I\to\mathbb{R}$ by $t\to\|h\|_s^{\Sigma_t}$ is measurable and essentially bounded.

$\tilde{E}_s$ endowed with the norm:

$$\|h\|_s=\operatorname{Ess}\sup_{t\in I}\|h\|_s^{\Sigma_t} \tag{2.2}$$

is a Banach space.

We will need the following multiplication properties.

Lemma 1. *If k, $l\geqq m$ and $k+l>m+\frac{n}{2}$ we have the continuous multiplication property*

$$\tilde{E}_k\times\tilde{E}_l\to\tilde{E}_m \tag{2.3}$$

by

$$(u, v) \to u \otimes v \tag{2.4}$$

i.e., there are constants $a_{k,l,m}$ *such that*

$$\|uv\|_m \leqq a_{k,l,m} \|u\|_k \cdot \|v\|_l . \tag{2.5}$$

Proof. On each Σ_t we can show that

$$\|uv\|_m^{\Sigma_t} \leqq a_{k,l,m} \|u\|_k^{\Sigma_t} \cdot \|v\|_l^{\Sigma_t} \tag{2.6}$$

in the same manner in which the analogous inequality is established for the corresponding Sobolev spaces over Σ_t [7, 8]. Taking the essential supremum in I we get (2.5). □

Corollary 1. *If* $k > \frac{n}{2}$, $\tilde{E}_k$ *is a Banach algebra.*

We shall write $a_{k,l,l}$ as $a_{k,l}$ and $a_{k,k}$ simply as a_k.

Lemma 2. *If* $s > \frac{n}{2}$, $\tilde{E}_s$ *is continuously imbedded in* $L^\infty(I, C_b^0(\mathbb{R}^n))$ *that is there exists a universal constant* c *such that* $\forall f \in \tilde{E}_s$

$$\operatorname*{Ess\,Sup}_{t \in I} \{|f|_{C_b^0}^{\Sigma_t}\} \leqq c \|f\|_s .$$

Lemma 3. *If the restriction* $(f)_{t_0} \in L^2(\Sigma_{t_0})$ *and* $Df \in \tilde{E}_0$ *then* $f \in \tilde{E}_1$ *and*

$$\|f\|_1^2 \leqq 2(|f|_{L^2}^{\Sigma_{t_0}})^2 + 2T^2 \cdot (\|Df\|_0)^2 .$$

Definition 2. Σ_t is uniformly space like with uniform lapse for a C^0 2-contra-variant symmetric tensor field γ on $I \times \mathbb{R}^3$ if there exist strictly positive constant a_0, A_0, a_1, A_1, B such that on Σ_t

$$\begin{aligned} a_0 &\leqq -\gamma^{00} \leqq A_0 \\ a_1 |\xi|^2 &\leqq \gamma^{ik} \xi_i \xi_k \leqq A_1 |\xi|^2 \\ |\gamma^{0i}| &\leqq B , \end{aligned}$$

where $\gamma^{\mu\nu}$ are the components of γ in the natural frame on $I \times \mathbb{R}^3$.

Remark. The above inequalities imply the analogous inequalities for $\gamma_{\mu\nu}$. The three conditions are equivalent to the following geometrical restrictions:

1. The normals to Σ_t lie always in a cone around $\frac{\partial}{\partial_t}$.
2. The opening of the light cone of the metric γ is bounded away from zero and infinity.
3. The lapse function on Σ_t is bounded and greater than a positive constant.

Definition 3. We say that a C^0 2-contra-variant tensorfield γ is a regularly hyperbolic metric on $I \times \mathbb{R}^3$ if Σ_t is uniformly spacelike with uniform lapse for each $t \in I$ and if the constants a_0, A_0, a_1, A_1, and B can be chosen independently of t.

Let $\Box_\gamma$ be the operator $\Box_\gamma = \gamma^{\mu\nu} D_\mu D_\nu$ and consider the equation on $I \times \mathbb{R}^3$

$$\Box_\gamma \phi = \varrho$$

with ϱ given, for Cauchy data $\phi = \partial_t \phi = 0$ on Σ_0. Denote by η the Minkowski metric on $I \times \mathbb{R}^3$, which is diagonal $(-1, +1, +1, +1)$ in the natural coordinates and let χ be the difference $\chi = \gamma - \eta$. The following theorem was established in [9].

Theorem 1

Suppose: 1. γ *is a regularly hyperbolic metric on* $I \times \mathbb{R}^3$.
2. $D\gamma \in L^\infty (I \times \mathbb{R}^3)$
3. $\chi \in \tilde{E}_k$
4. $\varrho \in \tilde{E}_{k-1}$

then there exists a solution ϕ *in* $\tilde{E}_k$ *to the above Cauchy problem, unique in* $\tilde{E}_1$, *satisfying the inequality:*

$$\|\phi\|_k \leqq c'_k \|\varrho\|_{k-1},$$

where constants c'_k *depend only on* a_0, A_0, a_1, A_1, B, $|D\gamma|_{L^\infty}$, $\|\chi\|_k$ *and* T.

