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Circuit Based Graphs for Renormalized Perturbation Theory*

Richard R. Hampton

St. Thomas Aquinas College, Sparkill, New York 10976, USA

Abstract. A generalization of graph theory is introduced and used to obtain Feynman parametric formulas relevant to renormalized amplitudes. The generalization of graph theory is based upon circuit coefficients instead of the usual incidence matrix. The parametric formulas presented are valid for amplitudes which have been renormalized, as in the Zimmermann formulation, by subtracting Taylor terms in momentum space.

0. Introduction

In their beautiful work on graph theory and Feynman amplitudes, Nakanishi [1] and Speer [2] were able to analyze the ultraviolet singularities of the unsubtracted Feynman amplitude by applying graph theory to obtain formulas for the integrand in a Feynman parametric space.

For each circuit, C, of the graph representing the amplitude, they assign circuit coefficients (set indicator) according to the rule

$$(C;\ell) = 1, \text{ if } \ell \text{ is a line of } C, \text{ and}$$
$$= 0, \text{ otherwise.}$$

These coefficients may be identified to within a sign with the numbers d_ℓ^i for which

$$p_\ell = \sum_{i=1}^{m} d_\ell^i k_i$$

gives the internal momentum of the ℓ-th line of the graph. We will call the d_ℓ^i momentum routing coefficients.

In the Zimmermann [3] formulation of the BPH [4, 5] subtraction procedure, renormalization is accomplished by subtracting certain Taylor terms in

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momentum space. The different singularity parts are organized into so-called forests. When there are overlapping divergent subgraphs, the variables associated with different forests are those of the various subgraphs. Under these circumstances, terms appear in which, in effect, the momentum routing coefficients have many values other than 0 or ± 1. The identification with circuit coefficients becomes in this case more obscure. There is apparently no choice of internal momenta variables which simplifies the values of the coefficients for all forests, although such simplifications are possible for the terms arising from any one forest [6].

To analyze such a (BPHZ) subtracted amplitude using exactly the graph theoretic approach would present apparently formidable obstacles involving insight into how the singularities cancel in α-space. Instead, we develop a generalization of graph theory.

A graph is usually defined as a collection of lines and vertices together with an incidence matrix which determines how the lines are to be connected. In the new version, the basic quantities are the lines and their momentum routing coefficients.

This generalized graph theory is then used to obtain formulas for those parameter functions which are important to the renormalized amplitudes.

The results of this investigation have already proven useful in analyzing point splitting (i.e.: Fourier transforming with respect to internal momenta) as a method of regularization [6]. They have also been used to investigate operator product light cone expansions [7].

1. A Generalized Graph Theory

We suppose that, in momentum space, the amplitude terms which are to be analyzed are of a form similar to those of the forest formula of BPHZ.

Definition 1.0. A *forest term* is a rational function of internal and external momenta in which the factors of the denominator are Feynman propagators, of the form given by Eq. (1.1), whose line momenta are linear combinations of some set of loop momenta and of external momenta.

The forest formula of BPHZ is a linear combination of forest terms. Any linear combination of forest terms may itself be considered as one such forest term by making a common denominator. The ℓ-th factor of this denominator is of the form

$$\Delta_\ell = \frac{i}{(p_\ell + q_\ell)^2 - m^2 + i\varepsilon((p_\ell + q_\ell)_E^2 + m^2)}, \tag{1.1}$$

where E is used to indicate a Euclidean metric on the term so subscripted, q_ℓ is a linear function of the external momenta, and p_ℓ is an internal momentum. If we take any linearly independent set of the internal momenta, $\{k_1, k_2, \dots, k_m\}$, that set defines momentum routing coefficients such that

$$p_\ell = \sum_{i=1}^{m} d_\ell^i k_i, \tag{1.2}$$

where we have rewritten the equation of the introduction in order to emphasize that the routing coefficients for a forest term have more generalized values than do those for the unsubtracted amplitude.

Definition 1.1. A *generalized graph* or *g-graph*, $\mathfrak{G}$, is a set of lines to each of which is assigned an m-dimensional vector called the *internal momentum* of that line. The set of vectors so assigned is taken to include a basis for m-dimensional space.

A representation of the internal momentum vectors, in a particular basis, is obtained in the momentum routing coefficients which, for fixed ℓ, form a "routing vector", or internal momentum vector,

$$(d_\ell^1, d_\ell^2, \ldots, d_\ell^m).$$

Definition 1.2. The *g-graph for a Feynman graph* is the g-graph, consisting of one line for each factor (of the form given by Eq. (1.1)) in the denominator of the forest term. The forest term may be the integrand for the amplitude corresponding to the graph or any of the subtraction terms. The momentum routing vectors assigned to the lines are the ones defined by the coefficients in Eq. (1.2).

