Commun. math. Phys. 64, 279—298 (1979)

Communications in
Mathematical
Physics

Dilation Analyticity in Constant Electric Field

I. The Two Body Problem*

Ira W. Herbst

Department of Mathematics, University of Virginia, Charlottesville, Virginia 22903, USA

Abstract. The resolvent of the operator $H_0(\varepsilon,\theta) = -\Delta e^{-2\theta} + \varepsilon x_1 e^{\theta}$ is not analytic in θ for θ in a neighborhood of a real point, if the electric field ε is nonzero. (One manifestation of this singular behavior is that for $0 < |\mathrm{Im}\,\theta| < \pi/3$, $H_0(\varepsilon,\theta)$ has no spectrum in the finite plane.) Nevertheless it is shown that the techniques of dilation analyticity still can be used to discuss the long-lived states (resonances) of a system described by a Hamiltonian of the form $H = -\Delta + \varepsilon x_1 + V(x)$.

I. Introduction

It is interesting that two of the first problems which arose in the early days of quantum mechanics, the Stark and Zeeman effects, have until recently remained largely unstudied from a mathematical point of view (and to some extent, for large fields at least, also from a physical point of view). Notable exceptions are contained in the work of Titchmarsh [33] and Riddell [29] on the Stark effect in hydrogen. More recent rigorous work on the Zeeman effect can be found in [6–10, 24] and on the Stark effect in [5, 16, 17, 34], however many interesting questions remain to be answered.

It is the purpose of this paper to discuss the long-lived states or resonances associated with systems described by Hamiltonians of the form $H = -\Delta + \varepsilon x_1 + V$ in $L^2(\mathbb{R}^n)$ (Stark Hamiltonians), where in this paper V is a multiplication operator which in some sense vanishes at ∞. In addition we wish to lay the groundwork for a study of the N-body Stark problem (e.g. atomic systems with N electrons). The resonances we will discuss are not solely associated with the operator H [21, 22, 23]. This can be understood by noting the fact that for a large class of potentials, V, (including $V(x) = -Z/|x|$ in $L^2(\mathbb{R}^3)$) H is for each $\varepsilon > 0$ unitarily equivalent to multiplication by x_1 [17]:

$$H = U x_1 U^{-1}$$

---

* Research supported by NSF Grant No. MCS 78-00101

0010-3616/79/0064/0279/$04.00

Thus the spectral analysis of H provides no information about so called resonances.

In [5], the concept of translation analyticity was introduced, modelled after the dilation analytic framework pioneered by Aguilar, Balslev and Combes [3, 11] (which was in turn designed to handle the $\varepsilon=0$ case). The idea is to consider potentials, $V(x_1, x_\perp)$, which are analytic in a strip $|\mathrm{Im}\, x_1|<a$ and such that (with

$$V^\lambda(x_1, x_\perp)=V(x_1+\lambda, x_\perp)) \quad \text{the operator} \quad V^\lambda(-\Delta+\varepsilon x_1+i)^{-1}$$

is compact in $|\mathrm{Im}\,\lambda|<a$. Then if $\varepsilon>0$ and $\mathrm{Im}\,\lambda>0$, the spectrum of

$$H_\lambda=-\Delta+\varepsilon x_1+\varepsilon\lambda+V^\lambda$$

consists of the line $z=\varepsilon\lambda+t$, $t\in\mathbb{R}$ along with discrete spectrum in the region $0\leqq \mathrm{Im}\, z<\varepsilon \mathrm{Im}\,\lambda$. The non-real eigenvalues of H_λ in the latter region are interpretable as resonances associated with the system described by H (the imaginary part being proportional to the inverse of the "lifetime" of the resonant state). The position of an eigenvalue E of H_λ turns out to be independent of λ as long as $\mathrm{Im}\, E<\varepsilon \mathrm{Im}\,\lambda$. For a discussion of the connection between these eigenvalues and lifetimes, at least in the case $\varepsilon=0$, see [30].

While translation analyticity seems appealing from a mathematical point of view, the method suffers from two defects. The first is a pragmatic physical one: the Coulomb potential is not translation analytic. From a computational point of view this is not very serious since if one replaces the point nuclear charge by say a gaussian charge density $\varrho(x)$, the Coulomb potential $|x|^{-1}$ transforms to $\varrho*|x|^{-1}$ which *is* translation analytic. The resonance eigenvalues should converge as $\varrho\to\delta$ (in fact they do as the results of Theorem III.5 show.) Nevertheless, this situation is still not entirely satisfactory. The second defect is a much more serious one. One would like to study the behavior of the eigenvalues of H_λ as $\varepsilon\downarrow 0$ and prove that they in fact converge to those of $-\Delta+V$. Here one is severely hampered by the dependence of the essential spectrum of H_λ on ε. As $\varepsilon\downarrow 0$ this spectrum converges downward to the real axis leaving very little room for the analysis of resonances.

Because of the singular nature of the perturbation εx_1, the dilation group does not act in a smooth way on the resolvent of H even when V is dilation analytic. Thus the idea of dilation analyticity was ignored in [5]. It was the detailed computer calculations of Stark resonance energies, using the complex scaling technique, presented by W. Reinhardt in a colloquium at the University of Virginia in the Fall of 1977, which motivated the research which led to this paper. (For the calculational aspects of the complex scaling technique, see [12, 13, 28] and the review article [14]. In the latter reference, a model calculation of resonance positions using the translation analytic technique is presented.) Independently and approximately simultaneously, Graffi and Grecchi [16] have analyzed the problem with $V(x)=-1/|x|$ using the fact that in this case, Schrödinger's equation is separable in parabolic coordinates. Our results overlap theirs in the case of hydrogen although the methods are very different.

The point of this paper is to demonstrate the applicability of complex scaling techniques to the Stark problem and to explore some of the unique properties of the scaled Hamiltonian. In Sect. II we consider the operator $-\Delta+\alpha x_1$ for complex

α, and in Sect. III the full problem is discussed. Here it is shown that resonances actually exist and converge as $\varepsilon \to 0$ to the eigenvalues of $-\Delta + V$. In Section IV some unsolved problems and speculations are discussed.

This is the first of a pair of papers, the second co-authored with B. Simon [19], will consider the N-body Stark problem along with analyticity and summability properties of the complex eigenvalues as a function of the electric field. (Some of the results of the present work and that in [19] have been announced in [18].)

II. The Operator $-\Delta + \alpha x_1$ for Complex α

Because of the unusual properties of the Hamiltonian operator for a particle in a complex electric field we will proceed cautiously. Define

$$h(\alpha) = -\Delta + \alpha x_1, \qquad \mathscr{D}(h(\alpha)) = \mathscr{S}(\mathbb{R}^n).$$

The object of this section is to describe the closure of $h(\alpha)$ in $L^2(\mathbb{R}^n, d^n x)$. Most of the important results are summarized in Theorems II.1, II.3, and II.5. We remind the reader that if α is real, $h(\alpha)$ is essentially self-adjoint and if in addition $\alpha \neq 0$ its closure has spectrum $(-\infty, \infty)$.

Theorem II.1. *Fix* α *with* $\operatorname{Im} \alpha \neq 0$.

a) *The numerical range,* $W(h(\alpha))$, *is the open half-plane*

$$S_\alpha = \{z : \operatorname{Re} z > (\operatorname{Re} \alpha / \operatorname{Im} \alpha) \operatorname{Im} z\}.$$

The operator $h(\alpha)$ *is closable. Denote its closure by* $\bar{h}(\alpha)$.

b) *The spectrum of* $\bar{h}(\alpha)$ *is empty.*

c) $\bar{h}(\alpha)^* = \bar{h}(\bar{\alpha})$.

At this stage we will only prove part a) of the theorem:

The numerical range of $h(\alpha)$ is clearly contained in S_α. (This has been noted previously in [13].) The fact that $W(h(\alpha)) = S_\alpha$ follows from a consideration of $(\psi, h(\alpha)\psi)$ for particular ψ in $\mathscr{S}$. First take $(\psi, x_1 \psi) = 0$ to show that $W(h(\alpha))$ contains $(0, \infty)$. Translating these $\psi(\psi(x) \to \psi(x-a))$ gives all of S_α. We refer to Kato [23] for the proof that an operator with numerical range in a half-plane is closable. However a shorter proof is given after Proposition II.2, along with the remainder of the proof of Theorem II.1. We mention that it does not follow from (a) of the theorem that $\sigma(\bar{h}(\alpha)) \subseteqq \bar{S}_\alpha$. For example, consider a closed symmetric operator A with both deficiency indices non-zero. Then $W(A) \subseteqq \mathbb{R}$ but $\sigma(A)$ is all of $\mathbb{C}$.

We remark at this point that by extending the graph of $\bar{h}(\alpha)$ one dimension at a time one can always find closed extensions of $h(\alpha)$ which have non-empty spectrum. However if a closed extension $h'(\alpha)$ of $h(\alpha)$ satisfies condition c) above, we must have $h'(\alpha) = \bar{h}(\alpha)$.