Remark. If $\chi \in \tilde{E}_k$ with $k \geqq 3$, Lemma 2 implies that condition 2 is satisfied and that

$$|\chi| \leqq c \|\chi\|_k$$

everywhere on $I \times \mathbb{R}^3$. On the other hand it is easy to check that γ is regularly hyperbolic if $|\chi| \leqq \text{const} < 1$. Hence if χ is contained in a closed ball of radius R in $\tilde{E}_k$ with $k \geqq 3$, such that:

$$Rc < 1, \tag{2.7}$$

condition 1 is also satisfied. We conclude that in this case the coefficients c'_k can be chosen to depend only on R and T. We therefore choose a fixed R, a priori, satisfying (2.7) and shall regard the c'_k as depending only on T.

3. The Convergent Iteration Method

Einstein's field equations for the metric g

$$R^{\mu\nu} = T^{\mu\nu} - \tfrac{1}{2} T g^{\mu\nu} \tag{3.1}$$

do not allow to prescribe $T^{\mu\nu}$ as a given field on a manifold because (3.1) implies

$$T^{\mu\nu}{}_{;\nu} = 0. \tag{3.2}$$

If one however imposes the condition of harmonic coordinates (3.1) becomes the reduced Einstein equation

$$R_h^{\mu\nu} = T^{\mu\nu} - \tfrac{1}{2} T g^{\mu\nu} = \tfrac{1}{2} \varrho^{\mu\nu}. \tag{3.3}$$

This equation has solutions for arbitrary prescribed $\varrho^{\mu\nu}$. The solution is however only a solution of (3.1) if 1) the data for (3.3) satisfy the constraints and 2) if the solution $g^{\mu\nu}$ and $T^{\mu\nu}$ defined from $\varrho^{\mu\nu}$ and $g^{\mu\nu}$ satisfy $T^{\mu\nu}{}_{;\nu} = 0$.

We are going to consider solutions of (3.3) with given ϱ and trivial Cauchy data, namely:

$$(g)_0=\eta\,,\qquad (\partial_t g)_0=0 \tag{3.4}$$

because we want to study how the source generates the field.

Expressed in terms of g (3.3) becomes

$$\Box_g g=f(g,Dg)+\varrho \tag{3.5}$$

with

$$f(g,Dg)=P_g(Dg,Dg)\,, \tag{3.6}$$

where P_g is a bilinear form with coefficients which are rational functions of $g^{\mu\nu}$.

In coordinates we have

$$g^{\alpha\beta}D_\alpha D_\beta g^{\mu\nu}=P^{\mu\nu\ \varrho\sigma}_{\alpha\beta\ \gamma\delta}D_\varrho g^{\alpha\beta}D_\sigma g^{\gamma\delta}+\varrho^{\mu\nu}\,. \tag{3.7}$$

We can write P_g in the following form:

$$P_g=I+H_h\,, \tag{3.8}$$

where $I=P_\eta$ is a combination of δ^μ_ν, $\eta^{\mu\nu}$ and $\eta_{\mu\nu}$ and we have defined $h=g-\eta$. We shall denote $|I|=p$.

Lemma 4. *If $h\in\tilde{E}_s$ with $s\geqq 2$, and $g=\eta+h$ is non-degenerate, then $H_h\in\tilde{E}_s$, and if in addition $\|h\|_s$ is sufficiently small, there exist constants q and r such that:*

i) $\|H_h\|_l\leqq q\|h\|_l$,

ii) $\|H_{h_1}-H_{h_2}\|_l<r\|h_1-h_2\|_l$

for each l: $0\leqq l\leqq s$.

Proof. First H_h is a rational function of h, namely a polynomial in h with denominator a power of $\det g$. Since g is non-degenerate, the denominator is different from zero. Therefore H_h is an analytic function and corollary 1 implies $H_h\in\tilde{E}_s$. Then i) and ii) follow from the fact that H_h contains no constant term and $\|h\|_s$ is sufficiently small. □

We associate with the reduced Einstein equations the linear system obtained by replacing g in all terms except the second derivatives by a given tensor field γ, having trivial Cauchy data. That is:

$$\Box_\gamma g=f(\gamma,D\gamma)+\varrho\,. \tag{3.9}$$

Let us set $\chi=\gamma-\eta$.

