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Exponential Clustering for Long-Range Integer-Spin Systems*

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Abstract. By using Kirkwood-Salsburg equations for classical spin systems with unbounded integer values we prove exponential decay (resp. power law decay) for exponential (resp. power law) decaying potentials. We use these results to prove the mass gap in the two-dimensional Higgs-Villain model in the weak coupling region.

1. Introduction

In a previous paper [1] we have worked out a Kirkwood-Salsburg equation for unbounded integer spin systems whose Hamiltonians are positive definite quadratic forms and investigated the conditions under which it led to a unique equilibrium state expressed as a convergent series in powers of the "Kirkwood-Salsburg" operator. We applied these results to the two-dimensional Higgs-Villain model, which is reduced to such a system by duality transformations [2–6].

In the present paper we use the series expansion of the equilibrium state to prove exponential (resp. power law) decay of truncated correlation functions when the interaction potential decays exponentially (resp. with a power law). Similar results for lattice gas and continuum systems are obtained in [7] and [8]. In fact, our method, especially in Theorem 2, is very similar to that of [7]. The results of [8] apply to very general systems, but only with finite range interactions.

We apply our results once more to the two-dimensional lattice Higgs-Villain model to prove that in a certain range of parameters (g^2/T large and g^2 small) truncated expectations of local observables exhibit exponential decay, i.e. there is a mass gap. Moreover our lower bound on the mass approaches the "bare mass" $gqT^{-1/2}$ as $T \to 0$ with g^2/T fixed. There is some question whether this mass gap should be interpreted as a Higgs mechanism. A massive photon would be associated with a Yukawa type potential between external charges, contradicting

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confinement [9]. Previous results on the Higgs mechanism in general lattice Higgs models were obtained in [10] using the cluster expansion. Further results are announced [4] and proved in detail in [11].

We recall notations and results on the Kirkwood-Salsburg equation for unbounded spin systems on the lattice [1]. The system has Hamiltonian $H_A = \sum_{x,y\in A} s_x V(x-y) s_y$ with the spins s_x belonging to $\mathbb{Z}$ and Gibbs measure $e^{-\beta H_A}$. The potential V is assumed to be real, symmetric, translation invariant (for the sake of simplicity) and positive definite with $\sum_{x,y\in A} s_x V(x-y) s_y \geqq \varepsilon \sum_{x\in A} s_x^2$ for some constant $\varepsilon > 0$. Q_X denotes a configuration $\{s_x\}_{x\in X}$ defined on X, with all $s_x \neq 0$. ϕ is the empty configuration. We write $v(Q_X) = \sum_{x\in X} s_x^2$. For any constant $r>0$, a norm on complex-valued functions on $\{Q_X\}$ is defined by $\|\varphi\|_r = \sup_{X,Q_X} |\varphi(Q_X)| e^{rv(Q_X)}$. The Banach space of functions $\varphi(Q_X)$ with $\|\varphi\|_r < \infty$ will be denoted by F_r.

The Kirkwood-Salsburg equation can be expressed as $\varrho = \delta + K\varrho$ where $\delta(\phi)=1$, $\delta(Q_X)=0$ for nonempty Q_X and $K\varrho(\phi)=0$

$$K\varrho(Q_X) = e^{-\beta W^x(Q_X)} \sum_{T\subset X^c} \sum_{Q_T} k(s_x, Q_T) R_x \varrho(Q_{X'} Q_T), \tag{1.1}$$

where the activity z which appears in [1] is absorbed in $V(0)$. Above

$$W^x(Q_X) = V(0) s_x^2 + 2 \sum_{y\in X'} s_x V(x-y) s_y\,; \qquad X' = X - \{x\}\,; x = x(Q_X) \in X \tag{1.2}$$

$$k(s_x, Q_T) = \prod_{y\in T} (e^{-2\beta s_x V(x-y) s_y} - 1)\,; \qquad k(s_x, \phi) = 1 \tag{1.3}$$

$$R_x \varrho(Q_Y) = \varrho(Q_Y) - \sum_{t\neq 0} \varrho(t_x Q_Y)\,. \tag{1.4}$$