Remark. The ways in which the lines of a g-graph might be connected remain obscure although we shall have no difficulty in defining circuits.

Definition 1.3. A set of lines of a g-graph is said to be independent if the internal momentum vectors assigned to these lines constitute a linearly independent set.

Remark. We are mainly concerned with g-graphs for a Feynman graph, but develop the theory in a slightly more general form both for convenience and because of mathematical interest.

Definition 1.4. A *generalized chord set* or *g-chord*, $\mathbb{T}^*$, is a maximal independent set of lines of a g-graph.

Definition 1.5. A *generalized tree* or *g-tree*, $\mathbb{T}$, is the complement in the g-graph of a generalized chord set.

Theorem 1.6. *The number of lines of any g-chord of a g-graph for a Feynman graph is always fixed and, for BPHZ, is the same as the number of independent momenta (or circuits) of the Feynman graph.*

Proof. In the case that the forest term is the integrand for the renormalized amplitude, the theorem is obvious since the lines of the Feynman graph are a subset of the lines of the g-graph. More generally it is proven elsewhere [6] that for BPHZ each forest term has as many independent internal line momenta as does the original unsubtracted amplitude. The independence of the line momenta is clearly equivalent to the independence of the lines (cf. Eq. 1.2). For other theories, Theorem 1.6 might have to be reexamined, but, in any case, the number of lines in any g-chord must be the same as the dimension of the space spanned by the momentum routing vectors.

Theorem 1.7. *Each g-chord, $\mathbb{T}^* = \{\ell_1, \ldots, \ell_m\}$, determines a unique set of momentum routing coefficients $\{d_\ell^{\mathbb{C}_1}, \ldots, d_\ell^{\mathbb{C}_m}\}$ such that for any line of the g-graph, $\ell \in \mathfrak{G}$,*

$$p_\ell = \sum_{i=1}^{m} d_\ell^{\mathbb{C}_i} p_{\ell_i} \tag{1.3}$$

Proof. The momenta of $\mathbb{T}^*$ are a basis, and the theorem merely restates this.

Definition 1.8, A *generalized circuit* or *g-circuit*, $\mathbb{C}_i$, is a maximal set of lines of a g-graph such that for some g-chord of the g-graph and for some i, the momentum routing coefficients of Theorem 1.7

$$d_\ell^{\mathbb{C}_i} \neq 0 \text{ for } \ell \in \mathbb{C}_i.$$

Theorem 1.9. *With* $\mathbb{T}^*$ *and* $\mathbb{C}_i$ *related by Theorem* 1.7 *and Definition* 1.8,

$$\mathbb{T}^* \cap \mathbb{C}_i = \{\ell_i\}, \text{ and } d_{\ell_j}^{\mathbb{C}_i} = \delta_j^i.$$

Proof. From Eq. 1.2 and the linear independence of the p_{ℓ_i}, we see that the momentum routing coefficient may be identified with the Kronecker delta. This, with Definition 1.8, implies the theorem.

Definition 1.10. A class of g-circuits, $\{\mathbb{C}_1, \ldots, \mathbb{C}_m\}$, *is called a fundamental set or f-set of g-circuits iff it is a maximal class of g-circuits for which*

$$(\forall k : 1 \leqq k \leqq m)\mathbb{C}_k \nsubseteq \bigcup_{\substack{j: \\ j \neq k}} \mathbb{C}_j.$$

Theorem 1.11. *The g-circuits determined in accordance with Theorem* 1.7 *by a g-chord constitute an f-set.*

Proof. Theorem 1.9 implies that each such circuit uniquely contains one line of the g-chord.

Theorem 1.12. *A g-tree is a maximal set of lines containing no g-circuits.*

Proof. Suppose $\mathbb{T}$ is a g-tree, $\mathbb{C}$ is a g-circuit, and $\mathbb{C} \subseteq \mathbb{T}$. Then $\mathbb{C} \cap \mathbb{T}^* = \varnothing$. In order for $\mathbb{C}$ to be a g-circuit, it must be generated, in the sense of Theorem 1.7 and Definition 1.8, by some other chord set $\mathbb{T}_1^*$. According to Theorem 1.9, $\mathbb{C}$ contains a line of $\mathbb{T}_1^*$. The internal momenta of the lines of $\mathbb{T}^*$ may be expressed in terms of those of $\mathbb{T}_1^*$. If $\mathbb{C} \cap \mathbb{T}^* = \varnothing$, then by Definition 1.8 the internal momenta of $\mathbb{T}^*$ may be expressed in terms of those of $\mathbb{T}_1^* - \mathbb{C}$. Since $\mathbb{C} \cap \mathbb{T}_1^* \neq \varnothing$, we have a contradiction with the fact that the internal momenta of $\mathbb{T}_1^*$ and of $\mathbb{T}^*$ constitute, in each case, a basis. Therefore $\mathbb{C} \cap \mathbb{T}^* \neq \varnothing$, and therefore $\mathbb{C} \nsubseteq \mathbb{T}$.