Lemma 5. *If $\chi\in\tilde{E}_s$ with $s\geqq 3$ and $\gamma=\eta+\chi$ non-degenerate, then $f(\gamma,D\gamma)\in\tilde{E}_{s-1}$, and if in addition $\|\chi\|_s$ is sufficiently small, it holds:*

$$\|f(\gamma,D\gamma)\|_{s-1}\leqq 2a_{s-1}p\|\chi\|_s^2\,.$$

Proof. We write $f(\gamma,D\gamma)$ as:

$$\begin{aligned} f(\gamma,D\gamma)&=P_\gamma(D\gamma,D\gamma)\\ &=I(D\chi,D\chi)+H_\chi(D\chi,D\chi)\,. \end{aligned}$$

As $D\chi \in \tilde{E}_{s-1}$ with $s-1 \geqq 2$, Corollary 1 together with Lemma 4 imply $f \in \tilde{E}_{s-1}$. Further, if $\|\chi\|_{s-1}$ is sufficiently small from Lemma 4i) we have:

$$\begin{aligned}\|f\|_{s-1} &\leqq a_{s-1}(p + qa_{s-1}\|\chi\|_{s-1})\|\chi\|_s^2 \\ &\leqq 2a_{s-1}p\|\chi\|_s^2\end{aligned}$$

provided that

$$qa_{s-1}\|\chi\|_{s-1} \leqq p. \quad \square \tag{3.10}$$

Lemma 6. *If χ_1, χ_2 are in the closed ball of radius b_s in $\tilde{E}_s$, $s \geqq 3$, and b_s sufficiently small, the following inequality holds for each l such that $1 \leqq l \leqq s$*

$$\|f(\chi_1, D\chi_1) - f(\chi_2, D\chi_2)\|_{l-1} \leqq 4pa_{s-1,l-1}b_s\|\chi_1 - \chi_2\|_l.$$

Proof. We write

$$\begin{aligned}f(\chi_1, D\chi_1) - f(\chi_2, D\chi_2) = I(D(\chi_1 - \chi_2), D(\chi_1 + \chi_2)) + H_{\chi_1}(D(\chi_1 - \chi_2), D(\chi_1 + \chi_2)) \\ + (H_{\chi_1} - H_{\chi_2})(D\chi_2, D\chi_2).\end{aligned} \tag{3.11}$$

We shall estimate the $\tilde{E}_{l-1}$-norm of each of the three terms in the above expression separately. For the first term using Lemma 1 $(\tilde{E}_{s-1} \times \tilde{E}_{l-1} \to \tilde{E}_{l-1})$

$$\begin{aligned}&\|I(D(\chi_1 - \chi_2), D(\chi_1 + \chi_2))\| \\ &\quad \leqq pa_{s-1,l-1}\|\chi_1 + \chi_2\|_s\|\chi_1 - \chi_2\|_l \leqq 2pa_{s-1,l-1}b_s\|\chi_1 - \chi_2\|_l.\end{aligned} \tag{3.12}$$

For the second term we use Lemma 1 $(\tilde{E}_{s-1} \times \tilde{E}_{l-1} \to \tilde{E}_{l-1})$ twice as well as Lemma 4i)

$$\begin{aligned}&\|H_{\chi_1}(D(\chi_1 - \chi_2), D(\chi_1 + \chi_2))\|_{l-1} \\ &\quad \leqq a_{s-1,l-1}\|H_{\chi_1}\|_{s-1}\|D(\chi_1 - \chi_2) \otimes D(\chi_1 + \chi_2)\|_{l-1} \\ &\quad \leqq q(a_{s-1,l-1})^2\|\chi_1\|_{s-1}\|\chi_1 + \chi_2\|_s\|\chi_1 - \chi_2\|_l \\ &\quad \leqq 2q(a_{s-1,l-1})^2 b_s^2\|\chi_1 - \chi_2\|_l.\end{aligned} \tag{3.13}$$

For the third term we use Lemma 1 $(\tilde{E}_{s-1} \times \tilde{E}_{l-1} \to \tilde{E}_{l-1})$, Corollary 1 $(\tilde{E}_{s-1} \times \tilde{E}_{s-1} \to \tilde{E}_{s-1})$ and Lemma 4ii) to get:

$$\begin{aligned}&\|(H_{\chi_1} - H_{\chi_2})(D\chi_2, D\chi_2)\|_{l-1} \\ &\quad \leqq a_{s-1,l-1}\|H_{\chi_1} - H_{\chi_2}\|_{l-1}\|D\chi_2 \otimes D\chi_2\|_{s-1} \\ &\quad \leqq ra_{s-1}a_{s-1,l-1}\|\chi_2\|_s^2\|\chi_1 - \chi_2\|_{l-1} \\ &\quad \leqq ra_{s-1}a_{s-1,l-1}b_s^2\|\chi_1 - \chi_2\|_l\end{aligned} \tag{3.14}$$

Combining (3.12)–(3.14) we obtain

$$\begin{aligned}&\|f(\chi_1, D\chi_1) - f(\chi_2, D\chi_2)\|_{l-1} \\ &\quad \leqq a_{s-1,l-1}b_s[2p + (2qa_{s-1,l-1} + ra_{s-1})b_s]\|\chi_1 - \chi_2\|_l,\end{aligned} \tag{3.15}$$

from which we recover the lemma if:

$$(2qa_{s-1,l-1} + ra_{s-1})b_s \leqq 2p. \tag{3.16}$$

Theorem 1 and Lemma 5 imply now that if $\varrho \in \tilde{E}_{k-1}$ and $\chi \in \tilde{E}_k$ with $k \geqq 3$ and if in addition $\|\chi\|_{\tilde{E}_k}$ is sufficiently small, then Eq. (3.9) has a solution $g = \eta + h$ for trivial Cauchy data such that $h \in \tilde{E}_k$. Furthermore, the solution is unique in $\tilde{E}_1$. Hence we can define a map $\Phi : \chi \mapsto h$ from a sufficiently small ball in $\tilde{E}_k$ into $\tilde{E}_k$, for $k \geqq 3$. We shall now prove:

Lemma 7. *If $\varrho \in \tilde{E}_{k-1}$, $k \geqq 3$ and b_k is sufficiently small, Φ maps the closed ball of radius b_k in $\tilde{E}_k$ into the closed ball of radius b'_k in the same space, where:*

$$b'_k = c'_k(2pa_{k-1}b_k^2 + \|\varrho\|_{k-1}). \tag{3.17}$$

Proof. From Theorem 1 we have:

$$\|h\|_k \leqq c'_k(\|f\|_{k-1} + \|\varrho\|_{k-1}),$$

which, taking into account Lemma 5 gives:

$$\|h\|_k \leqq c'_k(2pa_{k-1}\|\chi\|_k^2 + \|\varrho\|_{k-1}),$$

provided that (3.10) is satisfied. The above inequality implies the lemma. □

Corollary 2. *If $\varrho \in \tilde{E}_{k-1}$, $k \geqq 3$ and $\|\varrho\|_{k-1}$ is sufficiently small, Φ maps the closed ball of radius B_k in $\tilde{E}_k$ into itself, where*

$$B_k = \frac{1 - \sqrt{1 - 8(c'_k)^2 a_{k-1} p \|\varrho\|_{k-1}}}{4c'_k a_{k-1} p}. \tag{3.18}$$

Proof. From Lemma 7, $b'_k \leqq b_k$ if

$$c'_k(2pa_{k-1}b_k^2 + \|\varrho\|_{k-1}) \leqq b_k. \tag{3.19}$$

The above inequality can be satisfied provided that

$$8(c'_k)^2 a_{k-1} p \|\varrho\|_{k-1} \leqq 1. \tag{3.20}$$

Then the minimal b_k satisfying (3.19) is the smallest of the two roots of the corresponding equality, which is (3.18). Furthermore,

$$B_k \leqq 2c'_k \|\varrho\|_{k-1}. \tag{3.21}$$

Consequently, (3.10) is satisfied provided that

$$2qc'_k a_{k-1} \|\varrho\|_{k-1} \leqq p. \quad \square \tag{3.22}$$

Using the map Φ we define the sequence $\{h_n\}$ by:

$$h_0 = 0$$
$$h_{n+1} = \Phi(h_n).$$

Corollary 3. *If $\varrho \in \tilde{E}_{k-1}$, $k \geqq 3$ and $\|\varrho\|_{k-1}$ is sufficiently small, the sequence $\{h_n\}$ is contained in the closed ball of radius B_k in $\tilde{E}_k$.*

The sequence $\{h_n\}$ is shown to be a Cauchy sequence by demonstrating that Φ is a contracting map.

Lemma 8. *If χ_1, χ_2 are in a closed ball of radius b in $\tilde{E}_3$, b is sufficiently small and $\varrho \in \tilde{E}_2$, their images under Φ satisfy the following inequality:*

$$\|\Phi(\chi_1)-\Phi(\chi_2)\|_1 \leqq \lambda(b,b')\|\chi_1-\chi_2\|_1\,, \tag{3.23}$$

where:

$$\lambda(b,b')=c_1'(a_{1,1,0}b'+4pa_{2,0}b) \tag{3.24}$$

and b' is given in terms of b and $\|\varrho\|_2$ by (3.17) *for $k=3$.*

Proof. Let $h_1=\Phi(\chi_1)$ and $h_2=\Phi(\chi_2)$. Subtracting (3.9) for h_2 from the same equation for h_1 we obtain: ($\gamma_1=\eta+\chi_1$, $\gamma_2=\eta+\chi_2$)

$$\Box_{\gamma_1}h_1-\Box_{\gamma_2}h_2=f(\gamma_1,D\gamma_1)-f(\gamma_2,D\gamma_2) \tag{3.25}$$

or equivalently,

$$\Box_{\gamma_1}(h_1-h_2)=-(\Box_{\gamma_1}-\Box_{\gamma_2})h_2+f(\gamma_1,D\gamma_1)-f(\gamma_2,D\gamma_2)\,. \tag{3.26}$$

Applying Theorem 1 for $k=1$ to (3.26), where we consider h_1-h_2 as the unknown ϕ and the right-hand side as the given source ϱ, we obtain

$$\|h_1-h_2\|_1 \leqq c_1'\{\|(\chi_1-\chi_2)\cdot D^2h_2\|_0+\|f(\chi_1,D\chi_1)-f(\chi_2,D\chi_2)\|_0\}\,. \tag{3.27}$$