The proofs in this paper require the choice of the distinguished site x to be made in such a way that if $x_0 = x(Q_X Q_Y) \in X$, then also $x_0 = x(Q_X)$. It is not clear that our method of choice in the previous paper [1], leading to the bound $W^x(Q_X) \geqq (2\varepsilon - V(0)) s_x^2$, can satisfy the requirement. Instead, we choose $x(Q_X)$ to be the first point of X in some consistent ordering (e.g., lexicographic) for which $|s_x| = \max\{|s_y| : y\in X\}$. This leads to the bound

$$W^x(Q_X) \geqq V(0) s_x^2 - 2\left|\sum s_x V(x-y) s_y\right| \geqq (V(0) - 2S_1) s_x^2 \tag{1.5}$$

Therefore the following result is a slight modification of Theorem 2 in [1]:

Theorem 1. *Let* $S_1 = \sum_{x\neq 0} |V(x)|$ *and* $S_2 = \sum_{x\neq 0} V(x)^2$. *If* $r \geqq 1$ *and*

$$\beta(V(0) - 2S_1) - r - \beta^2 \frac{S_2}{r} - 2\beta \frac{S_1}{r} > \log 2\,, \tag{1.6}$$

then K *is an operator in* F_r *with norm* $\|K\|_r < 1$. *For* $1 \leqq r \leqq \beta\varepsilon$ *the unique solution of the* $K-S$ *equation gives the correlation functions of the theory.*

2. Exponential Clustering and the Kirkwood-Salsburg Equation

Our basic strategy is to use the expansion $\varrho = \sum_{n=0}^{\infty} K^n \delta$ so that

$$\varrho(Q_X Q_Y) - \varrho(Q_X)\varrho(Q_Y) = \sum_{n=2}^{\infty} \left[K^n \delta(Q_X Q_Y) - \sum_{j=1}^{n-1} K^j \delta(Q_X) K^{n-j} \delta(Q_Y)\right] \tag{2.1}$$

and to bound inductively each term on the right. In pulling out a factor such as $e^{-\mu d(X,Y)}$ from each term (where $d(X,Y)=\inf\{|x-y|:x\in X,\ y\in Y\}$) we obtain a quantity which can be bounded using the same estimates used in [1] to bound the operator K, but with a "comparison potential" $\tilde{V}$ instead of V. For example, we may take $\tilde{V}(x)=e^{\mu|x|}V(x)$.

Theorem 2. *Suppose β, V, and r satisfy* (1.6) *and* $|V(x)|\leqq$ const. $e^{-m|x|}$ *for some* $m>0$ *and $|x|$ is some norm on $\mathbb{R}^d$. Then there exist constants C and $\mu>0$ such that*

$$|\varrho(Q_XQ_Y)-\varrho(Q_X)\varrho(Q_Y)|\leqq Ce^{-rv(Q_XQ_Y)}e^{-\mu d(X,Y)}. \tag{2.2}$$

Proof. For $m>\mu\geqq 0$ let $\tilde{V}(x)=e^{\mu|x|}|V(x)|$ with $\tilde{S}_1=\sum_{x\neq 0}|\tilde{V}(x)|$ and $\tilde{S}_2=\sum_{x\neq 0}\tilde{V}(x)^2$. By continuity we can take $\mu>0$ such that

$$\tilde{L}=\beta(V(0)-2\tilde{S}_1)-r-\beta^2\frac{\tilde{S}_2}{r}-2\beta\frac{\tilde{S}_1}{r}>\log 2 \tag{2.3}$$

so that $\tilde{B}\equiv 2e^{-\tilde{L}}<1$. This is the condition on μ that we need for the proof.

As indicated above, it is enough to show that

$$|K^n\delta(Q_XQ_Y)-\sum_{j=1}^{n-1}K^j\delta(Q_X)K^{n-j}\delta(Q_Y)|\leqq C_n e^{-rv(Q_XQ_Y)}e^{-\mu d(X,Y)} \tag{2.4}$$

with $\sum_{n=1}^{\infty}C_n<\infty$. The proof will be by induction. For $n=1$ $K\delta(Q_XQ_Y)=0$ so $C_1=0$. Suppose (2.4) is true for n. Then (assuming $x=x(Q_XQ_Y)\in X$)