To prove Theorem II.1 and the remainder of the theorems of this section we will exploit a combination of two techniques. The first is based on an explicit formula for the semigroup generated by $i\alpha^{-1}\bar{h}(\alpha)$ and the second on a quadratic estimate. The latter is powerful enough to prove all of the theorems of this section, however the proofs are not as slick as those based on the semigroup formalism and hence we will rely heavily on this technique. The semigroup technique also plays an important role in [19] where a Weinberg-Van Winter equation for semigroups is introduced.

It is shown in [5] that for real α,

$$\exp(-it(p^2+\alpha x_1))=\exp(-it\alpha x_1)\exp(-itp^2+it^2p_1\alpha-i\alpha^2t^3/3)$$

where here and in the following we use the notation $p_j=-i\partial_j$, $p^2=\sum_{j=1}^{n} p_j^2=-\Delta$. The change of variable $t\alpha\to t$ motivates the definition (for $\operatorname{Im}\alpha>0$):

$$P_t=\exp(-itx_1)\exp(-it\alpha^{-1}(p^2-tp_1+t^2/3)); \quad t\geqq 0. \tag{2.1}$$

The case $\operatorname{Im}\alpha<0$ is handled by changing the sign of t.

Proposition II.2. *Suppose* $\operatorname{Im}\alpha>0$. *Then* P_t *is a strongly continuous semigroup satisfying* $\|P_t\|=\exp(-D(\alpha)t^3)$ *with* $D(\alpha)=\operatorname{Im}\alpha/12|\alpha|^2$. *If we define the generator of* P_t *by setting* $P_t=\exp(-tL(\alpha))$, *we have* $L(\alpha)=i\alpha^{-1}\bar{h}(\alpha)$.

Proof. Writing $p_1^2-tp_1+t^2/3=(p_1-t/2)^2+t^2/12$ we have

$$\|P_t\|=\sup_{q\in\mathbb{R}^n}|\exp(-it\alpha^{-1}(q_\perp^2+(q_1-t/2)^2+t^2/12))|=\exp(-D(\alpha)t^3))$$

where $q_\perp^2=q_2^2+\ldots+q_n^2$ and

$$D(\alpha)=\operatorname{Re}(i\alpha^{-1})/12=\operatorname{Im}\alpha/12|\alpha|^2.$$

A straightforward computation using

$$e^{isx_1}f(p_1)e^{-isx_1}=f(p_1-s)$$

gives the semigroup formula $P_tP_s=P_{t+s}$ while the strong continuity can be seen at a glance.

To prove $L(\alpha)=i\alpha^{-1}\bar{h}(\alpha)$ we use a technique now standard in semigroup theory [25, 27]. First note that $P_t:\mathscr{S}(\mathbb{R}^n)\to\mathscr{S}(\mathbb{R}^n)$ and that by a simple computation

$$\lim_{t\downarrow 0} t^{-1}(1-P_t)\psi=i\alpha^{-1}h(\alpha)\psi, \quad \psi\in\mathscr{S}(\mathbb{R}^n).$$

This shows that $L(\alpha)\supseteq i\alpha^{-1}h(\alpha)$. Let $l(\alpha)=i\alpha^{-1}h(\alpha)$. We show that $l(\alpha)+1$ has dense range: Suppose $(\phi,(l(\alpha)+1)\psi)=0$ all $\psi\in\mathscr{S}$. Then $f(t)=(\phi,P_t\psi)$ satisfies (for $\psi\in\mathscr{S}$) $f'(t)=-(\phi,L(\alpha)P_t\psi)=-(\phi,l(\alpha)P_t\psi)$ (since $P_t\psi\in\mathscr{S}$) $=f(t)$. Thus $f(t)=e^t(\phi,\psi)$; but since $\|P_t\|\leqq 1$, we must have $(\phi,\psi)=0$. Thus $\phi=0$. Since $l(\alpha)$ has a closed extension, namely $L(\alpha)$, it is closable and satisfies $\operatorname{Ran}(\bar{l}(\alpha)+1)=L^2(\mathbb{R}^n)$. If $\phi\in\mathscr{D}(L(\alpha))$ we thus have $(L(\alpha)+1)\phi=(\bar{l}(\alpha)+1)\psi$ for some $\psi\in\mathscr{D}(\bar{l}(\alpha))$. But since $L(\alpha)\supseteq\bar{l}(\alpha)$ this gives $(L(\alpha)+1)(\phi-\psi)=0$. But $\sigma(L(\alpha))\subseteq\{z:\operatorname{Re}z\geqq 0\}$ which implies $\phi=\psi$, and hence $\bar{l}(\alpha)\supseteq L(\alpha)$. This completes the proof.

We now finish the proof of Theorem II.1:

The fact that $\|P_t\|=\exp(-D(\alpha)t^3)$ implies that

$$R_T(z)\equiv i\alpha^{-1}\int_0^T P_t e^{i\alpha^{-1}tz}dt$$

has a norm convergent limit as $T\to\infty$ for all z:

$$\lim_{T\to\infty} R_T(z)=R(z).$$

But a short computation shows that for $\psi\in\mathscr{D}(\bar{h}(\alpha))$

$$(\bar{h}(\alpha)-z)R_T(z)\psi=R_T(z)(\bar{h}(\alpha)-z)\psi=\psi-e^{-i\alpha^{-1}zT}P_T\psi$$

and thus $(\bar{h}(\alpha)-z)R(z)\psi=R(z)(\bar{h}(\alpha)-z)\psi=\psi$. This implies as expected that $R(z)=(\bar{h}(\alpha)-z)^{-1}$. In particular $\sigma(\bar{h}(\alpha))=\emptyset$. An identical argument with $t\to -t$ proves this result for $\operatorname{Im}\alpha<0$.

To show that $\bar{h}(\alpha)^*=\bar{h}(\bar{\alpha})$, we start with the observation that P_t^* is a strongly continuous contraction semigroup with generator $L(\alpha)^*$. Simple computations show that $-i\bar{\alpha}^{-1}\bar{h}(\alpha)^*=L(\alpha)^*\supseteqq -i\bar{\alpha}^{-1}h(\bar{\alpha})$. An argument identical to the proof of $\bar{l}(\alpha)\supseteqq L(\alpha)$ shows that $L(\alpha)^*=-i\bar{\alpha}^{-1}\bar{h}(\bar{\alpha})$ and thus $\bar{h}(\alpha)^*=\bar{h}(\bar{\alpha})$. This completes the proof of Theorem II.1.

We now go on to determine the domain of $\bar{h}(\alpha)$.

Theorem II.3. *Suppose* $\operatorname{Im}\alpha\neq 0$. *Then*

a) $\mathscr{D}(\bar{h}(\alpha))=\mathscr{D}(-\Delta)\cap\mathscr{D}(x_1)$.

b) *If* $n=1$, *then* $(z-\bar{h}(\alpha))^{-1}$ *is compact.*

We note that b) follows from a) and the closed graph theorem, for by the latter $(p_1^2+|x_1|+1)(z-\bar{h}(\alpha))^{-1}$ is bounded and thus $(z-\bar{h}(\alpha))^{-1}=(p_1^2+|x_1|+1)^{-1}[(p_1^2+|x_1|+1)(z-\bar{h}(\alpha))^{-1}]$ is compact since $(p_1^2+|x_1|+1)^{-1}$ is compact. The proof of part a) rests on the quadratic estimate:

Proposition II.4. *Suppose that* $|\operatorname{Im}\alpha|>0$. *Then there are constants* $c(\alpha)$ *and* $\beta(\alpha)>0$ *so that for all* $\psi\in\mathscr{S}(\mathbb{R}^n)$.

$$\|h(\alpha)\psi\|^2+c(\alpha)\|\psi\|^2\geqq\beta(\alpha)(\|\Delta\psi\|^2+\|x_1\psi\|^2). \tag{2.2}$$

We can take $\beta(\alpha)=(1-|\cos\theta|)/2$ *and*

$$c(\alpha)=(1-|\cos\theta|)^{3/2}|\sin\theta|^{3/2}(3/2)|\alpha|^{4/3}$$

where $\theta=\arg\alpha$.

Proof. The following computations are to be interpreted as taking place on vectors in $\mathscr{S}(\mathbb{R}^n)$ and all operator inequalities as quadratic form inequalities between vectors in $\mathscr{S}(\mathbb{R}^n)$: We compute

$$(p^2+\bar{\alpha}x_1)(p^2+\alpha x_1)=p^4+2p_1\operatorname{Im}\alpha+\operatorname{Re}\alpha(p^2x_1+x_1p^2)+|\alpha|^2x_1^2.$$

The inequality $(p^2\beta^{-1}\pm\beta x_1)^2\geqq 0$ gives with $\beta=|\alpha|^{1/2}$

$$\operatorname{Re}\alpha(p^2x_1+x_1p^2)\geqq -|\alpha^{-1}\operatorname{Re}\alpha|(p^4+|\alpha|^2x_1^2)$$

and thus

$$(p^2+\bar{\alpha}x_1)(p^2+\alpha x_1)\geqq(1-|\alpha^{-1}\operatorname{Re}\alpha|)(p^4+|\alpha|^2x_1^2)+2p_1\operatorname{Im}\alpha.$$

Equation (2.2) follows after we use some of the p^4 to bound the term linear in p_1.