Using Lemma 1 ($\tilde{E}_1\times\tilde{E}_1\to\tilde{E}_0$) we estimate the first term on the right by:

$$\begin{aligned}\|(\chi_1-\chi_2)\cdot D^2h_2\|_0 &\leqq a_{1,1,0}\|\chi_1-\chi_2\|_1\|h_2\|_3\\ &\leqq a_{1,1,0}b'\|\chi_1-\chi_2\|_1\,,\end{aligned} \tag{3.28}$$

where we have applied Lemma 7 with $k=3$. To estimate the second term on the right in (3.27) we use Lemma 6 with $s=3$, $l=1$:

$$\|f(\chi_1,D\chi_1)-f(\chi_2,D\chi_2)\|_0 \leqq 4pa_{2,0}b\|\chi_1-\chi_2\|_1\,. \tag{3.29}$$

Substituting (3.28) and (3.29) in (3.27) we recover the lemma. □

Corollary 4. *If $\varrho\in\tilde{E}_2$ and $\|\varrho\|_2$ is sufficiently small, the map Φ is contractive in the $\tilde{E}_1$ norm on the closed ball of radius B in $\tilde{E}_3$.*

Proof. Under the hypotheses of the corollary, Corollary 2 ensures us that the closed ball of radius B [given by (3.18) with $k=3$] in $\tilde{E}_3$ is mapped into itself by Φ. The rest follows from Lemma 8 provided that:

$$\Lambda=\lambda(B,B)=c_1'(a_{1,1,0}+4pa_{2,0})B<1 \tag{3.30}$$

which, in view of (3.21) holds if

$$2c_1'c_3'(a_{1,1,0}+4pa_{2,0})\|\varrho\|_2<1\,. \quad \square \tag{3.31}$$

The contractiveness of Φ allows us to prove an existence and uniquess theorem.

Theorem 2. *If $\varrho\in\tilde{E}_{k-1}$ with $k\geqq 3$ and if $\|\varrho\|_{\tilde{E}_{k-1}}$ is sufficiently small, the reduced Einstein equations with source ϱ have one and only one solution g on $I\times\mathbb{R}^3$ with trivial Cauchy data, where $h=g-\eta$ is in $\tilde{E}_k$ and g is regularly hyperbolic.*

Proof. By Lemma 8, if the hypotheses of the theorem are satisfied, the map Φ is contractive in the $\tilde{E}_1$-norm on the closed ball of radius B in $\tilde{E}_3$. By the contraction mapping principle, Φ has a unique fixed point h in $\tilde{E}_1$. To show that h is also in $\tilde{E}_k$ we argue as follows: By Corollary 3, for each multi-index α with $0 \leqq |\alpha| \leqq k$, the sequence $(D^\alpha h_n)_t$ is uniformly (in n and t) bounded in the $L^2(\mathbb{R}^3)$-norm. Thus we can extract a subsequence which converges weakly to some $(h^\alpha)_t \in L^2(\mathbb{R}^3)$. It is easily seen that $h^\alpha = D^\alpha h^0$. Since the weak limit of a measurable map is measurable, we can show that $t \mapsto \|h^0\|_k^{\Sigma_t}$ is measurable, hence $h^0 \in \tilde{E}_k$. Finally, it is evident that $h^0 = h$ apart from subsets of zero measure in each Σ_t [9–11].

4. The Asymptotic Iteration Method

The convergent iteration method, which was analyzed in the previous sections, cannot be constructed in practice, since it requires the knowledge of the Green's function of a given metric for the Cauchy problem posed. This is the reason why the following iteration is used instead in actual calculations, which uses only the Green's function of the Minkowski metric, i.e., the truncated retarded integral. This is essentially the so-called "fast motion iteration method" [5, 12, 13]. This method is defined by associating with the reduced Einstein equations (3.5) the following linear system

$$\Box \bar{g} = f(\gamma, D\gamma) + \varrho - (\Box_\gamma - \Box)\gamma\,, \tag{4.1}$$

where γ is, as in Eq. (3.9), a given tensor field and $\Box \equiv \Box_\eta$.

Let $\varrho \in \tilde{E}_{k-2}$ and $\chi \in \tilde{E}_k$, $k \geqq 3$ such that $\gamma = \eta + \chi$ is non-degenerate. Then we know from Lemma 5 that $f(\gamma, D\gamma) \in \tilde{E}_{k-1}$. Further, the last term on the right which is $\chi \cdot D^2\chi \in \tilde{E}_{k-2}$. Hence the effective source in (4.1) is in $\tilde{E}_{k-2}$ and Theorem 1 (with $\gamma = \eta$) gives us a solution $\bar{h} = \bar{g} - \eta \in \tilde{E}_{k-1}$ having trivial Cauchy data, unique in $\tilde{E}_1$. Thus we can define a map $\Psi : \chi \mapsto \bar{h}$ which sends a sufficiently small ball in $\tilde{E}_k$ into $\tilde{E}_{k-1}$. This map is thus not, as Φ is, a map from one space into itself. The reason for this is the appearance of second derivatives on the right hand side of (4.1) which causes the loss of one order of differentiability.