$$\begin{aligned}
&K^{n+1}\delta(Q_XQ_Y)-\sum_{j=1}^{n}K^j\delta(Q_X)K^{n+1-j}\delta(Q_Y)\\
&\quad=e^{-\beta W^x(Q_XQ_Y)}\sum_{T\subset(X\cup Y)^c}\sum_{Q_T}k(s_x,Q_T)R_xK^n\delta(Q_{X'}Q_TQ_Y)\\
&\qquad-\sum_{j=0}^{n-1}e^{-\beta W^x(Q_X)}\sum_{T\subset X^c}\sum_{Q_T}k(s_x,Q_T)R_xK^j\delta(Q_{X'}Q_T)K^{n-j}\delta(Q_Y)\\
&\quad=T_1+T_2+T_3
\end{aligned}$$

where

$$T_1=(e^{-\beta W^x(Q_Y)}-1)e^{-\beta W^x(Q_X)})\sum_{T\subset(X\cup Y)^c}\sum_{Q_T}k(s_x,Q_T)R_xK^n\delta(Q_{X'}Q_TQ_Y)$$

(with $W^x(Q_Y)=2\sum_{y\in Y}s_xV(x-y)s_y$)

$$\begin{aligned}
T_2=e^{-\beta W^x(Q_X)}&\sum_{T\subset(X\cup Y)^c}\sum_{Q_T}k(s_x,Q_T)[R_xK^n\delta(Q_{X'}Q_TQ_Y)\\
&-\sum_{j=0}^{n-1}R_xK^j\delta(Q_{X'}Q_T)K^{n-j}\delta(Q_Y)]
\end{aligned}$$

$$T_3=-\sum_{j=0}^{n-1}e^{-\beta W^x(Q_X)}\sum_{\substack{T\subset X^c\\ T\cap Y\neq\phi}}\sum_{Q_T}k(s_x,Q_T)R_xK^j\delta(Q_{X'}Q_T)K^{n-j}\delta(Q_Y).$$

Using the inequality $|e^{\alpha t}-1|\leqq\alpha(e^{|t|}-1)$ for $0\leqq\alpha\leqq 1$, we have

$$|e^{-\beta W^x(Q_Y)}-1|\leqq e^{-\mu d(X,Y)}e^{\beta|\sum_{y\in Y}s_x\tilde{V}(x-y)s_y|}$$

So from the estimate

$$e^{2\beta\left|\sum_{y\in Y}s_x\tilde{V}(x-y)s_y\right|}e^{-\beta W^x(Q_X)}\leqq e^{-\beta(V(0)-2\tilde{S}_1)s_x^2},$$

following the proof of Theorem 2 in [1] we get

$$|T_1|\leqq\tilde{B}e^{-\mu d(X,Y)}e^{-rv(Q_XQ_Y)}\|K^n\delta\|_r.$$

If $T\subset(X\cup Y)^c$ is nonempty, choose $y_0\in T$ such that $|x-y_0|+d(X\cup T,Y)\geqq d(X,Y)$. In the estimate of T_2, we want to extract a factor $e^{-\mu|x-y_0|}$ from $k(s_x,Q_T)$ and $e^{-\mu d(X\cup T,Y)}$ from

$$R_xK^n\delta(Q_{X'}Q_TQ_Y)-\sum_{j=0}^{n-1}R_xK^j\delta(Q_{X'}Q_T)K^{n-j}\delta(Q_Y).$$

To do this we must consider together terms with s_{y_0} positive and negative. Let Q'_T be Q_T with the sign of s_{y_0} reversed. Then by the induction hypothesis (2.4), the definition (1.4) and the inequality $|\sinh\alpha t|\leqq\alpha|\sinh t|$ for $0\leqq\alpha\leqq 1$, it follows that

$$\begin{aligned}&|k(s_x,Q_T)|\,|R_xK^n\delta(Q_{X'}Q_TQ_Y)-\sum_{j=0}^{n-1}R_xK^j\delta(Q_{X'}Q_T)K^{n-j}\delta(Q_Y)|\\&\quad+\text{same expression with } Q'_T\\&\quad\leqq 4C_n|k(s_x,Q_{T\setminus\{y_0\}})|\sinh\beta|V(x-y_0)s_xs_{y_0}|e^{-\mu d(X\cup T,Y)}e^{-rv(Q_{X'}Q_TQ_Y)}\\&\quad\leqq 4C_n|k(s_x,Q_{T\setminus\{y_0\}})|\sinh\beta|\tilde{V}(x-y_0)s_xs_{y_0}|e^{-\mu d(X,Y)}e^{-rv(Q_{X'}Q_TQ_Y)}.\end{aligned}\tag{2.5}$$