We now complete the proof of Theorem II.3: It is clear that

$$\mathscr{D}(-\Delta)\cap\mathscr{D}(x_1)\subseteqq\mathscr{D}(\bar{h}(\alpha))$$

for if

$$\psi\in\mathscr{D}(-\Delta)\cap\mathscr{D}(x_1), \qquad \psi_t=e^{t\Delta}e^{-tx^2}\psi\in\mathscr{S}(\mathbb{R}^n)$$

for $t>0$ and

$$\|\psi_t-\psi\|+\|\Delta(\psi_t-\psi)\|+\|x_1(\psi_t-\psi)\|\to 0$$

as $t\to 0$ as a small computation shows. The fact that

$$\mathscr{D}(\bar{h}(\alpha))\subseteqq\mathscr{D}(-\Delta)\cap\mathscr{D}(x_1)$$

follows from the quadratic estimate for if $f\in\mathscr{D}(\bar{h}(\alpha))$ there is a sequence $f_m\in\mathscr{S}(\mathbb{R}^n)$ with $\|\bar{h}(\alpha)(f-f_m)\|^2+\|f-f_m\|^2\to 0$. But $\sup\|\bar{h}(\alpha)f_m\|<\infty$ implies

$$\sup(\|\Delta f_m\|+\|x_1 f_m\|)<\infty.$$

Fatou's lemma then gives $f\in\mathscr{D}(-\Delta)\cap\mathscr{D}(x_1)$.

We now go on to analyze the analyticity properties of $(\bar{h}(\alpha)-z)^{-1}$. We denote the distance from $z\in\mathbb{C}$ to $W(h(\alpha))$ by $d(z,\alpha)$. On the positive side we have

Theorem II.5. a) *The resolvent* $(z-\bar{h}(\alpha))^{-1}$ *is jointly analytic in the variables* (z,α) *for* $|\mathrm{Im}\,\alpha|>0$, $z\in\mathbb{C}$.
b) *If* $\mathrm{Im}\,\alpha_0=0$ *and* $\delta>0$ *then*

$$\operatorname*{s-lim}_{\substack{\alpha\to\alpha_0\\ d(z,\alpha)\geqq\delta}}(z-\bar{h}(\alpha))^{-1}=(z-\bar{h}(\alpha_0))^{-1}$$

Proof. Perhaps the simplest direct way of seeing a) is to first note that

$$F_t(\alpha,z)=i\alpha^{-1}P_t e^{-i\alpha^{-1}zt}$$

is analytic in the variables (z,α) for $\mathrm{Im}\,\alpha>0$ and $z\in\mathbb{C}$ and thus can be written as a double Cauchy integral

$$F_t(\alpha,z)=(2\pi i)^{-2}\oint\oint d\beta dw\frac{F_t(\beta,w)}{(\alpha-\beta)(z-w)}.$$

Integrating from 0 to ∞ on t gives the same formula for the resolvent from which the stated analyticity easily follows.

We now prove b). On vectors in $\mathscr{S}(\mathbb{R}^n)$ it is easy to derive the formula

$$(z-\bar{h}(\alpha))^{-1}-(z-\bar{h}(\alpha_0))^{-1}=(\alpha-\alpha_0)(z-\bar{h}(\alpha))^{-1}x_1(z-\bar{h}(\alpha_0))^{-1}.$$

We omit the easy proof which shows that $x_1(z-\bar{h}(\alpha_0))^{-1}(x_1+i)^{-1}$ is bounded (see [17]). This fact and the uniform bound $\|(z-\bar{h}(\alpha))^{-1}\|\leqq 1/d(z,\alpha)$, where $d(z,\alpha)$ is the distance from z to $W(h(\alpha))$, gives convergence on $\mathscr{S}$. The uniform bound converts convergence on a dense set to strong convergence.

On the negative side we have the following results, the last two of which show very vividly that $\bar{h}(\alpha)$ "remembers" its numerical range.

Proposition II.6. a) *If* $\operatorname{Im}\alpha_0=0$ *and z is fixed, then if* $\{\alpha_n\}_{n=1}^{\infty}$ *is any sequence with* $\alpha_n\to\alpha_0$ *and* $\alpha_n\neq\alpha_0$, *the resolvent* $(z-\bar{h}(\alpha_n))^{-1}$ *does not converge in norm to* $(z-\bar{h}(\alpha_0))^{-1}$. *In particular* $(z-\bar{h}(\alpha))^{-1}$ *is not analytic in any neighborhood of* α_0.

b) *Suppose* $\varepsilon>0$ *and* θ *is fixed with* $\theta\neq n\pi$ *and suppose* $z_0\in W(h(\varepsilon e^{i\theta}))$. *Then for* ε *small*

$$\|(z_0-\bar{h}(\varepsilon e^{i\theta}))^{-1}\|\geqq c\exp(\lambda/\varepsilon)$$

for some positive constants c and λ.

c) *Suppose* ε *and* z_0 *are fixed with* $\varepsilon>0$ *and* $\operatorname{Im}z_0<0$. *Then for* $\theta>0$ *and small*

$$\|(z_0-\bar{h}(\varepsilon e^{i\theta}))^{-1}\|\geqq c\exp(\lambda\theta^{-1/2})$$

for some positive constants c and λ.

Remarks. a) It may seem "obvious" that $(z-\bar{h}(\alpha_n))^{-1}$ cannot converge in norm to $(z-\bar{h}(\alpha_0))^{-1}$ as $\alpha_n\to\alpha_0$ with $\operatorname{Im}\alpha_n\neq 0$, for $\bar{h}(\alpha_n)$ has no spectrum (*ie* $(z-\bar{h}(\alpha_n))^{-1}$ is quasinilpotent) while $\bar{h}(\alpha_0)$ has spectrum equal to either $\mathbb{R}$ or $[0,\infty)$. However results of [4, 20] show that there exist sequences of *nilpotent* operators converging in norm to $(z-\bar{h}(\alpha_0))^{-1}$ for any $z\notin\sigma(\bar{h}(\alpha_0))$.

b) The last two results are perhaps surprising because z_0 can be chosen far from the spectrum of the "limiting" operators $\bar{h}(0)=-\Delta$ or $\bar{h}(\varepsilon)=-\Delta+\varepsilon x_1$.

Proof. To prove part a) we first reduce to the case $\operatorname{Im}\alpha_n\neq 0$: If the α_n are real and $\alpha_0=0$ then since $\sigma(\bar{h}(0))=[0,\infty)$ and $\sigma(\bar{h}(\alpha_n))=\mathbb{R}$, $\bar{h}(\alpha_n)$ cannot converge to $\bar{h}(0)$ in norm resolvent sense. If $\alpha_0\neq 0$, by scaling and reflecting if necessary we can take $\alpha_0=1$. By considering

$$U[(p^2+x_1+i)^{-1}-(p^2+\alpha_n x_1+i)^{-1}]U^{-1}$$

with $U=\exp(-ip_1^3/3)$, norm resolvent convergence implies

$$\|(p_\perp^2+x_1+i)^{-1}-((1-\alpha_n)p_1^2+p_\perp^2+\alpha_n x_1+i)^{-1}\|\to 0\,.$$

But if $\chi(x)$ is the characteristic function of a ball of radius 1, $\chi((1-\alpha_n)p_1^2+p^2+\alpha_n x_1+i)^{-1}$ is compact [5] while $\chi(p_\perp^2+x_1+i)^{-1}$ is not and hence the latter cannot be the norm limit of the former.

Now consider the case $\operatorname{Im}\alpha_n\neq 0$ all n. Let ϕ be a fixed nonzero vector in $L^2(\mathbb{R}^{n-1},d^{n-1}x)$ and define $\mathscr{H}_1=\{f\otimes\phi : f\in L^2(\mathbb{R},dx_1)\}$; let P be the orthogonal projection onto $\mathscr{H}_1$. Consider the map $K_\alpha:\mathscr{H}_1\to\mathscr{H}_1$ defined by

$$K_\alpha=P(z-\bar{h}(\alpha))^{-1}1\mathscr{H}_1$$

If $\operatorname{Im}\alpha\neq 0$, then K_α is compact since by Theorem II.3 $(p_1^2+|x_1|+1)K_\alpha$ is bounded. If $(z-\bar{h}(\alpha_n))^{-1}\xrightarrow{\|\cdot\|}(z-\bar{h}(\alpha_0))^{-1}$ then K_{α_0} is also compact. But this is impossible because K_{α_0} commutes with the group of unitaries

$$\begin{aligned} U(a)&=e^{ip_1^3/3\alpha_0}e^{iax_1}e^{-ip_1^3/3\alpha_0} && \alpha_0\neq 0\\ &=e^{iap_1} && \alpha_0\neq 0\,. \end{aligned}$$

(Since $U(a)\xrightarrow{w}0$ as $a\to 0$ while $\|K_{\alpha_0}U(a)\psi\|=\|K_{\alpha_0}\psi\|$, K_{α_0} cannot be compact.)