Using the map Ψ we define the sequence $\{\bar{h}_n\}$ by

$$\begin{aligned} \bar{h}_0 &= 0 \\ \bar{h}_{n+1} &= \Psi(\bar{h}_n)\,. \end{aligned} \tag{4.2}$$

Lemma 9. *The map Ψ sends the ball of radius b_k in $\tilde{E}_k$, $k \geqq 3$, into the ball of radius b''_{k-1} in $\tilde{E}_{k-1}$, where:*

$$b''_{k-1} = c_{k-1}\{\|\varrho\|_{k-2} + (2a_{k-1}p + a_{k-1,k-2})b_k^2\}\,.$$

The coefficients c_k are the coefficient c'_k of Theorem 1 in the special case $\gamma = \eta$.

Proof. Applying Theorem 1 with $\gamma = \eta$ gives us

$$\|\bar{h}\|_{E_{k-1}} \leqq c_{k-1}\{\|\varrho\|_{k-2} + \|f(\chi, D\chi)\|_{k-2} + \|\chi \cdot D^2\chi\|_{k-2}\}\,. \tag{4.3}$$

Using Lemma 1 ($\tilde{E}_{k-1} \times \tilde{E}_{k-2} \to \tilde{E}_{k-2}$), $k \geqq 3$

$$\|\chi \cdot D^2\chi\|_{k-2} \leqq a_{k-1,k-2}\|\chi\|_{k-1}\|\chi\|_k \leqq a_{k-1,k-2}\|\chi\|_k^2\,. \tag{4.4}$$

The term in f is estimated using Lemma 5.

$$\|f(\chi, D\chi)\|_{k-2} \leqq \|f(\chi, D\chi)\|_{k-1} \leqq 2a_{k-1} p \|\chi\|_k^2 . \tag{4.5}$$

Substituting (4.5), (4.4) into (4.3) we get Lemma 9. □

Corollary 5. *If* $\varrho \in \tilde{E}_{n+k-3}$, $k \geqq 3$ *and* $\|\varrho\|_{n+k-3}$ *is sufficiently small then* $\bar{h}_n \in \tilde{E}_{k-1}$ *and*

$$\|\bar{h}_n\|_{k-1} \leqq 2c_{k-1} \|\varrho\|_{n+k-3} .$$

The proof is by induction on n using Lemma 9.

The analog for Ψ of the contraction property of Φ is:

Lemma 10. *If* χ_1, χ_2 *are in a sufficiently small ball of radius* b_k *in* $\tilde{E}_k$, $k \geqq 3$ *their images* $\bar{h}_1$, $\bar{h}_2$ *under* Ψ *satisfy:*

$$\|\bar{h}_1 - \bar{h}_2\|_{k-1} \leqq \mu_k \|\chi_1 - \chi_2\|_k ,$$

where: $\mu_k = 2c_{k-1}(2pa_{k-1} + a_{k-1,k-2}) b_k$.

Proof. $\bar{h}_1$, $\bar{h}_2$ satisfy the Eq. (4.1), hence their difference satisfies

$$\square(\bar{h}_1 - \bar{h}_2) = f(\chi_1, D\chi_1) - f(\chi_2, D\chi_2) - (\chi_1 \cdot D^2\chi_1 - \chi_2 \cdot D^2\chi_2). \tag{4.6}$$

We write the second term as

$$\chi_1 \cdot D\chi_1 - \chi_2 \cdot D^2\chi_2 = (\chi_1 - \chi_2) \cdot D^2\chi_1 + \chi_2 \cdot D^2(\chi_1 - \chi_2) .$$

Using Lemma 1 $(\tilde{E}_{k-1} \times \tilde{E}_{k-2} \to \tilde{E}_{k-2})$, $k \geqq 3$ we get

$$\|(\chi_1 - \chi_2) \cdot D^2\chi_1\|_{k-2} \leqq a_{k-1,k-2} \|\chi_1 - \chi_2\|_{k-1} \|\chi_1\|_k$$

$$\|\chi_2 \cdot D^2(\chi_1 - \chi_2)\|_{k-2} \leqq a_{k-1,k-2} \|\chi_2\|_{k-1} \|\chi_1 - \chi_2\|_k .$$

Hence combining both gives

$$\|\chi_1 \cdot D^2\chi_1 - \chi_2 \cdot D^2\chi_2\|_{k-2} \leqq 2a_{k-1,k-2} b_k \|\chi_1 - \chi_2\|_k . \tag{4.7}$$

The terms in f are estimated using Lemma 6 for $l = s = k$:

$$\begin{aligned} \|f(\chi_1, D\chi_1) - f(\chi_2, D\chi_2)\|_{k-2} &\leqq \|f(\chi_1, D\chi_1) - f(\chi_2, D\chi_2)\|_{k-1} \\ &\leqq 4pa_{k-1} b_k \|\chi_1 - \chi_2\|_k . \end{aligned} \tag{4.8}$$