Following again [1], we obtain

$$|T_2|\leqq\tilde{B}C_ne^{-\mu d(X,Y)}e^{-rv(Q_XQ_Y)}.$$

Similarly we can extract a factor $e^{-\mu|x-y|}$ with $y\in T\cap Y$ from $k(s_x,Q_T)$ in each term of T_3, obtaining

$$|T_3|\leqq(n-1)\tilde{B}e^{-\mu d(X,Y)}e^{-rv(Q_XQ_Y)}\|K^n\delta\|_r$$

Thus we have (2.4) with $C_{n+1}\leqq\tilde{B}(n\|K\|_r^n+C_n)$. Since $\|K\|_r<\tilde{B}<1$, it is easily seen that $\sum C_n<\infty$.

The short distance behavior of $V(x)$ will have a strong influence on the allowable values of μ in Theorem 2. For example, in section 4 we will have $V(x)=(-\Delta+m^2)^{-1}(x)$ *which has the asymptotic form const.* $\frac{e^{-m|x|}}{\sqrt{m|x|}}$. *However,* $V(1)\approx m^{-4}\approx m^{-2}V(0)$ for large m, so in order to have $V(0)>2\tilde{S}_1$, as is necessary for (2.3) to hold, we will need $\mu<2\log m$. We would instead like to have μ (the lower bound on the "physical mass") of the same order of magnitude as m (the "bare mass"). In fact, we will show that (2.2) holds with any $\mu<m$ if β is sufficiently large. The strategy is to modify the definition of $\tilde{V}$ at short distances. This will make it necessary to have $\tilde{B}$ small enough, which will be achieved for large enough β.

Theorem 3. *Suppose* $\sum |V(x)|e^{\mu|x|} < \infty$ *and* $V(0) - 2S_1 > 2\sqrt{S_2}$. *Then there are some* β_0 *and* r_0 *such that for* $\beta > \beta_0$ *and* $r = \beta r_0$ (2.2) *holds (with* $C \to 0$ *as* $\beta \to \infty$*).*

Proof. The proof of this theorem is a combination of the proof above and that of the Corollary in [1]. Let us take $r = \beta r_0$. Then

$$\tilde{L} = \beta(V(0) - 2\tilde{S}_1 - r_0 - \tilde{S}_2/r_0) - 2\tilde{S}_1/r_0 .$$

For any μ satisfying $\sum |V(x)|e^{\mu|x|} < \infty$, we define $\tilde{V}(x) = |V(x)|$ if $|x| \leqq R$ and $\tilde{V}(x) = |V(x)|e^{\mu|x|}$ if $|x| > R$, where R is chosen so that the condition $V(0) - 2\tilde{S}_1 > 2\sqrt{\tilde{S}_2}$ holds. Then there exists r_0 such that $V(0) - 2\tilde{S}_1 - r_0 - \tilde{S}_2/r_0 > 0$ and therefore a β_0 such that, for $\beta > \beta_0$, $\tilde{B} = 2e^{-\tilde{L}} < e^{-\mu R}$. Now we imitate the proof of Theorem 2. If $d(X, Y) \leqq R$ we use

$$|K^{n+1}\delta(Q_X Q_Y) - \sum_{j=1}^{n} K^j \delta(Q_X) K^{n+1-j} \delta(Q_Y)| \leqq (n+1) \|K\|_r^{n+1} e^{-rv(Q_X Q_Y)}$$

$$\leqq (n+1) \|K\|_r^{n+1} e^{\mu R} e^{-rv(Q_X Q_Y)} e^{-\mu d(X, Y)} .$$

If $d(X, Y) > R$ we have the same estimate on T_1 and T_3 as in Theorem 2. If there is $y_0 \in T$ with $|x - y_0| > R$, we again have the estimate (2.5). If $|x - y_0| \leqq R$ for all $y_0 \in T$, then $d(X \cup T, Y) \geqq d(X, Y) - R$ so