To prove parts b) and c), we first reduce the problem to a one-dimensional one by noting that if $N^{-1}=\|(p_\perp^2+p_1^2+\alpha x_1-z_0)^{-1}\|$, $|\mathrm{Im}\,\alpha|\neq 0$, then

$$N^{-1}=\inf_\phi \|(p_\perp^2+p_1^2+\alpha x_1-z_0)\phi\|$$

where the infimum is taken over all $\phi\in\mathscr{D}(-\Delta)\cap\mathscr{D}(x_1)$ with $\|\phi\|=1$. Thus

$$\begin{aligned} N^{-1} &\leqq \inf_{\phi\otimes\psi} \|(p_\perp^2+p_1^2+\alpha x_1-z_0)\phi\otimes\psi\| \\ &\leqq \inf_\phi \|(p_1^2+\alpha x_1-z_0)\phi\| + \inf_\psi \|p_\perp^2\psi\| \\ &= \inf_\phi \|(p_1^2+\alpha x_1-z_0)\phi\| = N_1^{-1}, \end{aligned}$$

where $\phi\in L^2(\mathbb{R}, dx_1)$, $\psi\in L^2(\mathbb{R}^{n-1}, d^{n-1}x)$, $\|\phi\|=\|\psi\|=1$, and

$$N_1=\|(p_1^2+\alpha x_1-z_0)^{-1}\|.$$

The estimates in b) and c) are both consequences of the estimate

$$\|(p_1^2+i\beta^{-1}x_1-1)^{-1}\|\geqq c(\mathrm{Re}\,\beta)^{1/4}\exp\left(\frac{2\,\mathrm{Re}\,\beta}{3}\right) \tag{2.3}$$

as $\mathrm{Re}\,\beta\nearrow\infty$. Equation (2.3) is proved by first Fourier transforming the operator $\frac{-d^2}{dx^2}+i\beta^{-1}x-1$ to get the unitarily equivalent operator $x^2-1+\beta^{-1}\frac{d}{dx}$. Let $\phi_0=e^{-\beta(x^3/3-x)}$ and note that $\left(x^2-1+\beta^{-1}\frac{d}{dx}\right)\phi_0=0$. However since $\phi_0\notin L^2$ we use the trial function $f\phi_0$ for suitable f.

We have

$$\begin{aligned} &\left\|\left(x^2-1+\beta^{-1}\frac{d}{dx}\right)f\phi_0\right\|^2/\|f\phi_0\|^2 \\ &\quad=|\beta|^{-2}\int|f'(x)|^2|\phi_0(x)|^2dx/\int|f(x)|^2|\phi_0(x)|^2dx. \end{aligned} \tag{2.4}$$

Choosing $g\in C^\infty(\mathbb{R})$ with $g(x)=0$ if $x\leqq 0$ and $g(x)=1$ if $x\geqq 1$, we take $f(x)=g(x\,\mathrm{Re}\,\beta)$. The integrals in Eq. (2.4) can be estimated with the help of [1] giving Eq. (2.3) with $c>0$.

It remains only to see how b) and c) follow from Eq. (2.3): By translating we see that $p_1^2+\varepsilon e^{i\theta}x_1-z_0$ and $p_1^2+\varepsilon e^{i\theta}x_1-(z_0+a\varepsilon e^{i\theta})$ are unitarily equivalent. By choosing a to be dependent on $\varepsilon e^{i\theta}$, in b) we can choose a so that $z_0+a\varepsilon e^{i\theta}=r_0>0$, where r_0 is fixed, while in c) $r_0=\mathrm{Re}\,z+|\mathrm{Im}\,z|\mathrm{ctn}\,\theta$ can be realized, as a little geometry shows. Complex conjugation shows that we can assume in b) that $\sin\theta>0$. A scale transformation shows that for $r_0>0$

$$\|(p_1^2-r_0+\varepsilon e^{i\theta}x_1)^{-1}\|=r_0^{-1}\|(p_1^2-1+\varepsilon e^{i\theta}r_0^{-3/2}x_1)^{-1}\|.$$

Thus in comparing to Eq. (2.3) we have $\mathrm{Re}\,\beta=\varepsilon^{-1}\sin\theta r_0^{3/2}$. In b), $\mathrm{Re}\,\beta\sim c\varepsilon^{-1}$ while in c) $\mathrm{Re}\,\beta\sim c\theta^{-1/2}$. This gives the result.

III. Resonances

In this section we will be concerned with making sense out of the formal operator

$$H(\varepsilon,\theta)=-\Delta e^{-2\theta}+\varepsilon x_1 e^{\theta}+V(xe^{\theta}) \tag{3.1}$$

for complex θ, and discussing its properties. We will first introduce an assumption concerning V which allows us to make sense out of $H(\varepsilon,\theta)$ for $0<|\mathrm{Im}\,\theta|<\theta_0$ and all real ε and for $\varepsilon=0$ and $0\leqq|\mathrm{Im}\,\theta|<\theta_0$. This hypothesis (see below) will lead to discrete eigenvalues of $H(\varepsilon,\theta)$ which converge to those of $-\Delta+V$ as $\varepsilon\downarrow 0$ (see Theorems III.2 and III.3). We will then make a weak hypothesis which allows us to prove self-adjointness of $H(\varepsilon)=-\Delta+\varepsilon x_1+V$ on $\mathscr{D}(-\Delta+\varepsilon x_1)$ and which further provides a connection between the spectrum of $H(\varepsilon,\theta)$ for $\mathrm{Im}\,\theta\neq 0$ and the singularities in analytically continued matrix elements of $(z-H(\varepsilon))^{-1}$. This will also give further information about $\sigma(H(\varepsilon,\theta))$. (see Theorem III.4). Finally we show in Theorem III.5 that if V is replaced by $\varrho * V$ where ϱ is a normalized Gaussian, the resulting potential is translation analytic [5] and the resulting complex eigenvalues converge to those of $H(\varepsilon,\theta)$ $(\mathrm{Im}\,\theta\neq 0)$ as ϱ converges to a delta function. Thus translation analyticity may be useful in situations where V is not translation analytic.

Our Hilbert space $\mathscr{H}$ will be $L^2(\mathbb{R}^n,d^nx)$ and will also need $\mathscr{H}_{+1}=\mathscr{D}(-\Delta)$ normed with $\|\phi\|_{+1}=\|(-\Delta+1)\phi\|$. We introduce the dilation group

$$U(\theta)f(x)=e^{n\theta/2}f(e^{\theta}x)\,,\qquad \theta\in\mathbb{R}$$

and for a self-adjoint multiplication operator M we use the notation $M(\theta)=U(\theta)MU(-\theta)$.

We make the following standard looking assumption [3, 11] concerning the potential V which we will make use of without further reference:

V is a self-adjoint multiplication operator with $\mathscr{D}(V)\supseteqq\mathscr{D}(-\Delta)$. As a map from $\mathscr{H}_{+1}$ to $\mathscr{H}$, $V(\theta)$ is compact and extends to a compact analytic operator valued function of θ in the strip $|\mathrm{Im}\,\theta|<\theta_0$ for some $\theta_0\in(0,\pi/3]$.

The suitability of this assumption for our problem will become clear in what follows. For $\varepsilon\geqq 0$ we use the notation

$$H_0(\varepsilon,\theta)=e^{-2\theta}\bar{h}(\varepsilon e^{3\theta})=-\Delta e^{-2\theta}+\varepsilon x_1 e^{\theta}$$
$$H_0(\theta)=H_0(0,\theta)\,.$$

From the last section it follows that for $0<|\mathrm{Im}\,\theta|<\pi/3$, $\sigma(H_0(\varepsilon,\theta))=\emptyset$. We use the notation $W(\theta)$ for the numerical range of $H_0(1,\theta)$, and $d(z,\theta)$ for the distance from z to $W(\theta)$.

All of the properties of the operator in (3.1) will follow from an analysis of the operator $K(\varepsilon,\theta;z)=V(\theta)(H_0(\varepsilon,\theta)-z)^{-1}$. This operator is analyzed in the following Proposition. We write $K(\theta;z)\equiv K(0,\theta;z)$.

Proposition III.1. (i) *The operator $K(\varepsilon,\theta;z)$ is compact and jointly analytic in z and θ for $0<|\mathrm{Im}\,\theta|<\theta_0$ and $\varepsilon>0$.*

(ii) $K(\varepsilon,\theta;z)\xrightarrow[\varepsilon\downarrow 0]{\|\cdot\|}K(\theta;z)$ *uniformly for* (θ,z) *in compacts of* $\{(\theta,z):d(z,\theta)>0,0<|\mathrm{Im}\,\theta|<\theta_0\}$.

(iii) $\lim_{d(z,\theta)\to\infty}\|K(\varepsilon,\theta;z)\|=0$ *for each* θ *with* $0<|\mathrm{Im}\,\theta|<\theta_0$.

Proof. We write (for $0<|\mathrm{Im}\,\theta|<\theta_0$)

$$(H_0(\varepsilon,\theta)-z)^{-1}=(2\pi i)^{-2}\oint\oint d\phi dw(H_0(\varepsilon,\phi)-w)^{-1}(w-z)^{-1}(\phi-\theta)^{-1}$$

where the integral is over a suitable product of circles. From the quadratic estimate it follows that $J(\phi,w)\equiv(-\Delta+1)(H_0(\varepsilon,\phi)-w)^{-1}$ is bounded and it is thus easy to justify the formula

$$J(\theta,z)=(2\pi i)^{-2}\oint\oint d\phi dw(w-z)^{-1}(\phi-\theta)^{-1}J(\phi,w).$$

This implies that J is analytic in the set $0<|\mathrm{Im}\,\theta|<\theta_0$, $z\in\mathbb{C}$. Since $V(\theta)(H_0(\varepsilon,\theta)-z)^{-1}=V(\theta)(-\Delta+1)^{-1}J(\theta,z)$, part (i) is proved.