That $\mathbb{T}$ is a maximal such set is clear since any larger set of lines contains the lines of $\mathbb{T}$ and at least one line of $\mathbb{T}^*$. By Theorem 1.9, this set then contains the entire circuit associated with this line.

Corollary. *Every g-circuit intersects every g-chord in at least one line.*

Theorem 1.13. *Every f-set of g-circuits* $\{\mathbb{C}_1, \ldots, \mathbb{C}_m\}$ *determines one or more g-chords,* $\mathbb{T}^* = \{\ell_1, \ldots, \ell_m\}$, *such that* $(\forall i : 1 \leqq i \leqq m)\mathbb{C}_i \cap \mathbb{T}^* = \{\ell_i\}$. *In turn, the f-set of g-circuits determined, in the sense of Theorem 1.7. and Definition 1.8, by* $\mathbb{T}^*$ *is* $\{\mathbb{C}_1, \ldots, \mathbb{C}_m\}$.

Proof. Choose any line ℓ_i, from each $\mathbb{C}_i$, such that

$$\ell_i \in \mathbb{C}_i - \bigcup_{\substack{j:\\ j\neq i}} \mathbb{C}_j,$$

and define $\mathbb{T}^* = \{\ell_1, \ldots, \ell_m\}$. For each i, let $\mathbb{T}_i^*$ be a g-chord that, in the sense of Theorem 1.7 and Definition 1.8, determines an f-set of g-circuits, $\{\mathbb{C}_{i1}, \ldots, \mathbb{C}_{im}\}$ such that $\mathbb{C}_{ii} = \mathbb{C}_i$. The internal momenta, k_{ℓ_i}, of $\mathbb{T}^*$ may be expressed in terms of the internal momenta, $k_{rj} = p_{\ell_{rj}}$ of $\mathbb{T}_r^* = \{\ell_{r1}, \ldots, \ell_{rm}\}$. As a result, if $\{a_1, \ldots, a_m\}$ is any set of numbers such that

$$\sum_{i=1}^{m} a_i p_{\ell_i} = 0, \tag{1.4}$$

then, for every r, i.e., for every choices of basis lines, $\mathbb{T}_r^*$,

$$0 = \sum_{i=1}^{m} a_i p_{\ell_i} = \sum_{i=1}^{m} a_i \sum_{j=1}^{m} d_{\ell_i}^{\mathbb{C}_{rj}} k_{rj}, \tag{1.5}$$

Since $\mathbb{T}_r^*$ is a g-chord, $\{k_{r1}, \ldots, k_{rm}\}$ is linearly independent. Therefore, Eq. 1.5 implies that for every r and for every j,

$$\sum_{i=1}^{m} a_i d_{\ell_i}^{\mathbb{C}_{rj}} = 0. \tag{1.6}$$

Taking $r = j$ and using our construction of $\mathbb{T}^*$ and the definition of a g-circuit,

$$0 = \sum_{i=1}^{m} a_i d_{\ell_i}^{\mathbb{C}_{jj}} = a_j d_{\ell_j}^{\mathbb{C}_j}, \tag{1.7}$$

since $d_{\ell_i}^{\mathbb{C}_j} = 0$ for $\ell_i \in \mathbb{C}_i - \bigcup_{\substack{r:\\ r\neq i}} \mathbb{C}_r$ unless $i = j$. Finally, by the construction of $\mathbb{T}^*$, $\ell_j \in \mathbb{C}_j$ implies $d_{\ell_j}^{\mathbb{C}_j} \neq 0$ implies $a_j = 0$. Therefore $\{p_{\ell_1}, \ldots, p_{\ell_m}\}$ is linearly independent and this implies $\mathbb{T}^*$ is a g-chord, as stated in the theorem.