$$|R_x K^n \delta(Q_{X'} Q_T Q_Y) - \sum_{j=0}^{n-1} R_x K^j \delta(Q_{X'} Q_T) K^{n-j} \delta(Q_Y)|$$

$$\leqq 2C_n e^{-\mu d(X, Y)} e^{\mu R} e^{-rv(Q_{X'} Q_T Q_Y)}$$

This leads to the estimate $|T_2| \leqq \tilde{B} C_n e^{\mu R} e^{-rv(Q_X Q_Y)} e^{-\mu d(X, Y)}$. Thus

$$C_{n+1} \leqq \max\{\tilde{B}(n\|K\|_r^n + e^{\mu R} C_n), (n+1)\|K\|_r^{n+1} e^{\mu R}\} .$$

It is easily seen that $C_n \leqq D\tilde{B}^n e^{n\mu R}$ for some D independent of β. So $C = \sum_{n=2}^{\infty} C_n$ is finite if $\tilde{B} < e^{-\mu R}$, and as $\beta \to \infty$ we have $\tilde{B} \to 0$ and so $C \to 0$.

Similar results to Theorems 2 and 3 can be obtained for power law, rather than exponential, decays by using the metric $\delta(x, y) = \log(1 + |x - y|)$ instead of $|x - y|$. Thus if $V(0) - 2S_1 > 2\sqrt{S_2}$ and $\sum |V(x)| \, |x|^\gamma < \infty$, then for β sufficiently large $|\varrho(Q_X Q_Y) - \varrho(Q_X)\varrho(Q_Y)| < C d(X, Y)^{-\gamma} e^{-rv(Q_X Q_Y)}$ for some C and r.

3. Exponential Clustering of Observables

Once we have obtained exponential clustering of the correlation functions $\varrho(Q_X)$ by Theorem 2 or 3, we may look at cluster properties of more general observables. Suppose f is a function on the configuration space $\mathbb{Z}^X$ for a region X on the lattice, with the property that

$$f(\{s_x\}) = 0 \quad \text{if any} \quad s_x = 0 \quad \text{for} \quad x \in X . \qquad (3.1)$$

Then $\langle f\rangle=\sum_{Q_X} f(Q_X)\varrho(Q_X)$ (assuming that the sum converges absolutely; since $\varrho\in F_r$ for some $1\leqq r\leqq\beta\varepsilon$, this is a very mild condition on the grouth of f). For two such functions f and g on $\mathbb{Z}^X$ and $\mathbb{Z}^Y$ respectively, with X and Y disjoint,

$$\langle fg\rangle-\langle f\rangle\langle g\rangle=\sum_{Q_X}\sum_{Q_Y} f(Q_X)g(Q_Y)(\varrho(Q_XQ_Y)-\varrho(Q_X)\varrho(Q_Y))$$

so that if ϱ satisfies (2.2) we obtain

$$|\langle fg\rangle-\langle f\rangle\langle g\rangle|\leqq C\Big(\sum_{Q_X} f(Q_X)e^{-rv(Q_X)}\Big)\Big(\sum_{Q_Y} g(Q_Y)e^{-rv(Q_Y)}\Big)e^{-\mu d(X,Y)}.$$

If f does not have the property (3.1) we can write $f=\sum_{T\subset X} P_Tf$ where P_Tf depends only on the spins in T, and vanishes if any of these is zero. In fact $P_Tf(\{s_x\})=\sum_{S\subset T}(-1)^{|T\setminus S|}f(\{0\}_{X\setminus S},\{s_x\}_{x\in S})$. For $T=\phi$, $P_\phi f$ is the constant $f(\{0\})$. From the clustering of $P_{T_1}f$ and $P_{T_2}g$, we obtain that of f and g.

