Commun. Math. Phys. 67, 199–203 (1979)

Communications in
Mathematical
Physics

A Note on the Vacuum Structure of an SU(2) Yang-Mills Theory

D.H. Mayer[1]* and K.S. Viswanathan[2]**

[1] Institut für Theoretische Physik, E, RWTH Aachen, D-5100 Aachen, Federal Republic of Germany
[2] Physics Department, Simon Fraser University, Burnaby, B. C., Canada V5A 1S6

Abstract. We discuss different compactifications of the spacial part $\mathbb{R}^3$ of Minkowski space and give classifications of the vacuum structure for a Yang-Mills theory.

1. Introduction

The possible vacua in an SU(2) Yang-Mills theory and their physical implications have been discussed in several papers [1]–[6]. It turns out that a classification of this vacuum structure can be given by using homotopy theory of the underlying topological spaces. If we restrict ourselves to the so called $A^0=0$ gauge, the vacuum configurations are given by pure gauge fields $\mathbf{A}(\mathbf{x})=g^{-1}(\mathbf{x})\nabla g(\mathbf{x})$, where $g:\mathbb{R}^3\to \mathrm{SU}(2)$ is some mapping of the spacial part $\mathbb{R}^3$ of Minkowski space into the gauge group G, which for simplicity we take to be SU(2). To get the above mentioned classification one proceeds as follows: one compactifies $\mathbb{R}^3$ to some compact space K and studies then the continuous mappings of K into the gauge group G. Commonly one takes for K the one-point compactification S^3 and gets then a vacuum classification, for instance, via $\pi_3(\mathrm{SU}(2))$ which is isomorphic to $\mathbb{Z}$. Therefore an infinite sequence $\mathbf{A}_n, n=0,\pm 1,\ldots,$ of vacua arises in an SU(2) theory.

In terms of the mappings $g:\mathbb{R}^3\to\mathrm{SU}(2)$ the one-point compactification can be described also by allowing only those mappings g which have the property that $\lim_{\mathbf{x}\to\infty} g(\mathbf{x})=\text{const.}$ independent of the direction in which one goes to infinity.

Now Gribov [5] found that the physical properties of such a theory are in a great deal affected also by mappings g which have a more complicated behaviour at infinity. It is therefore natural to look for different compactifications of $\mathbb{R}^3$ which allow also such mappings.

Instead of giving a compactification of the space $\mathbb{R}^3$ in terms of a certain topological compact space K in which it can be embedded we use another comple-

* Partially supported by DFG and by Simon Fraser University
** Partially supported by National Research Council of Canada grant 751-010

0010-3616/79/0067/0199/$01.00

tely equivalent approach [7]: the space K can be described more indirectly by the set $\mathscr{C}(K, \mathrm{SU}(2))$ of all continuous mappings $\bar{g}: K \to \mathrm{SU}(2)$. If K should be a compactification of $\mathbb{R}^3$ then all these $\bar{g}$ arise from mappings $g: \mathbb{R}^3 \to \mathrm{SU}(2)$. It is clear that a compactification K is therefore also determined by the set of those g's which we want to be extendible in the above sense to continuous mappings $\bar{g}: K \to \mathrm{SU}(2)$.

The procedure we want to follow therefore is to choose a certain set τ_0 of mappings $g: \mathbb{R}^3 \to \mathrm{SU}(2)$ to which the Gribov mappings belong and classify these mappings via homotopy theory. In a second section we show how the set τ_0 is related to a natural compactification K of the space $\mathbb{R}^3$ in an SO(3) Yang-Mills theory, which is topologically just real projective 3-space $\mathbb{P}^3$.

2. The τ_0 Compactification of $\mathbb{R}^3$

Let be given a vacuum field $\mathbf{A}(\mathbf{x})$ such that

$$\mathbf{A}(\mathbf{x}) = g^{-1}(\mathbf{x})\nabla g(\mathbf{x}), \tag{1}$$

where $g: \mathbb{R}^3 \to \mathrm{SU}(2)$ is some continuous mapping of $\mathbb{R}^3$ into the gauge group SU(2) which is diffeomorphic topologically to the 3-sphere S^3. We denote by $f(\hat{\mathbf{x}})$ the function $f(\hat{\mathbf{x}}) = \lim_{\substack{\mathbf{x}\to\infty \\ \hat{\mathbf{x}}=\mathbf{x}/\|\mathbf{x}\|}} g(\mathbf{x})$ which we assume to exist. Then define the set τ by

$$\tau := \{g \in \mathscr{C}(\mathbb{R}^3, S^3) : f \in \mathscr{C}(S^2, S^3)\}. \tag{2}$$

The compactification K of $\mathbb{R}^3$ which allows all $g \in \tau$ to be extendible to $\bar{g} \in \mathscr{C}(K, S^3)$ is just the unit 3-ball $\bar{B}_3$ in $\mathbb{R}^3$, where $\bar{B}_3 := \{\mathbf{x} \in \mathbb{R}^3 : \|\mathbf{x}\| \leqq 1\}$. Because $\bar{B}_3$ is contractible, any continuous mapping $\bar{g}: K \to S^3$ can be continuously deformed into the constant mapping. The vacuum structure of an SU(2) theory with this compactification is therefore trivial.