We procede to show that the f-set of g-circuits generated by $\mathbb{T}^*$ is the same as the one we started out with. By definition, $\ell \in \mathbb{C}_i$ is equivalent to $d_\ell^{\mathbb{C}_{ii}} \neq 0$. Then, since $\ell_i \in \mathbb{C}_i$,

$$p_{\ell_i} = \sum_{j=1}^{m} d_{\ell_i}^{\mathbb{C}_{ij}} k_{ij} \tag{1.8}$$

implies

$$k_{ii} = (1/d_{\ell_i}^{\mathbb{C}_{ii}})(p_{\ell_i} - \sum_{\substack{j:\\ j\neq i}} d_{\ell_i}^{\mathbb{C}_{ij}} k_{ij}). \tag{1.9}$$

It is now easy to see that $(\mathbb{T}_i^* - \mathbb{C}_i) \cup \{\ell_i\}$ is a g-chord, one of whose f-set of circuits is $\mathbb{C}_{ii} = \mathbb{C}_i$. In fact, for any line, ℓ, of the g-graph, Eq. 1.9 implies

$$p_\ell = \frac{d_\ell^{\mathbb{C}_{ii}}}{d_{\ell_i}^{\mathbb{C}_{ii}}} p_{\ell_i} + \sum_{\substack{j:\\ j\neq i}} \left(d_\ell^{\mathbb{C}_{ij}} - d_\ell^{\mathbb{C}_{ii}} \frac{d_{\ell_i}^{\mathbb{C}_{ij}}}{d_{\ell_i}^{\mathbb{C}_{ii}}} \right) k_{ij}, \tag{1.10}$$

and it is clear that the coefficient of p_{ℓ_i} vanishes iff $d_\ell^{\mathbb{C}_{ii}}$ vanishes, and therefore $\mathbb{C}_{ii}$ is one of the g-circuits of this new g-chord. We re-label $(\mathbb{T}_i^* - \mathbb{C}_i) \cup \{\ell_i\}$ as $\mathbb{T}_i^*$. We also re-label the momentum routing coefficients so that $d_\ell^{\mathbb{C}_{ij}}$ is the coefficient of k_{ij} in p_ℓ with, by the re-definition of $\mathbb{T}_i^*, k_{ii} = p_{\ell_i}$. Then, if we take $d_\ell^{\mathbb{C}_i}$ to be the momentum routing coefficients generated by $\mathbb{T}^* = \{\ell_1, \ell_2, \ldots, \ell_m\}$, where perhaps $d_\ell^{\mathbb{C}_i} \neq d_\ell^{\mathbb{C}_{ii}}$, we obtain

$$0 = p_\ell - p_\ell = (d_\ell^{\mathbb{C}_i} - d_\ell^{\mathbb{C}_{ii}}) p_{\ell_i} + \sum_{\substack{j: \\ j \neq i}} (d_\ell^{\mathbb{C}_j} p_{\ell_j} - d_\ell^{\mathbb{C}_{ij}} k_{ij}). \tag{1.11}$$

Since the k_{ij}, for fixed i, constitute a basis, the p_{ℓ_j} may be expressed in terms of them. For $r \neq i$,

$$p_{\ell_r} = \sum_{j=1}^{m} d_{\ell_r}^{\mathbb{C}_{ij}} k_{ij} = \sum_{\substack{j: \\ j \neq i}} d_{\ell_r}^{\mathbb{C}_{ij}} k_{ij}, \tag{1.12}$$

where the last equation follows from $\ell_r \notin \mathbb{C}_{ii}$ for $r \neq i$, as required by our construction of $\mathbb{T}^*$. The linear independence of the k_{ij} with fixed i and $p_{\ell_i} = k_{ii}$ then requires that Eq. 1.11 can not be true unless $(\forall \ell) d_\ell^{\mathbb{C}_i} = d_\ell^{\mathbb{C}_{ii}}$, which requires that $\mathbb{C}_i = \mathbb{C}_{ii}$, where the first set is generated $\mathbb{T}^*$ and the second is the original g-circuit generated by $\mathbb{T}_i^*$. This completes the proof of Theorem 1.13.

At this point we can see that our g-graphs have structural characteristics quite analogous to those of graphs. There remains a question of possible mathematical interest. To what extent can knowledge of the circuits be used to determine an incidence matrix? In other words, can g-graphs be represented as graphs?

We will not further pursue these questions here, but choose instead to limit our attention to an application: the generalization of the earlier results [1] for the parametric formulas associated with the Feynman integral.

2. Parametric Formulas

We recall from Definition 1.0 that a forest term may comprise any of the subtraction terms for an amplitude or even the entire renormalized amplitude integrand in momentum space. In any case, the g-graph for the Feynman graph (Definition 1.2) is generated by combining terms so that there is one denominator.

When the resulting integrand is reexpressed as an integral over the Feynman parameters and is then integrated over the space of internal momenta, the behaviour of the final α-space (i.e., Feynman parameter space) integrand is determined by the properties of a certain matrix [1, 2, 4, 6], A, whose elements are

$$A_{ij} = \sum_{\ell \in \mathfrak{G}} d_\ell^{\mathbb{C}_i} \alpha_\ell d_\ell^{\mathbb{C}_j}. \tag{2.1}$$

The following theorem was obtained by Lowenstein [8].