To prove (ii) we let χ_r be multiplication by the characteristic function of the ball $|x|\leqq r$, and first show that for fixed r

$$\chi_r(K(\varepsilon,\theta;z)-K(\theta;z))$$

converges to zero with the stated uniformity. The norm of this operator is less than

$$c\varepsilon\|\chi_r V(\theta)(z-H_0(\theta))^{-1}x_1(z-H_0(\varepsilon,\theta))^{-1}\|. \tag{3.2}$$

Using $[(z-H_0(\theta))^{-1},x_1]=(z-H_0(\theta))^{-1}(-2ie^{-2\theta}p_1)(z-H_0(\theta))^{-1}$ and the fact that $\|(H_0(\varepsilon,\theta)-z)^{-1}\|\leqq(d(z,\theta))^{-1}$ we see that (3.2) is bounded by

$$c\varepsilon\|(1+|x_1|)\chi_r K(\theta;z)\|$$

and thus converges to zero. The proof is complete if we can show that

$$\lim_{r\to\infty}\|(1-\chi_r)(K(\varepsilon,\theta;z)-K(\theta;z))\|=0$$

with the stated uniformity in (θ,z) and in addition, uniformly in ε as $\varepsilon\downarrow 0$. Firstly note that by the quadratic estimate, $\|(-\Delta+1)(z-H_0(\varepsilon,\theta))^{-1}\|$ is uniformly bounded in the stated region and thus it suffices to prove

$$\lim_{r\to\infty}\|(1-\chi_r)V(\theta)(-\Delta+1)^{-1}\|=0. \tag{3.3}$$

Denote by B, the unit sphere in $\mathscr{H}$, and suppose C is a compact subset of $\{\theta:0<|\mathrm{Im}\,\theta|<\theta_0\}$. Then the set

$$S=\bigcup_{\theta\in C}V(\theta)(-\Delta+1)^{-1}B$$

has compact closure so that the convergence of $1-\chi_r$ to zero is uniform on S. This proves (3.3).

To prove (iii), note that if z_0 is fixed and

$$G(z)=(H_0(\varepsilon,\theta)-z_0)(H_0(\varepsilon,\theta)-z)^{-1}$$

then $G(z)^* \xrightarrow{s} 0$ as $d(z,\theta)\to\infty$. This follows from the standard theory of generators of contraction semigroups [35]. Thus

$$K(\varepsilon,\theta;z)=K(\varepsilon,\theta;z_0)G(z)$$

so that $\|K(\varepsilon,\theta;z)\|=\|G^*(z)K(\varepsilon,\theta;z_0)^*\|\to 0$. This completes the proof of Proposition III.1.

The last proposition leads directly to interesting properties of the operator $H(\varepsilon,\theta)$ which we at last define for $0<|\mathrm{Im}\,\theta|<\theta_0$ and $\varepsilon\geqq 0$:

$$H(\varepsilon,\theta)\equiv H_0(\varepsilon,\theta)+V(\theta), \qquad \mathscr{D}(H(\varepsilon,\theta))\equiv\mathscr{D}(H_0(\varepsilon,\theta)).$$

Theorem III.2. *Suppose* $0<\mathrm{Im}\,\theta<\theta_0$ *and* $\varepsilon>0$.

a) $H(\varepsilon,\theta)$ *is closed.*

b) *The family of operators* $\{H(\varepsilon,\theta):0<\mathrm{Im}\,\theta<\theta_0\}$ *is an analytic family of type A (see* [23] *for a definition).*

c) *The spectrum of* $H(\varepsilon,\theta)$ *is discrete (i.e. consists of eigenvalues of finite multiplicity) and independent of* θ. *Further, the multiplicity of each eigenvalue is independent of* θ.

Remark. Given (b) of the Theorem and the methods of [3, 11], part (c) follows immediately. We give a short proof for the convenience of the reader.

Proof. The fact that $H(\varepsilon,\theta)$ is closed follows from (iii) of Proposition III.1 which says that $V(\theta)$ is $H_0(\varepsilon,\theta)$-bounded with relative bound zero. The spectrum of $H(\varepsilon,\theta)$ is easily seen to be the set of z for which $1+K(\varepsilon,\theta;z)$ is not invertible. (This follows from the formula $H(\varepsilon,\theta)-z=(1+K(\varepsilon,\theta;z))(H_0(\varepsilon,\theta)-z)$.) By the analytic Fredholm theorem [27], $1+K(\varepsilon,\theta;z)$ is invertible except for a discrete set of z (note that from (iii) of Proposition III.1, if $d(z,\theta)$ is large, $\|K(\varepsilon,\theta;z)\|<1$). If $z_0\in\sigma(H(\varepsilon,\theta_1))$ then since $U(\phi)K(\varepsilon,\theta_1;z_0)U(-\phi)=K(\varepsilon,\theta_1+\phi;z_0)$, we have $z_0\in\sigma(H(\varepsilon,\theta_1+\phi))$ for all real ϕ. Again by the analytic Fredholm theorem (and the analyticity of $K(\varepsilon,\theta;z_0)$ in the variable θ), since the set of all θ for which $1+K(\varepsilon,\theta;z_0)$ is not invertible is not a discrete set, it must be all of $\{\theta:0<\mathrm{Im}\,\theta<\theta_0\}$.

The analyticity of $(H(\varepsilon,\theta)-z)^{-1}$ in θ clear and the fact that $\{H(\varepsilon,\theta)\}$ is type A, i.e. the constancy of the domain of $H(\varepsilon,\theta)$, follows from the quadratic estimate.

The analytic Fredholm theorem also tells us that if $E_0\in\sigma(H(\varepsilon,\theta))$ then $(1+K(\varepsilon,\theta;z))^{-1}=\sum_{n=1}^{N}(z-E_0)^{-n}F_n(\varepsilon,\theta)+F(\varepsilon,\theta;z)$ where F is analytic in a neighborhood of E_0 and F_n has finite rank. Thus for $\delta>0$ and small enough so that $\sigma(H(\varepsilon,\theta))\cap\{z:|z-E_0|\leqq\delta\}=\{E_0\}$, the projection

$$P(\varepsilon,\theta;E_0)=(2\pi i)^{-1}\oint_{|z-E_0|=\delta}(z-H(\varepsilon,\theta))^{-1}dz$$

associated with the point $E_0\in\sigma(H(\varepsilon,\theta))$ has finite rank. Thus E_0 is an eigenvalue of $H(\varepsilon,\theta)$ of finite multiplicity. Since clearly $P(\varepsilon,\theta;E_0)$ is analytic in θ (in $0<\mathrm{Im}\,\theta<\theta_0$) the dimension of its range is constant [23]. This completes the proof of Theorem III.2.

We interpret the eigenvalues of $H(\varepsilon,\theta)$ as resonances which result when an electric field of strength ε is applied in the x_1 direction to a system described by the

Hamiltonian $-\Delta+V$. It is interesting to note that even if $-\Delta+V$ has an infinite set of eigenvalues accumulating at the origin, there are only a finite number of resonances in any bounded region of the complex plane. This makes good physical sense since the imaginary parts of the resonance eigenvalues are inversely proportional to the lifetimes of the resonance "states", and the weakly bound states with energies close to zero should have very short lifetimes.

In order to make the connection between the eigenvalues of $-\Delta+V$ and those of $H(\varepsilon,\theta)$ more precise, we prove the following "stability" theorem and perturbation theoretic result. The proof is a direct translation of ideas in [8, 10] to the Stark case.

Theorem III.3. *Suppose E_0 is a negative eigenvalue of $-\Delta+V$ of multiplicity j. Then for $\varepsilon>0$ and small, there are exactly j eigenvalues, counting multiplicity, of $H(\varepsilon,\theta)$ ($\mathrm{Im}\,\theta>0$) nearby and as $\varepsilon\downarrow 0$, these eigenvalues converge to E_0.*

If E_0 is non-degenerate, then the Rayleigh-Schrödinger series for the nearby resonance is asymptotic and the remainder after N is bounded by $(C\varepsilon)^{N+1}(N+1)!$

Remarks. a) The multiplicity of the resonance eigenvalue referred to in the theorem is the algebraic multiplicity. Jordan blocks are not ruled out.

b) If the degeneracy is caused by a symmetry of $-\Delta+V$ which is also a symmetry of $H(\varepsilon,\theta)$, (for example, possibly rotations about the 1-axis) then if by restricting to an invariant subspace, this degeneracy is removed, the second part of the theorem still holds without change. We hope to come back to this general problem in [19].

c) Since the Rayleigh-Schrödinger series is real, the imaginary part of the nearby resonance eigenvalue (in the non-degenerate case) evidently vanishes faster than any power of ε as $\varepsilon\downarrow 0$.