A particularly important example is where $f=\exp\Big(i\sum_{x\in X} a_xs_x\Big)$ and $g=\exp\Big(i\sum_{y\in Y} b_ys_y\Big)$. Then $P_Tf=\prod_{x\in T}(e^{ia_xs_x}-1)$, and similarly for g. Thus if ϱ satisfies (2.2)

$$\begin{aligned}
&|\langle fg\rangle-\langle f\rangle\langle g\rangle|\\
&=\Big|\sum_{T_1\subset X}\sum_{T_2\subset Y}\sum_{Q_{T_1}}\sum_{Q_{T_2}}\prod_{x\in T_1}\prod_{y\in T_2}(e^{ia_xs_x}-1)(e^{ib_ys_y}-1)\\
&\quad\cdot(\varrho(Q_{T_1}Q_{T_2})-\varrho(Q_{T_1})\varrho(Q_{T_2}))\Big|\\
&\leqq Ce^{-\mu d(X,Y)}\sum_{T_1T_2}\sum_{Q_{T_1}Q_{T_2}}\Big(\prod_{x\in T_1}|e^{ia_xs_x}-1|e^{-rs_x^2}\Big)\Big(\prod_{y\in T_2}|e^{ib_ys_y}-1|e^{-rs_y^2}\Big)\\
&=Ce^{-\mu d(X,Y)}\prod_{x\in X}\Big(1+\sum_{s\neq 0}|e^{ia_xs_x}-1|e^{-rs_x^2}\Big)\prod_{y\in Y}\Big(1+\sum_{s\neq 0}|e^{ib_ys_y}-1|e^{-rs_y^2}\Big)\\
&\leqq Ce^{\gamma\Sigma|a_x|}e^{\gamma\Sigma|b_y|}e^{-\mu d(X,Y)}
\end{aligned}\tag{3.2}$$

where $\gamma=2\sum_{s=1}^{\infty}|s|e^{-rs^2}$.

We can accomodate sums and integrals of such functions. Suppose $f=\int d\nu_1(a)e^{i\Sigma a_xs_x}$ and $g=\int d\nu_2(b)e^{i\Sigma b_ys_y}$ for complex measures $d\nu_1$ and $d\nu_2$ on $[-\pi,\pi]^X$ and $[-\pi,\pi]^Y$ respectively. Then if ϱ satisfies (2.2)

$$|\langle fg\rangle-\langle f\rangle\langle g\rangle|\leqq C(\int d\nu_1(a)e^{\gamma\Sigma|a_x|})(\int d\nu_2(b)e^{\gamma\Sigma|b_y|})e^{-\mu d(X,Y)}.$$

Finally, we note a further refinement that will be useful in the application to the Higgs-Villain model. Suppose again $f=\exp(i\sum a_xs_x)$ and $g=\exp(i\sum b_ys_y)$, but instead of being strictly local they satisfy conditions $\sum_x|a_x|e^{\mu d(x,X)}<\infty$ and $\sum_y|b_y|e^{\mu d(y,Y)}<\infty$. Let $x_0\in T_1$, $y_0\in T_2$ with $d(T_1,T_2)=|x_0-y_0|$. Then

$$\sum_{Q_{T_1}}\prod_{x\in T_1}|e^{ia_xs_x}-1|e^{-rs_x^2}\leqq e^{-\mu d(x_0,X)}\sum_{Q_{T_1}}\prod_{x\in T_1}|a_xe^{\mu d(x,X)}s_x|e^{-rs_x^2}.$$

Since $d(Y,X) \leqq d(x_0,X)+d(y_0,Y)+|x_0-y_0|$ we obtain

$$|\langle fg\rangle - \langle f\rangle\langle g\rangle| \leqq C\exp(\gamma \sum |a_x| e^{\mu d(x,X)} + \gamma \sum |b_y| e^{\mu d(y,Y)}) e^{-\mu d(X,Y)}. \tag{3.3}$$

4. The Mass Gap in the Higgs-Villain Model

We want to use the results of the previous sections to prove exponential clustering of localized observables in the two-dimensional Higgs-Villain model. This model has angle variables ϕ_x and $A_{x\mu}$ associated with sites in $\mathbb{Z}^2$ and nearest-neighbor bonds respectively ($x\mu$ is the bond between x and $x+\hat{\mu}$, $\mu=1$ or 2, where $\hat{1}=(1,0)$ and $\hat{2}=(0,1)$). The partition function and Gibbs equilibrium states are $Z=\int d\sigma$ and $\langle\!\langle f\rangle\!\rangle = Z^{-1}\int f\,d\sigma$ respectively, with

$$d\sigma = \prod_{x\mu} \sum_{m_{x\mu}\in\mathbb{Z}} \exp -\frac{1}{2T}(\partial_\mu\phi_x - qA_{x\mu} - 2\pi m_{x\mu})^2$$

$$\cdot \prod_x \sum_{m_x\in\mathbb{Z}} \exp -\frac{1}{2g^2}(\varepsilon_{\mu\nu}\partial_\mu A_{x\nu} - 2\pi m_x)^2 \prod_x \frac{d\phi_x}{2\pi} \prod_{x\mu} \frac{dA_{x\mu}}{2\pi}.$$