Theorem 2.1. *If A is given by Eq. 2.1 and $\mathscr{T}^*(\mathfrak{G})$ is the class of all g-chords in the g-graph $\mathfrak{G}$, and $d_{\mathbb{T}^*}^2$ are certain positive definite numbers defined by Eq. 2.4,* *then*

$$\det A = \sum_{\substack{\mathbb{T}^*: \\ \mathbb{T}^* \in \mathscr{T}^*(\mathfrak{G})}} d^2_{\mathbb{T}^*} \prod_{\in \mathbb{T}^*} \alpha_\ell .$$

Proof. Using the elementary properties of determinants, with $L =$ the number of lines of $\mathfrak{G}, m =$ the number of loops of Γ, and, for convenience, setting

$$d_\ell^{\mathbb{C}_i} = d_\ell^i,$$

we obtain

$$\begin{aligned} \det A &= \sum_{\ell_1=1}^{L} \sum_{\ell_2=1}^{L} \cdots \sum_{\ell_n=1}^{L} |d^i_{\ell_i} \alpha_{\ell_i} d^j_{\ell_i}| \\ &= \sum_{\ell_1=1}^{L} \cdots \sum_{\ell_m=1}^{L} \left(\prod_{r=1}^{m} \alpha_{\ell_r} d^r_{\ell_r} \right) |d^j_{\ell_i}|. \end{aligned} \tag{2.2}$$

Clearly those terms which involve linearly dependent set of lines vanish. Therefore,

$$\det A = \sum_{\substack{\mathbb{T}^*: \\ \mathbb{T}^* \in \mathscr{T}^*(\mathfrak{G})}} \left(\prod_{\ell \in \mathbb{T}^*} \alpha_\ell \right) d^2_{\mathbb{T}^*} \tag{2.3}$$

with

$$\begin{aligned} d^2_{\mathbb{T}^*} &= \sum_{\ell_1 \in \mathbb{T}^*} \cdots \sum_{\ell_2 \in \mathbb{T}^*} |d^i_{\ell_i} d^j_{\ell_i}| \\ &= \left| \sum_{\ell_i \in \mathbb{T}^*} d^i_{\ell_i} d^j_{\ell_i} \right| \\ &= (\det(d^i_{\ell_j}))^2 > 0. \end{aligned}$$

The last inequality follows from the fact that $\mathbb{T}^*$ is a g-chord, i.e., a linearly independent set of lines. This completes the proof of Theorem 2.1.

Definition 2.2. For any $\mathbb{B}$ such that $\mathbb{B} \subseteq \mathfrak{G}$, the *reduced g-graph* $\mathfrak{G}/\mathbb{B}$, is defined as $\mathfrak{G} - \mathbb{B}$, with it understood that all momentum routing coefficients remain the same, except for the deletion of the entries whose subscripts are lines of $\mathbb{B}$; the momentum assignments remain unchanged.

Theorem 2.3. *Let $\mathbb{C}_1$ be the g-circuit for which the momentum routing coefficients are $d_\ell^{\mathbb{C}_1} = d_\ell^1$. Let $\alpha(\mathbb{C}) = 0$ mean that $\ell \in \mathbb{C}$ implies $\alpha = 0$. Then the cofactors of A obey*

$$[A]^{ij}_{\alpha(\mathbb{C}_1)=0} = \sum_{\substack{\mathbb{T}^*: \\ \mathbb{T}^* \in \mathscr{T}^*(\mathfrak{G}/\mathbb{C}_1)}} d^2_{\mathbb{T}^*} \prod_{\ell \in \mathbb{T}^*} \alpha_\ell \delta_{i1} \delta_{j1},$$

where $d^2_{\mathbb{T}^*}$ is a positive function of the momentum routing coefficients.

Proof. If either $i \neq 1$ or $j \neq 1$, then one of the rows or columns of the matrix obtained from A by deleting the i-th column and the j-th row has elements

$$\sum_{\ell \in \mathfrak{G}} d_\ell^1 \alpha_\ell d_\ell^j .$$

For $\ell \notin \mathbb{C}_1, d_\ell^1 = 0$. The remaining terms vanish if $\alpha(\mathbb{C}_1) = 0$. The theorem for this case then follows directly.

Otherwise, setting $\alpha(\mathbb{C}_1) = 0$ has the same effect as deleting all the lines of $\mathbb{C}_1$. To form this cofactor, we also delete the first row and first column of A. We are then left with the A matrix for $\mathfrak{G}/\mathbb{C}_1$. Theorem 2.1 now gives the desired result, and Eq. 2.4, identifies $d_{\mathbb{T}^*}^2$. This completes the proof of Theorem 2.3.