To get a nontrivial structure we have to make some restrictions on the allowed mappings $g: \mathbb{R}^3 \to S^3$. This is generally done by introducing equivalence relations on $\bar{B}_3$: the S^3 compactification corresponds to the equivalence relation where all points on the boundary $\partial\bar{B}_3$ are identified to one point. In terms of the mappings g such an equivalence relation can be described by demanding that g takes on the same values at points which should be identified. What we want to do is still a little bit more general: we prescribe certain relations between the values the mapping g takes on at different points. To formulate this more precisely we note that the set τ in (2) can be described also by the set $\bar{\tau}$ which is given by

$$\bar{\tau} = \{\bar{g}: \bar{B}_3 \to S^3 : \bar{g} \text{ is induced by a } g \in \tau\},$$

and this set is just $\mathscr{C}(\bar{B}_3, S^3)$. We define then the sets $\tau_0 \subset \tau$ and $\bar{\tau}_0 \subset \bar{\tau}$ as follows:

$$\tau_0 = \{\bar{g} \in \tau : f(\hat{\mathbf{x}}) = \pm f(-\hat{\mathbf{x}}),$$

$$\bar{\tau}_0 = \{\bar{g} \in \bar{\tau} : \bar{g}(\mathbf{x}) = \pm \bar{g}(-\mathbf{x}) \text{ for } \mathbf{x} \in \partial\bar{B}_3\}. \tag{3}$$

The mappings $\bar{g} \in \bar{\tau}_0$ can be classified now via homotopy theory. Denote by $\bar{\tau}_\pm$

the sets

$$\bar{\tau}_{\pm} = \{\bar{g} \in \bar{\tau}_0 : \bar{g}(\mathbf{x}) = \pm \bar{g}(-\mathbf{x}) \text{ for } \mathbf{x} \in \partial \bar{B}_3\}. \tag{4}$$

Then we have

Lemma 1. *The homotopy classification of the set $\bar{\tau}_+$ is given by the cohomotopy set $\pi^3(\mathbb{P}^3)$ which is in one to one correspondence with $H^3(\mathbb{P}^3) \cong \mathbb{Z}$.*

Proof. Because real projective 3-space $\mathbb{P}^3$ is diffeomorphic to the unit 3-ball $\bar{B}_3$ with antipodal points on the boundary $\partial\bar{B}_3$ identified [8], every $\bar{g} \in \bar{\tau}_+$ induces a mapping of $\mathbb{P}^3$ into S^3. The homotopy classification of these mappings is given by the third cohomotopy set $\pi^3(\mathbb{P}^3)$ [9] of $\mathbb{P}^3$. The Hopf Theorem [10] then gives the one-to-one correspondence with the third cohomology group of $\mathbb{P}^3$ with integer coefficients, which turns out to be $\mathbb{Z}$.

This solves the classification of the set $\bar{\tau}_+$. It is infact given by the degree of the mappings $\bar{g} \in \bar{\tau}_+$. We mention only briefly that every $\bar{g} \in \bar{\tau}$ with $\bar{g}(\mathbf{x}) = \text{const.}$ for $\mathbf{x} \in \partial \bar{B}_3$ belongs to $\bar{\tau}_+$. It induces even a mapping $\bar{g}: S^3 \to S^3$ and can be classified also by $\pi_3(S^3)$. The classification is again via the degree and gives the same result.

We are left with the classification of the set $\bar{\tau}_-$. But this can be given quite easily, as one can see best in the one dimensional case. Here one has to consider mappings of the closed interval $[-1, +1]$ into S^1 such that $\bar{g}(1) = -\bar{g}(-1)$. This forces the image to cover the unit-circle half-integer times. It is straightforward to see that exactly the same happens in higher dimensions. So, if we define the number q as

$$q = -\frac{1}{2\pi^2} \int_{\bar{g}(\bar{B}_3)} d\Omega, \tag{5}$$

then this q is a topological invariant and takes on all half integers $\pm\frac{1}{2}, \pm\frac{3}{2}, \ldots$.

Any element of the class $[\pm\frac{1}{2}]$ generates all other classes in the set $\bar{\tau}_0$.

To summerize we have

Theorem 1. *The homotopy classification of the set $\bar{\tau}_0$ and therefore also of the set τ_0 is in one-to-one correspondence with the group $\frac{1}{2}\mathbb{Z}$.*

A mathematical problem which is left at this stage is whether a compact space K exists such that $\bar{\tau}_0 = \mathscr{C}(K, S^3)$ and if this space can be characterized in a simple way. It is certainly not a quotient space of the unit 3-ball B_3. In the next section we discuss the relation between $\bar{\tau}_0$ and the vacuum classification problem in an SO(3) Yang-Mills theory.