Proof. Suppose $\delta>0$ is such that the disk $D_\delta=\{z:|z-E_0|\leqq\delta\}$ satisfies $D_\delta\cap\sigma(-\Delta+V)=\{E_0\}$, $D_\delta\cap W(\theta_1)=\emptyset$ for some θ_1 with $0<\mathrm{Im}\,\theta_1<\theta_0$. We will show that there is an $\varepsilon_\delta>0$ so that for $0<\varepsilon\leqq\varepsilon_\delta$ there are exactly j eigenvalues (counting multiplicity) of $H(\varepsilon,\theta_1)$ inside D_δ. This will prove the first part of the theorem because the fact that δ can be taken arbitrarily small means that these eigenvalues converge to E_0 as $\varepsilon\downarrow 0$, and because the algebraic multiplicity of an eigenvalue of $H(\varepsilon,\theta)$ is independent of θ.

First note that [3, 11] E_0 is an eigenvalue of multiplicity j of $H_0(\theta_1)+V(\theta_1)$ and that $\sigma(H_0(\theta_1)+V(\theta_1))\cap D_\delta=\{E_0\}$. Since $\|K(\theta_1;z)-K(\varepsilon,\theta_1;z)\|\to 0$ as $\varepsilon\downarrow 0$ uniformly for $z\in\partial D_\delta$ (part (ii) of Proposition III.1) and since $(1+K(\theta_1;z))^{-1}$ is uniformly bounded on this circle, $1+K(\varepsilon,\theta_1;z)$ is invertible for $z\in\partial D_\delta$ if $0<\varepsilon\leqq\varepsilon'_\delta$. This means $\sigma(H(\varepsilon,\theta_1))\cap\partial D_\delta=\emptyset$ for $0<\varepsilon\leqq\varepsilon'_\delta$. The algebraic multiplicity of the eigenvalues of $H(\varepsilon,\theta_1)$ inside D_δ is just the dimension of the projection

$$P(\varepsilon,\theta_1)=(2\pi i)^{-1}\oint_{\partial D_\delta}(z-H(\varepsilon,\theta_1))^{-1}dz\,.$$

It is thus sufficient to prove that $P(\varepsilon,\theta_1)$ converges in norm to

$$P(\theta_1)=(2\pi i)^{-1}\oint_{\partial D_\delta}(z-H_0(\theta_1)-V(\theta_1))^{-1}dz$$

as $\varepsilon\downarrow 0$ for if $\|P(\varepsilon,\theta_1)-P(\theta_1)\|<1$, the two projections have the same dimension. Now

$$\begin{aligned}(z-H(\varepsilon,\theta_1))^{-1}&=(z-H_0(\varepsilon,\theta_1))^{-1}(1+K(\varepsilon,\theta_1;z))^{-1}\\&=(z-H_0(\varepsilon,\theta_1))^{-1}-(z-H_0(\varepsilon,\theta_1))^{-1}K(\varepsilon,\theta_1;z)(1+K(\varepsilon,\theta_1;z)^{-1}\end{aligned}$$

while

$$\begin{aligned}&(z-H_0(\varepsilon,\theta_1))^{-1}K(\varepsilon,\theta_1;z)(1+K(\varepsilon,\theta_1;z))^{-1}\\&\xrightarrow[\varepsilon\downarrow 0]{\|\cdot\|}(z-H_0(\theta_1))^{-1}K(\theta_1;z)\cdot(1+K(\theta_1;z)^{-1}.\end{aligned}$$

The latter follows from $K(\varepsilon,\theta_1;z)\xrightarrow{\|\cdot\|}K(\theta_1;z)$ and the strong convergence of $(z-H_0(\varepsilon,\theta_1))^{-1}$ to $(z-H_0(\theta_1))^{-1}$. Since $\oint_{\partial D_\delta}(z-H_0(\varepsilon,\theta_1))^{-1}dz=0$ and similarly $\int_{\partial D_\delta}(z-H_0(\theta_1))^{-1}dz=0$,

$$\begin{aligned}\|P_0(\varepsilon,\theta_1)-P(\theta_1)\|=\frac{1}{2\pi}\Big\|\int_{\partial D_\delta}&[(z-H_0(\varepsilon,\theta_1))^{-1}K(\varepsilon,\theta_1;z)(1+K(\varepsilon,\theta_1;z))^{-1}\\&-(z-H_0(\theta_1))^{-1}K(\theta_1;z)(1+K(\theta_1;z))^{-1}]dz\Big\|\to 0\end{aligned}$$

as $\varepsilon\downarrow 0$. This completes the proof of the first part of the theorem.

To prove the second part of the theorem we write the resonance eigenvalue close to E_0 as the ratio of two functions of ε:

$$\begin{aligned}E(\varepsilon)&=f(\varepsilon)/g(\varepsilon)\\f(\varepsilon)&=\oint_{|z-E_0|=\delta}zdz(\psi_{\bar\theta},(H(\varepsilon,\theta)-z)^{-1}\psi_\theta)\\g(\varepsilon)&=\oint_{|z-E_0|=\delta}dz(\psi_{\bar\theta},(H(\varepsilon,\theta)-z)^{-1}\psi_\theta).\end{aligned}$$

Here $(-\Delta+V)\psi=E_0\psi$, $\psi_\theta=U(\theta)\psi$ with $\operatorname{Im}\theta>0$, and δ is small enough so that $D_\delta\cap\sigma(H(\varepsilon,\theta))=E(\varepsilon)$, $D_\delta\cap W(\theta)=\emptyset$. Note that ψ_θ is analytic in θ for $|\operatorname{Im}\theta|<\theta_0$ and that $f(\varepsilon)/g(\varepsilon)=(\psi_{\bar\theta},H(\varepsilon,\theta)P(\varepsilon,\theta)\psi_{\bar\theta})/(\psi_\theta,P(\varepsilon,\theta)\psi_\theta)=E(\varepsilon)$. Here we have assumed that ε is small enough that the denominator does not vanish. $\Big($We have $\lim_{\varepsilon\downarrow 0}(\psi_{\bar\theta},P(\varepsilon,\theta)\psi_\theta)=(\psi_{\bar\theta},P(\theta)\psi_\theta)=(\psi_{\bar\theta},\psi_\theta)=(\psi,\psi).\Big)$

We now expand the resolvent up to N terms and keep the remainder:

$$\begin{aligned}(z-H(\varepsilon,\theta))^{-1}=(z-H_0(\theta)-V(\theta))^{-1}\Bigg(&\sum_{n=0}^{N}[\varepsilon x_1e^\theta(z-H_0(\theta)-V(\theta))^{-1}]^n\\&+(z-H(\varepsilon,\theta))^{-1}[\varepsilon x_1e^\theta(z-H_0(\theta)-V(\theta))^{-1}]^{N+1}\Bigg).\end{aligned}$$

We first bound $\|(x_1(z-H_0(\theta)-V(\theta))^{-1})^n\psi_\theta\|$ by $c^n n!$: From the work of Combes and Thomas [15] it follows that for some $\alpha>0$, $\psi_\theta\in\mathscr{D}(e^{\alpha|x_1|})$ and $e^{\beta x_1}(z-H_0(\theta)-V(\theta))^{-1}e^{-\beta x_1}$ is uniformly bounded in β, z and θ for $|\beta|\leqq\alpha$, $z\in\partial D_\delta$ and θ such that $|\mathrm{Re}\,\theta|$ is small and $|\mathrm{Im}\,\theta|$ is small enough that D_δ is bounded away from $W(\theta)$. Writing $R=(z-H_0(\theta)-V(\theta))^{-1}$, $j(x_1)=e^{-\alpha|x_1|/n}$ we have

$$(x_1R)^n\psi_\theta=(x_1j)(j^{-1}Rj)(x_1j)(j^{-2}Rj^2)$$
$$\cdot(x_1j)(j^{-3}Rj^3)\dots(x_1j)(j^{-n}Rj^n)\cdot e^{\alpha|x_1|}\psi_\theta.$$

Since $\|e^{k\alpha|x_1|/n}Re^{-k\alpha|x_1|/n}\|\leqq c$ if $0\leqq k\leqq n$ and $|x_1e^{-\alpha|x_1|/n}|\leqq cn$ we have the desired bound.

We thus have

$$E(\varepsilon)=\left(\sum_{n=0}^{N}a_n\varepsilon^n+R_{N+1}\varepsilon^{N+1}\right)\Big/\left(\sum_{n=0}^{N}b_n\varepsilon^n+R'_{N+1}\varepsilon^{N+1}\right)$$

where by analytic continuation back to $\theta=0$

$$a_n=(2\pi i)^{-1}\oint_{|z-E_0|=\delta}zdz(\psi,(z-E_0)^{-1}(x_1(z-\Delta-V)^{-1})^n\psi)$$

$$b_n=(2\pi i)^{-1}\oint_{|z-E_0|=\delta}dz(\psi,(z-E_0)^{-1}[x_1(z-\Delta-V)^{-1}]^n\psi)$$

and $|R_N(\varepsilon)|+|R'_N(\varepsilon)|\leqq c^{N+1}N!$. It is easy to see directly that the a_n and b_n are real and that $b_0=\|\psi\|^2$. If $\sum_{n=0}^{\infty}c_nx^n=\sum_{n=0}^{\infty}a_nx^n/\sum_{n=0}^{\infty}b_nx^n$ as formal power series, it can be shown that $\left|f(\varepsilon)/g(\varepsilon)-\sum_{n=0}^{N}c_n\varepsilon^n\right|\leqq\beta^{N+1}(N+1)!\varepsilon^{N+1}$. The proof is not completely trivial. We refer the reader to [31] where the result is stated and some hints for a proof are given. This completes the proof of Theorem III.3.