Theorem 2.4. *Let "splitting" parameters $\xi_1, \xi_2, \ldots, \xi_{L(\mathfrak{G})}$ be assigned to the lines of $\mathfrak{G}$. Then*

$$\sum_{\substack{i,j\\=1}}^{m} (\xi d^i) A_{ij}^{-1} (\xi d^j) = \frac{1}{\det A} \sum_{\substack{\mathbb{C}:\\ \mathbb{C} \in \mathscr{C}(\mathfrak{G})}} \left\{ \frac{(\xi d^{\mathbb{C}})^2}{(\det T_{\mathbb{C}})^2} \sum_{\substack{\mathbb{T}^*:\\ \mathbb{T}^* \in \mathscr{T}^*(\mathfrak{G}/\mathbb{C})}} d_{\mathbb{T}^*}^2 \left[\prod_{\ell \in \mathbb{T}^*} \alpha_\ell \right] \right\},$$

where line indices are suppressed to indicate the usual scalar or dot product, $\mathscr{C}$ is the class of all g-circuits of $\mathfrak{G}$, $d_{\mathbb{T}^*}^2$ is positive definite and is defined by Eq. 2.4, and $T_{\mathbb{C}}$ is the non-singular matrix which changes the basis from the one of the original g-chord to one which includes $\mathbb{C}$ as the first of the corresponding f-set of g-circuits.

Remark 1. The formulas involving splitting parameters are important for understanding the effects of numerators in forest terms. This is because polynomials in the internal momenta may be replaced by derivatives with respect to splitting parameters [2, 6]. In this case the splitting parameters are four component vectors, the result of the dot product with respect to the suppressed line index is a four vector and four vector products are computed according to Euclidean or Minkowskian metric, as may be appropriate.

Remark 2. The special case of assigning splitting parameters to Γ only is easily recovered if the lines of Γ are a subset of the lines of $\mathfrak{G}$. One merely assigns the other parameters the value of zero.

Remark 3. For convenience, we will often refer to the lines and their internal momenta interchangeably.

Proof of Theorem 2.4. The theorem is the generalized version of the result found in Nakanishi [1], and is proven in virtually the same way.

Let $A_{\mathbb{T}^*}$ be the A matrix generated by using the internal momenta of the lines of $\mathbb{T}^*$ as a basis. Then

$$(A_{\mathbb{T}^*})_{ij} = \sum_{\ell=1}^{L(\mathfrak{G})} d_\ell^{\mathbb{C}_i} \alpha_\ell d_\ell^{\mathbb{C}_j}, \tag{2.4}$$

with the g-circuits $\mathbb{C}_i$ determined by Theorem 1.7. Let the different sets of momentum routing coefficients be related by the transformation $T_{\mathbb{T}^*}$:

$$d_\ell^{\mathbb{C}_i} = \sum_{j=1}^{m(\mathfrak{G})} (T_{\mathbb{T}^*})_{ij} d_\ell^j, \tag{2.5}$$

where d_ℓ^j are the momentum routing coefficients in the initially chosen basis.

Eqs. 2.4 and 2.5 imply

$$A_{\mathbb{T}^*} = T_{\mathbb{T}^*} A \tilde{T}_{\mathbb{T}^*}. \tag{2.6}$$

Therefore

$$A_{\mathbb{T}^*}^{-1} = \tilde{T}_{\mathbb{T}^*}^{-1} A^{-1} T_{\mathbb{T}^*}^{-1}, \tag{2.7}$$

and this implies, in an obvious notation,

$$(\widetilde{\xi d}) A^{-1} (\xi d) = (\widetilde{\xi d^{\mathbb{C}}}) \tilde{T}_{\mathbb{T}^*}^{-1} A^{-1} T_{\mathbb{T}^*}^{-1} (\xi d^{\mathbb{C}}). \tag{2.8}$$

Eq. 2.6 implies

$$\det A_{\mathbb{T}^*} = (\det T_{\mathbb{T}^*})^2 \det A, \tag{2.9}$$

and det $T_{\mathbb{T}^*} \neq 0$ because $T_{\mathbb{T}^*}$ is a change of basis.