Here ∂_μ is the finite difference operator $\partial_\mu f(x) = f(x+\hat{\mu}) - f(x)$, $\varepsilon_{\mu\nu}$ is the antisymmetric tensor $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and repeated Greek indices are summed. Thus $\varepsilon_{\mu\nu}\partial_\mu A_{x\nu}$ is the plaquette variable $A_{x1}+A_{(x+\hat{1})2}-A_{(x+\hat{2})1}-A_{x2}$. We start with a finite lattice and assume periodic boundary conditions, in order to ensure translation invariance. As in the above sections, our results hold as well in the thermodynamic limit. We will actually work not with $\langle\!\langle\cdot\rangle\!\rangle$, but with a θ vacuum $\langle\!\langle\cdot\rangle\!\rangle_\theta$, where θ is an integer from 0 to $q-1$ [1].

Let X be a bounded region in $\mathbb{R}^2$, and f a function of the variables ϕ_x and $A_{x\mu}$ for $x\in X$ and $x\mu\subset X$ respectively. We will assume that f has an absolutely convergent Fourier series $\sum c_{st}\exp i((s,\phi)+(t_\mu,A_\mu))$. Here s_x and $t_{x\mu}$ are integer-valued functions on sites and bonds respectively, and we write (g,h) for the l^2 inner product $\sum \overline{g(x)}h(x)$. Unless $(s,\phi)+(t_\mu,A_\mu)$ is gauge-invariant, $\langle\!\langle \exp i((s,\phi)+(t_\mu,A_\mu))\rangle\!\rangle_\theta = 0$. Now the gauge-invariant terms $(s,\phi)+(t_\mu,A_\mu)$ are the linear combinations with integer coefficients of $\partial_\mu\phi_x - qA_{x\mu}$ and $\varepsilon_{\mu\nu}\partial_\mu A_{x\nu}$. This may involve terms with $x\notin X$ unless we assume X is simply connected, and thus contains any site enclosed by a path in X. Discarding all but gauge-invariant terms, we may write $f=\sum a_{uv}e_{uv}$ where $e_{uv}=\exp i((u,\varepsilon_{\mu\nu}\partial_\mu A_\nu)+(v_\mu,\partial_\mu\phi - qA_\mu))$. We let $\|f\|_\zeta = \sum |a_{uv}| \exp\zeta((u,u)+(v_\mu,v_\mu))$ for $\zeta>0$. We will prove the following theorem:

Theorem 4. *Suppose* (1.6) *holds with* $V(x)=(-\Delta+m^2)^{-1}(x)$, $\beta=\frac{2\pi^2}{T}$ *and* $m^2 = \frac{g^2q^2}{T}$. *Then there are positive numbers* μ, ζ, C *such that, for any functions* f_X *and* f_Y *of the variables in two disjoint simply-connected regions* X *and* Y,

$$|\langle\!\langle f_X f_Y\rangle\!\rangle_\theta - \langle\!\langle f_X\rangle\!\rangle_\theta \langle\!\langle f_Y\rangle\!\rangle_\theta| \leqq C\|f_X\|_\zeta \|f_Y\|_\zeta e^{-\mu d(X,Y)}. \tag{4.1}$$

Moreover as $T\to 0$ with m fixed, we may take $\mu\to m$.