After having shown that for small ε, at least certain of the eigenvalues of $H(\varepsilon,\theta)$ have something to do with the negative eigenvalues of $-\Delta+V$, we should also show that these eigenvalues have some connection with the operator $H(\varepsilon)=-\Delta+\varepsilon x_1+V$. With a strong enough assumption, we can make this connection:

Theorem III.4. *Suppose* $\varepsilon>0$ *and* $V(-\Delta+\varepsilon x_1+i)^{-1}$ *is compact. Then*

a) $H(\varepsilon)=-\Delta+\varepsilon x_1+V$ *is self-adjoint on* $\mathscr{D}(-\Delta+\varepsilon x_1)$.

b) *If* $\mathrm{Im}\,\theta>0$ *all eigenvalues of* $H(\varepsilon,\theta)$ *lie in* $\{z:\mathrm{Im}\,z\leqq0\}$.

c) *If* ϕ,ψ *are dilation analytic vectors then* $f_{\phi,\psi}(z)=(\phi,(z-H(\varepsilon))^{-1}\psi)$ *has a meromorphic continuation from* $\mathrm{Im}\,z>0$ *to all of* $\mathbb{C}$. *The only poles possible are the eigenvalues of* $H(\varepsilon,\theta)$, $\mathrm{Im}\,\theta>0$. *The set of singularities* $\{z:f_{\phi,\psi}$ *has a pole at* z *for some dilation analytic vectors* ϕ *and* $\psi\}=\sigma(H(\varepsilon,\theta))$.

d) *If E is real and* $E \in \sigma(H(\varepsilon, \theta))$ *then E is an eigenvalue of* $H(\varepsilon)$.

e) $H(\varepsilon)$ *has no singular continuous spectrum. The point spectrum of* $H(\varepsilon)$ *has no point of accummulation.*

Remark. For "reasonable" potentials, $H(\varepsilon)$ will have no eigenvalues (see [2, 17, 33] and [19]). In the latter reference a proof of the absence of eigenvalues in certain regions is given in the N-body case using dilation analytic machinery.

Part (a) of Theorem III.4 is standard. The rest of the proof rests of Lemma III.5 (which is trivial in the $\varepsilon = 0$ case). After one has the lemma the remainder of the proof is a somewhat lengthy direct transcription of [3, 11] and thus we refer the reader to the latter papers.

Lemma III.5. *Suppose* $V(-\Delta + \varepsilon x_1 + i)^{-1}$ *is compact and* ϕ, ψ *are dilation analytic vectors with* ϕ_θ *and* ψ_θ *analytic in* $|\mathrm{Im}\,\theta| < \gamma$ *where* $\gamma \leqq \theta_0$. *Then if* $\mathrm{Im}\, z_0 > 0$, $0 < \mathrm{Im}\,\theta < \gamma$, *and* $z_0 \notin \sigma(H(\varepsilon, \theta))$ *we have*

$$(\phi, (z_0 - H(\varepsilon))^{-1}\psi) = (\phi_{\bar{\theta}}, (z_0 - H(\varepsilon, \theta))^{-1}\psi_\theta) \tag{3.4}$$

where for a dilation analytic vector f we write $f_\theta = U(\theta) f$.

Remark. We do not prove that $(z - H(\varepsilon, \theta))^{-1} \xrightarrow{s} (z - H(\varepsilon))^{-1}$ as $\theta \to 0$ with $\mathrm{Im}\,\theta > 0$. This seems to require mildly stronger assumptions about V than we have made. If strong convergence is true then Lemma III.5 has a trivial proof.

Proof of Lemma III.5. The proof proceeds by showing that for a smaller class of potentials, one can prove Eq. (3.4) by showing that $(z - H(\varepsilon, \theta))^{-1} \xrightarrow{s} (z - H(\varepsilon))^{-1}$ and then by approximating V by such potentials. We do not attempt to find a large class of V for which strong convergence as $\theta \to 0$ holds.

Let $V_\delta(x) = e^{-\delta|x|} V(x)$ for $\delta > 0$. We will show that for $\mathrm{Im}\, z > 0$

(i) $V_\delta(\theta)(H_0(\varepsilon, \theta) - z)^{-1} \xrightarrow{\|\cdot\|} V_\delta(-\Delta + \varepsilon x_1 - z)^{-1}$ as $\theta \to 0$ with $\mathrm{Im}\,\theta > 0$.

(ii) $V_\delta(\theta)(H_0(\varepsilon, \theta) - z)^{-1} \xrightarrow{\|\cdot\|} V(\theta)(H_0(\varepsilon, \theta) - z)^{-1}$ as $\delta \downarrow 0$ if $z \notin W(\theta)$ and θ is fixed with $\theta_0 > |\mathrm{Im}\,\theta| \geqq 0$.

Given (i) and (ii) the proof of Eq. (3.4) is easy. For suppose ϕ and ψ are dilation analytic in $|\mathrm{Im}\,\theta| < \gamma$. Let $H_\delta(\varepsilon, \theta) = H_0(\varepsilon, \theta) + V_\delta(\theta)$ and $H_\delta(\varepsilon) = -\Delta + \varepsilon x_1 + V_\delta$. Then because of (ii), for small anough δ, $z_0 \notin \sigma(H_\delta(\theta, \varepsilon))$ and thus

$$f_\delta(\theta) = (\phi_{\bar{\theta}}, (z_0 - H_\delta(\varepsilon, \theta))^{-1}\psi_\theta)$$

is analytic in θ for $0 < \mathrm{Im}\,\theta < \gamma$. However f_δ is also independent of θ because if $\phi \in \mathbb{R}$, $U(\phi)$ is unitary and thus $f_\delta(\theta + \phi) = f_\delta(\theta)$. From (i) it is easy to see that

$$(z_0 - H_\delta(\varepsilon, \theta))^{-1} = (z_0 - H_0(\varepsilon, \theta))^{-1} \cdot (1 + V_\delta(\theta)(z_0 - H_0(\varepsilon, \theta))^{-1})^{-1} \xrightarrow{s} (z_0 - H_\delta(\varepsilon))^{-1}$$

as $\theta \to 0$ with $\text{Im}\,\theta > 0$ and thus $(\phi, (z_0 - H_\delta(\varepsilon))^{-1}\psi) = (\phi_{\bar\theta}, (z_0 - H_\delta(\varepsilon, \theta))^{-1}\psi_\theta)$. Now we use (ii) to take the limit $\delta \downarrow 0$ on both sides of the above equation. The result is Eq. (3.4).

It remains to prove (i) and (ii). To see (i) we write

$$V_\delta(\theta)(H_0(\varepsilon,\theta)-z)^{-1} = V_\delta(\theta)(x_1+i)(-\Delta+1)^{-1}B(\theta)$$

where $B(\theta) = (-\Delta+1)(x_1+i)^{-1}(H_0(\varepsilon,\theta)-z)^{-1}$. Note that

$$V_\delta(\theta)(x_1+i)(-\Delta+1)^{-1} \xrightarrow{\|\cdot\|} V_\delta(x_1+i)(-\Delta+1)$$

because $V_\delta(\theta)(x_1+i)(-\Delta+1)^{-1}$ is analytic in θ for $|\text{Im}\,\theta| < \theta_0$. Since $V_\delta(x_1+i)(-\Delta+1)^{-1}$ is compact it is enough to prove that $B(\theta)^* \xrightarrow{s} B(0)^*$. Now for $f \in \mathcal{S}$, it is clear that $B(\theta)^* f \to B(0)^* f$ because of the strong convergence of $(H_0(\varepsilon,\theta)-z)^{-1}$ to $(-\Delta+\varepsilon x_1 - z)^{-1}$ (Theorem II.5). It remains to prove that $\|B(\theta)\|$ is uniformly bounded. The proof is exactly the same as in the Appendix of [5].

To prove (ii) for any θ with $|\text{Im}\,\theta| < \theta_0$ we need only remark that $V(\theta)(H_0(\varepsilon,\theta)-z)^{-1}$ is a fixed compact operator while $\exp(-\delta|x|e^\theta) \xrightarrow{s} 1$ as $\delta \downarrow 0$ as a multiplication operator.

This completes the proof of Lemma III.5.

We end this section by showing that translation analyticity can be useful as a calculational tool in problems where the potential is not translation analytic but is dilation analytic. We use the notation $f^a(x_1, x_2, \ldots, x_n) = f(x_1+a, x_2, \ldots, x_n)$ and $H_0(\varepsilon) = -\Delta + \varepsilon x_1$.