By combining Eqs. 2.8 and 2.9 with Theorem 2.3, we obtain

$$\sum_{\substack{i,j\\=1}}^{m} (\xi d^i)[A]^{ij}(\xi d^j) = (\det T_{\mathbb{T}^*}^{-2}) \sum_{\substack{i,j\\=1}}^{m} (\xi d^{\mathbb{C}_i})[A_{\mathbb{T}^*}]^{ij}(\xi d^{\mathbb{C}_j}) \tag{2.10}$$

implies

$$\left.\sum_{\substack{i,j\\=1}}^{m} (\xi d^i)[A]^{ij}(\xi d^j)\right|_{\alpha(\mathbb{C})=0} = (\det T_{\mathbb{T}^*}^{-2})(\xi d^{\mathbb{C}})^2 \sum_{\substack{\mathbb{T}_1^*:\\ \mathbb{T}_1^* \in \mathscr{T}^*(\mathfrak{G}/\mathbb{C})}} \left[d_{\mathbb{T}_1^*}^2 \left(\prod_{\ell \in \mathbb{T}_1^*} \alpha_\ell \right) \right] \tag{2.11}$$

where $\mathbb{T}^*$ is any particular g-chord which contains $\mathbb{C}$ as the first of its f-set of g-circuits, and $d_\ell^{\mathbb{C}}$ are the momentum routing coefficients in this basis. We choose one such g-chord for every g-circuit in $\mathfrak{G}$, and we define $T_{\mathbb{T}^*} = T_{\mathbb{C}}$ for this g-chord. We then examine

$$R = \sum_{\substack{i,j\\=1}}^{m} (\xi d^i)[A]^{ij}(\xi d^j) - \sum_{\substack{\mathbb{C}:\\ \mathbb{C} \in \mathscr{C}(\mathfrak{G})}} \frac{(\xi d^{\mathbb{C}})^2}{(\det T_{\mathbb{C}})^2} \sum_{\substack{\mathbb{T}_1^*:\\ \mathbb{T}_1^* \in \mathscr{T}^*(\mathfrak{G}/\mathbb{C})}} \left[d_{\mathbb{T}_1^*}^2 \left(\prod_{\ell \in \mathbb{T}_1^*} \alpha_\ell \right) \right]. \tag{2.12}$$

Suppose $\mathbb{C}'$ and $\mathbb{C}$ are any two different g-circuits in $\mathfrak{G}$. We will show that $\mathbb{C}' - \mathbb{C}$ is a g-circuit in $\mathfrak{G}/\mathbb{C}$. In fact, whatever g-chord generates $\mathbb{C}'$ in $\mathfrak{G}$, the lines of $\mathbb{C}' - \mathbb{C}$ will be the only lines of $\mathfrak{G}/\mathbb{C} = \mathfrak{G} - \mathbb{C}$ at which $d_\ell^{\mathbb{C}'}$ does not vanish, so that $\mathbb{C}' - \mathbb{C}$ does indeed seem to be a g-circuit in $\mathfrak{G}/\mathbb{C}$. The only question is whether there is a chord set in $\mathfrak{G}/\mathbb{C}$ that generates $\mathbb{C}' - \mathbb{C}$. If $\mathbb{T}^*$ generates $\mathbb{C}'$ in $\mathfrak{G}$, and $\mathbb{T}^* \cap \mathbb{C}' \subseteq \mathbb{C}$, then $\mathbb{T}^* - \mathbb{C}$ is a g-chord in $\mathfrak{G}/\mathbb{C}$, since $\mathbb{T}^*$ and therefore $\mathbb{T}^* - \mathbb{C}$ consists of independent lines. This set provides the same basis for internal line momenta in $\mathfrak{G}/\mathbb{C}$ as it did in $\mathfrak{G}$. Therefore, in this case $\mathbb{T}^* - \mathbb{C}$ is a g-chord of $\mathfrak{G}/\mathbb{C}$, one of whose g-circuits is $\mathbb{C}' - \mathbb{C}$.

In any case, Theorem 1.13 insures that $\mathbb{C}'$ and $\mathbb{C}$ can be considered the first two of an f-set of circuits generated by some g-chord which has one member in $\mathbb{C}' - \mathbb{C}$. We take $\mathbb{T}^*$ of Eq. 2.11 to be exactly this g-chord, and apply the agrument of the preceding paragraph. Eqs. 2.11 and 2.12 then imply

$$R\Big|_{\alpha(\mathbb{C}')=0} = \sum_{\substack{\mathbb{C}:\\ \mathbb{C}\in\mathscr{C}(\mathfrak{G})\\ \text{and } \mathbb{C}\neq\mathbb{C}'}} \frac{(\xi d^{\mathbb{C}})^2}{(\det T_{\mathbb{C}})^2} \sum_{\substack{\mathbb{T}_1^*:\\ \mathbb{T}_1^*\in\mathscr{T}^*(\mathfrak{G}/\mathbb{C})}} d^2_{\mathbb{T}_1^*}\left(\prod_{\ell\in\mathbb{T}_1^*}\alpha_\ell\right)\Big|_{\alpha(\mathbb{C}')=0} \tag{2.13}$$