Proof. It is sufficient to show that, for u and v_μ (resp. u' and v'_μ) supported in X (resp. Y)

$$|\langle\!\langle e_{uv}e_{u'v'}\rangle\!\rangle_\theta - \langle\!\langle e_{uv}\rangle\!\rangle_\theta \langle\!\langle e_{u'v'}\rangle\!\rangle_\theta| \leqq Ce^{\zeta((u,u)+(u',u')+(v_\mu,v_\mu)+(v'_\mu,v'_\mu))-\mu d(X,Y)}. \tag{4.2}$$

Using the duality transformations of $\{1\}$ we obtain

$$\langle\!\langle e_{uv}\rangle\!\rangle_\theta = \exp\left(\frac{1}{2T}(w,(-\Delta+m^2)^{-1}w) - \frac{T}{2}(v_\mu,v_\mu) - \frac{g^2}{2}(u,u) + g^2\theta \sum u_x\right) \cdot \left\langle \exp\frac{2\pi i}{T}(s,(-\Delta+m^2)^{-1}w)\right\rangle_\theta$$

and similar expressions for $\langle\!\langle e_{uv}e_{u'v'}\rangle\!\rangle_\theta$ and $\langle\!\langle e_{u'v'}\rangle\!\rangle_\theta$, where $-\Delta = \partial^*_\mu \partial_\mu$, $w = T\varepsilon_{\mu\nu}\partial^*_\nu v_\mu + g^2 qu$, and

$$\langle f\rangle_\theta = \frac{\sum f(s)\exp\left(\frac{2\pi^2}{T}(s,(-\Delta+m^2)^{-1}s) + \frac{2\pi i\theta}{q}\sum s_x\right)}{\sum \exp\left(\frac{2\pi^2}{T}(s,(-\Delta+m^2)^{-1}s) + \frac{2\pi i\theta}{q}\sum s_x\right)}.$$

The imaginary "external field" in this model can be dealt with by a slight modification of the definition of K, as in $\{1\}$, leaving unchanged all the estimates of the preceding sections. Thus by (3.3) we have

$$|\langle e^{i(s,\lambda+\lambda')}\rangle_\theta - \langle e^{i(s,\lambda)}\rangle_\theta \langle e^{i(s,\lambda')}\rangle_\theta| \leqq Ce^{-\mu d(X,Y)} e^{A(\Sigma|w_x| + \Sigma|w'_x|)} \tag{4.3}$$

with $\lambda = \frac{2\pi}{T}(-\Delta+m^2)^{-1}w$, $\lambda' = \frac{2\pi}{T}(-\Delta+m^2)^{-1}w'$, C and μ as in Theorem 2 or Theorem 3, and $A = \frac{2\pi\gamma}{T}\tilde{S}_1$, noting that

$$\sum_x |\lambda_x| e^{\mu d(x,X)} \leqq \frac{2\pi}{T}\sum_{xy}|w_y V(x-y)| e^{\mu|x-y|} = \frac{2\pi}{T}\tilde{S}_1 \sum_x |w_x|.$$

Moreover, we can use the inequality $|e^{\alpha t}-1| \leqq \alpha e^{|t|}$ for $0 \leqq \alpha \leqq 1$ to prove

$$|e^{T^{-1}(w,(-\Delta+m^2)^{-1}w')} - 1| \leqq e^{-\mu d(X,Y)} \exp\frac{1}{T}\sum_{xy} V(x-y)e^{\mu|x-y|}|w_x w'_y|$$

so that

$$\begin{aligned}
&\left|\exp\frac{1}{2T}(w+w',(-\Delta+m^2)^{-1}(w+w')) \right.\\
&\qquad \left. - \exp\frac{1}{2T}(w,(-\Delta+m^2)^{-1}w) + (w',(-\Delta+m^2)^{-1}w')\right| \\
&\leqq \exp\frac{1}{2T}((w,(-\Delta+m^2)^{-1}w) + (w',(-\Delta+m^2)^{-1}w') \\
&\qquad + 2\tilde{S}_1(w,w)^{1/2}(w',w')^{1/2})e^{-\mu d(X,Y)} \\
&\leqq \exp B((w,w)+(w',w'))e^{-\mu d(X,Y)}
\end{aligned} \tag{4.4}$$

with $B=\frac{1}{2T}(m^{-2}+\tilde{S}_1)$. Putting together (4.3), (4.4) and easy bounds on w and w' in terms of u, v_μ, u' and v'_μ, and noting that $\sum |u_x| \leqq (u,u)$ because the u_x are integers (and similarly for u', v_μ and v'_μ), we obtain (4.2).

For this model the condition (10.6) is true if $m^2 > 10.4$ (so that $V(0)-2S_2 > 2\sqrt{S_2}$ holds) and g^{-2} is sufficiently large (with a bound approaching 0.2 as $m \to \infty$).

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