3. The Vacua in an SO(3) Theory

Because $\mathrm{SO}(3) \cong \mathrm{SU}(2)/\mathbb{Z}_2$ and $\mathbb{P}^3 \cong S^3/\mathbb{Z}_2$ we have $\mathrm{SO}(3) \cong \mathbb{P}^3$. The vacuum structure in such a theory is therefore related to the homotopy classification of mappings $g: \mathbb{R}^3 \to \mathbb{P}^3$. We consider in analogy to the previous case the set $\sigma := \{g \in \mathscr{C}(\mathbb{R}^3, \mathbb{P}^3) : f(\hat{\mathbf{x}}) \in \mathscr{C}(S^2, \mathbb{P}^3)\}$. Furthermore denote by $\sigma_+ = \{g \in \sigma : f(\hat{\mathbf{x}}) = f(-\hat{\mathbf{x}})\}$. The mappings $g \in \sigma_+$ induce continuous mappings $\bar{g}: \mathbb{P}^3 \to \mathbb{P}^3$. Their classification is given by a Theorem of Olum [11] and Wada [12]:

Theorem 2. *The homotopy classes of all mappings* $\bar{g}: \mathbb{P}^3 \to \mathbb{P}^3$ *are in one-to-one correspondence with the group*

$$G = H^1(\mathbb{P}^3, \mathbb{Z}_2) \oplus H^3(\mathbb{P}^3, \mathbb{Z}) \cong \mathbb{Z}_2 \oplus \mathbb{Z}.$$

H^1 *denotes the first cohomology group of* $\mathbb{P}^3$ *with coefficients in* $\mathbb{Z}_2$ *and* H^3 *the third cohomology group with integer coefficients.*

To get the relation of the classification of $\bar{\tau}_0$ with the above result we proceed as follows:
For $\bar{g} \in \bar{\tau}_0$ define the mapping $\bar{f}: \bar{B}_3 \to \mathbb{P}^3$ by

$$\bar{f} := \pi \circ \bar{g} \tag{6}$$

where $\pi: S^3 \to \mathbb{P}^3$ is the canonical projection. Because $\pi(y) = \pi(-y)$ for all $y \in S^3$ we get also $\bar{f}(\mathbf{x}) = \bar{f}(-\mathbf{x})$ for all $\mathbf{x} \in \partial \bar{B}_3$. Therefore $\bar{f}$ induces a mapping $\bar{\bar{f}}: \mathbb{P}^3 \to \mathbb{P}^3$. It is also clear that $\bar{g}$ and $-\bar{g}$ induce the same $\bar{\bar{f}}$.

Every $\bar{\bar{f}}: \mathbb{P}^3 \to \mathbb{P}^3$ induces a homomorphism $\bar{\bar{f}}_*: \pi_1(\mathbb{P}^3) \to \pi_1(\mathbb{P}^3)$ of the fundamental group of $\mathbb{P}^3$. Because $\pi_1(\mathbb{P}^3) \cong \mathbb{Z}_2$ there exist only two homomorphisms:

$$\begin{aligned} h_+ &= 0 \quad \text{(trivial map)} \\ h_- &= id \quad \text{(identity map).} \end{aligned} \tag{7}$$

Therefore all mappings $\bar{\bar{f}}: \mathbb{P}^3 \to \mathbb{P}^3$ fall trivially into two sets $\xi_\pm$ defined as

$$\xi_\pm = \{\bar{\bar{f}}: \mathbb{P}^3 \to \mathbb{P}^3 : \bar{\bar{f}}_* = h_\pm\}. \tag{8}$$

This reflects just the group $\mathbb{Z}_2$ in the Theorem of Wada and Olum.* A little reflection shows also that every $\bar{g} \in \bar{\tau}_\pm$ induces a mapping $\bar{\bar{f}}_\pm$ such that $\bar{\bar{f}}_\pm \in \xi_\pm$.
Because the group $\mathbb{Z}$ in Theorem 2 corresponds just to the degree of the mappings $\bar{\bar{f}}$ [11] we have

Theorem 3. *The homotopy classification of the set* $\bar{\tau}_+$ *is in one-to-one correspondence with the one of the set* ξ_+. *The classification of the set* $\bar{\tau}_-/\{[\pm\frac{1}{2}]\}$ *corresponds in a one-to-one way to that of the set* $\xi_-/[(\frac{1}{2}, 0)]$. *The classes* $[\pm\frac{1}{2}]$ *correspond to the single class* $[(\frac{1}{2}, 0)]$ *in* ξ_-.
(We denote the elements of $\mathbb{Z}_2$ by 0 and $\frac{1}{2}$).

Therefore the two Gribov vacua in an SU(2) theory belong to the same homotopy class in an SO(3) theory where the space $\mathbb{R}^3$ is compactified to projective 3-space $\mathbb{P}^3$.

An extended version of this work will appear elsewhere.

Acknowledgements. The paper was written during the visit of one of the authors (D.M.) at the Physics Department of Simon Fraser University. He wants to thank the members of the Department for their kind hospitality and the grant which made the visit possible. It is a pleasure to acknowledge useful discussions with Jae Hyung Yee. One of us (K.S.V.) would like to thank National Research Council of Canada for partial financial support.

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Communicated by R. Stora

Received July 8, 1978; in revised form August 24, 1978