With our choice of $\mathbb{T}^*$ to supply the basis in $\mathfrak{G}$, it is now clear, by the previous argument, that $\mathbb{C}' - \mathbb{C}$ is a circuit in $\mathfrak{G}/\mathbb{C}$. We then conclude from the corollary to Theorem 1.12 that $\mathbb{C}' - \mathbb{C}$ intersects every chord set of $\mathfrak{G}/\mathbb{C}$. Therefore

$$R\,|_{\alpha(\mathbb{C}')=0} = 0 \tag{2.14}$$

holds for any g-circuit, $\mathbb{C}'$, of $\mathfrak{G}$.

Suppose that R is not identically zero. Then R consists of terms of the form

$$c\prod_{i=1}^{m-1}\alpha_{\ell_i}$$

where c is independent of α, and where $I = \{\ell_1, \ell_2, \dots, \ell_{m-1}\}$ is not a g-chord in $\mathfrak{G}$ because I has at most $m-1$ lines, while every g-chord has m lines.

We take any maximal set of independent lines of I and complete this with the lines of $\mathfrak{G}$ to form some g-chord, $\mathbb{T}^*$, in the usual way that we complete an independent set to make a basis. By this construction, $I - \mathbb{T}^*$ consists of lines whose internal momenta way be expressed as linear combinations of those of $\mathbb{T}^*\cap I$. Suppose the f-set of g-circuits generated by $\mathbb{T}^*$ is $\{\mathbb{C}_1, \mathbb{C}_2, \dots, \mathbb{C}_m\}$, with the first r of these, in the order listed, generated by the lines of $\mathbb{T}^*\cap I$. Since I has at most $m-1$ independent lines, it must be that $r \leqq m-1$. On the other hand, the lines of $I - \mathbb{T}^*$ are all in one or more of the first r g-circuits, and not in any of the remaining $m-r$ g-circuits. Combining this result with Theorem 1.9, we conclude that the last $m-r$ g-circuits are all entirely contained in $I^* = \mathfrak{G} - I$. Therefore there are $m-r$, that is, at least one g-circuit $\mathbb{C}\subseteq I^*$. In Eq. 2.14, we take $\mathbb{C}'$ to be the same as the g-circuit, $\mathbb{C}$, of the present discussion. But Eq. 2.14 then requires that the constant, c, of the typical term in R vanish, and this requires that R identically vanish. This completes the proof of Theorem 2.4.

Theorem 2.5. *If the matrix A is defined by Eq. 2.1, and the momentum routing coefficients d^i_ℓ, used to define A are those for which $\mathbb{T}^* = \{1, 2, \dots, m\}$, by labeling, if necessary, then*

$$A^{-1}_{ij} = \frac{1}{\det A}\sum_{\substack{\mathbb{C}:\\ \mathbb{C}\in\mathscr{C}(\mathfrak{G})}} \frac{d_i^{\mathbb{C}} d_j^{\mathbb{C}}}{(\det T_{\mathbb{C}})^2} \sum_{\substack{\mathbb{T}^*:\\ \mathbb{T}^*\in\mathscr{T}^*(\mathfrak{G}/\mathbb{C})}} \left[d^2_{\mathbb{T}^*}\prod_{\ell\in\mathbb{T}^*}\alpha_\ell\right].$$

Proof. We define $\chi^r_\ell = \delta^r_\ell$, and use the formula of Theorem 2.4 to evaluate

$$4A^{-1}_{ij} = \sum_{\substack{r,s\\=1}}^{m}\left[((\chi^i+\chi^j)d^r)A^{-1}_{rs}((\chi^i+\chi^j)d^s) - ((\chi^i-\chi^j)d^r)A^{-1}_{rs}((\chi^i-\chi^j)d^s)\right].$$

We note that

$$\chi^i d^{\mathbb{C}} = d_i^{\mathbb{C}}, \text{ and}$$

$$\tfrac{1}{4}(d_i^{\mathbb{C}} + d_j^{\mathbb{C}})^2 - (d_i^{\mathbb{C}} - d_j^{\mathbb{C}})^2 = d_i^{\mathbb{C}} d_j^{\mathbb{C}}.$$

This completes the proof of Theorem 2.5.

Readers who are interested in applying these formulas are encouraged to consult the articles by Zimmermann [3] for a clear review of Taylor subtractive procedures.

Using these formulas, it is possible to prove that point splitting provides a method of regularization for which the regulator and propagator epsilon limits may be interchanged. The author hopes to publish his research on this subject.

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