Applying Theorem 1 to (4.6) and using estimates (4.7) and (4.8) for the effective source we obtain Lemma 10. □

Corollary 6. *If* $\varrho \in \tilde{E}_{n+k-3}$, $n \geqq 1$, $k \geqq 3$ *and* $\|\varrho\|_{n+k-3}$ *sufficiently small then*

$$\|\bar{h}_n - \bar{h}_{n-1}\|_{k-1} \leqq e_{n,k-1} \|\varrho\|_{n+k-3}^n ,$$

where the coefficients $e_{n,k}$ *are given by the recursion relation*

$$e_{n+1,k-1} = 4c_{k-1} c_k (2pa_{k-1} + a_{k-1,k-2}) e_{n,k}$$

with

$$e_{1,k}=c_k\,.$$

The proof is by induction on n using Lemma 10.

Lemmas 9 and 10 show that there is a loss of 1 order of differentiability at each step of the iteration method of this section. Therefore the sequence $\{\bar{h}_n\}$ can not be expected to converge in general, even if ϱ is in $\tilde{E}_k$ for all k. It will however be shown in Sect. 6 that it is asymptotic to the exact solution.

5. Comparison Between the Convergent and Asymptotic Iteration

In the previous sections we discussed the convergent and asymptotic iteration methods separately. In this section we shall fulfill the aim of this paper by establishing relations between the two iteration methods.

Lemma 11. *Let χ be in a ball of radius b in $\tilde{E}_3$, with b sufficiently small. Then*

$$\|\Phi(\chi)-\Psi(\chi)\|_1\leqq vb\|\chi-\Psi(\chi)\|_2\,,$$

where $v=c_1'a_{2,0}$.

Proof. Writing $\Phi(\chi)=h$, $\Psi(\chi)=\bar{h}$ the following equations hold [(3.9), (4.1)]

$$\Box_\gamma h=f(\chi,D\chi)+\varrho\,, \tag{5.1}$$

$$\Box\bar{h}=f(\chi,D\chi)+\varrho-\chi\cdot D^2\chi\,, \tag{5.2}$$

where $\gamma=\eta+\chi$.

Subtracting (5.2) from (5.1) and adding on both sides of the resulting equation the quantity $(\Box-\Box_\gamma)\bar{h}=-\chi\cdot D^2h$ one gets

$$\Box_\gamma(h-\bar{h})=\chi\cdot D^2(\chi-\bar{h})\,. \tag{5.3}$$

Theorem 1 implies

$$\begin{aligned}\|h-\bar{h}\|_1&\leqq c_1'\|\chi\cdot D^2(\chi-\bar{h})\|_0\\&\leqq c_1'a_{2,0}\|\chi\|_2\|\chi-\bar{h}\|_2\,,\end{aligned}$$

from which we recover the lemma. □

We shall use the above lemma in

Lemma 12. *Let χ_1, χ_2 be in a ball of radius b in $\tilde{E}_3$ with b sufficiently small and let ϱ be in $\tilde{E}_2$. Then*

$$\|\Phi(\chi_1)-\Psi(\chi_2)\|_1\leqq\lambda(b,b')\|\chi_1-\chi_2\|_1+vb\|\chi_2-\Psi(\chi_2)\|_2\,,$$

where $\lambda(b,b')$ is defined in (3.24).

Proof. We have

$$\|\Phi(\chi_1)-\Psi(\chi_2)\|_1\leqq\|\Phi(\chi_1)-\Phi(\chi_2)\|_1+\|\Phi(\chi_2)-\Psi(\chi_2)\|_1\,.$$

Estimating the first term on the right using Lemma 8 and the second using Lemma 11, we recover the lemma.

Theorem 3. *Let ϱ be in $\tilde{E}_m$, $m \geqq 2$ and $\|\varrho\|_m$ sufficiently small. Then the difference between the iterative $\bar{h}_m$ and the exact solution h satisfies the following estimate:*

$$\|h - \bar{h}_m\|_1 \leqq j_m \|\varrho\|_m^{m+1},$$

where the j_m's are constants which depend only on the size T of the interval I.

Proof. We substitute in Lemma 12 for χ_1, χ_2 the iterates h_n and $\bar{h}_n$ respectively, and hence for $\Phi(\chi_1)$, $\Psi(\chi_2)$ their successors h_{n+1}, $\bar{h}_{n+1}$ in the corresponding sequences. The assumptions of the lemma require h_n and $\bar{h}_n$ to be contained in a ball of radius b in $\tilde{E}_3$. From Corollary 3, if $\varrho \in \tilde{E}_2$ and $\|\varrho\|_2$ is sufficiently small, we have:

$$\|h_n\|_3 \leqq 2c_3' \|\varrho\|_2 \tag{5.4}$$

[taking into account (3.21)]. On the other hand if $\varrho \in \tilde{E}_{n+1}$, $n \geqq 1$ with $\|\varrho\|_{n+1}$ sufficiently small, Corollary 5 gives us:

$$\|\bar{h}_n\|_3 \leqq 2c_3 \|\varrho\|_{n+1}. \tag{5.5}$$

From (5.4) and (5.5) we conclude that both h_n and $\bar{h}_n$ are contained in the ball of radius

$$b = 2c_3' \|\varrho\|_{n+1} \tag{5.6}$$

in $\tilde{E}_3$. Then (3.17) gives

$$b' = c_3'(2pa_2 b^2 + \|\varrho\|_2) \leqq 2c_3' \|\varrho\|_{n+1} = b, \tag{5.7}$$

where the last step holds if

$$8pa_2 (c_3')^2 \|\varrho\|_{n+1} \leqq 1.$$

Substituting (5.6) and (5.7) in (3.24) we obtain

$$\lambda(b, b') \leqq \alpha \|\varrho\|_{n+1}, \tag{5.8}$$

where:

$$\alpha = 2c_1' c_3' (a_{1,1,0} + 4pa_{2,0}). \tag{5.9}$$

Further

$$\nu b = \beta \|\varrho\|_{n+1}, \tag{5.10}$$

where:

$$\beta = 2c_1' c_3' a_{2,0}. \tag{5.11}$$

On the other hand Corollary 6 implies

$$\|\chi_2 - \Psi(\chi_2)\|_2 = \|\bar{h}_n - \bar{h}_{n+1}\|_2 \leqq e_{n+1,2} \|\varrho\|_{n+1}^{n+1}. \tag{5.12}$$

With the above substitutions Lemma 12 becomes the recursion relation

$$\|h_{n+1} - \bar{h}_{n+1}\|_1 \leqq \alpha \|\varrho\|_{n+1} \|h_n - \bar{h}_n\|_1 + e_{n+1,2} \|\varrho\|_{n+1}^{n+2}. \tag{5.13}$$

Taking into account the fact that $h_1 = \bar{h}_1$, if $\varrho \in \tilde{E}_m$, $m \geqq 2$ and $\|\varrho\|_m$ sufficiently small, using induction on n for $1 \leqq n \leqq m-1$, (5.13) leads to:

$$\|h_m - \bar{h}_m\|_1 \leqq f_m \|\varrho\|_m^{m+1}, \tag{5.14}$$

where:

$$f_m = \sum_{i=2}^{m} \beta e_{i,2} \alpha^{m-i}. \tag{5.15}$$

The difference between the exact solution h and the iterate h_n of the convergent sequence satisfies the estimate:

$$\|h - h_n\|_1 \leqq \frac{\Lambda^n}{1-\Lambda} \|h_1 - h_0\|_1 = \frac{\Lambda^n}{1-\Lambda} \|h_1\|_1, \tag{5.16}$$

where Λ is the contraction constant, $\Lambda < 1$, given by (3.30):

$$\Lambda \leqq \alpha \|\varrho\|_2. \tag{5.17}$$

The estimate (5.16) follows from:

$$\|h_{n+1} - h_n\|_1 \leqq \Lambda \|h_n - h_{n-1}\|_1$$

(contractiveness of Φ). If then

$$2\alpha \|\varrho\|_2 \leqq 1,$$

we get from (5.16): (since $\|h_1\|_1 \leqq c_1 \|\varrho\|_0$)

$$\|h - h_n\|_1 \leqq 2c_1 \alpha^n \|\varrho\|_2^{n+1}. \tag{5.18}$$

Combining finally (5.14) with (5.18) we obtain:

$$\|h - \bar{h}_m\|_1 \leqq j_m \|\varrho\|_m^{m+1},$$

where:

$$j_m = f_m + 2c_1 \alpha^m. \qquad \text{Q.E.D.}$$

6. Conclusions

Theorem 3 of the previous section shows that the sequence $\{\bar{h}_n\}$ calculated by the "fast motion" iteration method which uses only the Minkowski Green's function is asymptotic to the exact solution h. That is, for fixed n, if $\varrho \to 0$ in the $\tilde{E}_n$ norm then $\bar{h}_n \to h$ in the $\tilde{E}_1$ norm as $\|\varrho\|_n^{n+1}$. Furthermore we deduced precise error estimates for the sequence $\{\bar{h}_n\}$ in $\tilde{E}_1$ norm. In applying the estimates given to physical cases, a choice of length scale should beforehand be made, since neither the constants c_k, c'_k nor the $\tilde{E}_k$ norms are scale invariant. Perhaps the most convenient such choice is to take the time interval $T = 1$.

On the other hand domain of dependence considerations indicate that the sequence $\{\bar{h}_n\}$ does not converge to h, except in the case where ϱ is analytic.

Several generalizations of our results are possible using essentially the same methods: First, we can admit general Cauchy data instead of trivial. Second, we

can give error estimates in the $\tilde{E}_l$ norm, with $l \geqq 1$. Third, the treatment applies to any weakly coupled quasilinear hyperbolic system in any dimension, which has the same structure as the reduced Einstein equations.

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