Theorem III.5. *Let* $\varrho(x) = (2\pi\sigma^2)^{-n/2} e^{-x^2/2\sigma^2}$, $\sigma > 0$. *Suppose* $V(H_0(\varepsilon)+i)^{-1}$ *is compact and* $V(\theta)(-\Delta+1)^{-1}$ *is compact and analytic in* $|\text{Im}\,\theta| < \theta_0$ *with* $\theta_0 < \pi/4$. *Then* $\varrho * V$ *also satisfies these conditions. In addition,* $(\varrho * V)^z (H_0(\varepsilon)+i)^{-1}$ *is an entire compact operator valued function of z. If* $a < 0$, *the spectrum of* $H_0(\varepsilon) + i\varepsilon a + (\varrho * V)^{ia}$ *in* $Q_a = \{z : \varepsilon a < \text{Im}\, z\}$ *coincides with* $\sigma(H_0(\varepsilon,\theta) + (\varrho * V)(\theta)) \cap Q_a$ *if* $0 < \text{Im}\,\theta < \theta_0$, *and an eigenvalue of the complex translated Hamiltonian has the same algebraic multiplicity as that of the dilated Hamiltonian.*

If E_0 *is an eigenvalue of* $H_0(\varepsilon,\theta) + V(\theta)$ *of multiplicity j, then for small* σ *and for* $-a$ *large enough, there are exactly j eigenvalues (counting multiplicity) of* $H_0(\varepsilon) + i\varepsilon a + (\varrho * V)^{ia}$ *near* E_0 *and as* $\sigma \to 0$, *these eigenvalues converge to* E_0.

Proof. We first show that $\varrho * V$ has the same properties as V: It is easy to see that

$$(\varrho * V)(\theta)(-\Delta+1)^{-1} = \int \varrho_\theta(y) T(y) V(\theta)(-\Delta+1)^{-1} T(-y)\,dy \tag{3.5}$$

where $\varrho_\theta(x) = e^{n\theta}\varrho(e^\theta x)$, and $T(y)f(x) = f(x-y)$. If $|\text{Im}\,\theta| < \theta_0$, $(\varrho * V)(-\Delta+1)^{-1}$ is a norm convergent integral of compact operators and thus compact. It is clearly analytic in θ for $|\text{Im}\,\theta| < \theta_0$.

Similarly

$$\varrho * V(H_0(\varepsilon)+i)^{-1} = \int \varrho(y) T(y) V(H_0(\varepsilon)+i+\varepsilon y_1)^{-1} T(-y)\,dy$$

is also a norm convergent integral of compacts and thus compact. To prove analyticity and compactness of $(\varrho * V)^z(H_0(\varepsilon)+i)^{-1}$ we write

$$(\varrho * V)^z(H_0(\varepsilon)+i)^{-1}=\int \varrho^z(y)T(y)V(H_0(\varepsilon)+i+\varepsilon y_1)^{-1}T(-y)dy\,.$$

By the same argument as above this operator is compact and is clearly an entire function of z.

That the spectra of $H_0(\varepsilon)+i\varepsilon a+(\varrho * V)^{ia}$ and $H_0(\varepsilon,\theta)+(\varrho * V)(\theta)$ are the same in the relevant region follows from the existence of a result analogous to Theorem III.4 (c) in the translation analytic case [5]. To see that an eigenvalue of each operator has the same algebraic multiplicity note that if P is the spectral projection associated to eigenvalue E_0 of the operator $H_0(\varepsilon)+i\varepsilon a+(\varrho * V)^{ia}$ and similarly Q for $H_0(\varepsilon,\theta)+(\varrho * V)(\theta)$, then by integrating $(\phi,(z-H(\varepsilon))^{-1}\psi)$ around a small circle centered at E_0 we have

$$(\phi^{-ia},P\psi^{ia})=(\phi_{\bar\theta},Q\psi_\theta)$$

for a dense set of vectors from which ϕ and ψ can be chosen (say linear combinations of Hermite functions). If $\operatorname{Ran}P$ has dimension N, then there exist N linearly independent vectors $\psi_n=(f_n)^{ia}$ and $\phi_n=(g_n)^{-ia}$ from this set such that $\det\{(\phi_n,P\psi_l)\}\neq 0$. Then $\psi'_n=(f_n)_\theta$, $\phi'_n=(g_n)_{\bar\theta}$ satisfy $\det\{(\phi'_n,Q\psi'_l)\}\neq 0$ and hence $\operatorname{Ran}Q$ has dimension at least N. Reversing the argument gives $\dim(\operatorname{Ran}P)=\dim(\operatorname{Ran}Q)$. We remark that this same type of argument can be used to show that for any E and $\operatorname{Im}\theta_1>0$, $\operatorname{Im}\theta_2>0$, the operators $H(\varepsilon,\theta_1)P(\varepsilon,\theta_1;E)$ and $H(\varepsilon,\theta_2)P(\varepsilon,\theta_2;E)$ are similar. Here

$$P(\varepsilon,\theta;E)=(2\pi i)^{-1}\oint_{|z-E|=\delta}(z-H(\varepsilon,\theta))^{-1}dz$$

where $\sigma(H(\varepsilon,\theta))\cap\{z:0<|z-E|\leqq\delta\}=\emptyset$.

To see the stability result we proceed by using the same method as in the proof of Theorem III.3: Since as we have just shown the spectrum and algebraic multiplicity of the eigenvalues of $H_0(\varepsilon)+i\varepsilon a+(\varrho * V)^{ia}$ are the same as that of $H_0(\varepsilon,\theta)+(\varrho * V)(\theta)$ (in the relevant region) it is enough to prove the same result with $H_0(\varepsilon)+i\varepsilon a+(\varrho * V)^{ia}$ replaced by $H_0(\varepsilon,\theta)+(\varrho * V)(\theta)$. A glance at at the proof of Theorem III.3 shows that it is sufficient to show

$$\|((\varrho * V)(\theta)-V(\theta))(-\Delta+1)^{-1}\|\to 0$$

as $\sigma\downarrow 0$ for fixed θ with $0<\operatorname{Im}\theta<\theta_0$. From Eq. (3.5) it follows that

$$((\varrho * V)(\theta)-V(\theta))(-\Delta+1)^{-1}$$

$$=\int \varrho_\theta(y)[T(y)V(\theta)(-\Delta+1)^{-1}T(-y)-V(\theta)(-\Delta+1)^{-1}]dy\,.$$

Since $V(\theta)(-\Delta+1)^{-1}$ is compact, it follows that

$$\lim_{|y|\to 0}\|T(y)V(\theta)(-\Delta+1)^{-1}T(-y)-V(\theta)(-\Delta+1)^{-1}\|=0\,.$$

Since $\int_{|y|>\delta} |\varrho_\theta(y)|dy \to 0$ as $\sigma \downarrow 0$ for each $\delta > 0$ the result follows by a simple argument.

We remark that a Gaussian ϱ is clearly only one of a large class of smoothing functions which will perform the desired function.

IV. Discussion

To most physicists, the word "resonance" evokes thoughts of exponential decay and poles in scattering amplitudes. The former is an experimentally observed fact: Suppose a system described by a Hamiltonian H with an isolated eigenvalue E_0 and eigenvector ψ is perturbed to $H+\varepsilon W$, and suppose E_0 dissolves into continuous spectrum as soon as $\varepsilon > 0$. Then for small ε if a state corresponding to ψ is prepared at time 0 the probability that the state at time t is still ψ, i.e. the number $|(\psi, e^{-it(H+\varepsilon W)}\psi)|^2$, behaves as $e^{-\Gamma t}$.

Unfortunately, this cannot be an exact result for large t in most situations. For if $H+\varepsilon W$ is semibounded then the Paley-Wiener theorem prevents a bound of the form

$$|(\psi, e^{-it(H+\varepsilon W)}\psi)|^2 \leqq ce^{-\Gamma t} \tag{4.1}$$

from being true [32]. However in the Stark case, $H(\varepsilon)$ (for $\varepsilon > 0$) is not semibounded. In fact as has been stated, in many cases one can prove that $H(\varepsilon)$ is unitarily equivalent to multiplication by x_1 on $L^2(\mathbb{R}^n)$. In this case it is easy to display vectors ψ for which the bound (4.1) is satisfied. The question which the author finds interesting is whether the eigenvectors of the hydrogenic Hamiltonian, $H = -\Delta - Z/|x|$, satisfy (4.1) when $W = x_1$. This problem and its extension to more general interactions seems to involve a much more thorough understanding of the structure of the resolvent $(z - H(\varepsilon, \theta))^{-1}$ than has been given here. In particular one needs to have a more detailed analysis of $\sigma(H(\varepsilon, \theta))$. This leads to another question. In Theorem III.1 it was noted that even though Hydrogen has a infinite set of eigenvalues, $H(\varepsilon, \theta)$ has only a finite number of eigenvalues in any compact set. A conservative hypothesis might be that $H(\varepsilon, \theta)$ has an infinite number of eigenvalues $E_n(\varepsilon)$ in one-one correspondence with those of hydrogen and that $|\mathrm{Im}\, E_n(\varepsilon)| \to \infty$ as $n \to \infty$ for fixed $\varepsilon > 0$. It is entirely conceivable to this author, however, (based on some very imprecise physical intuition) that in fact $\sigma(H(\varepsilon, \theta))$ is finite.

An answer to these two questions seems to require a much more global analysis of the problem than we have presented here.

Let us return to the second response of the physicist: poles in the scattering amplitude. There is a large literature which shows that in many cases, complex eigenvalues of $-\Delta e^{-2\theta} + V(\theta)$ ($\mathrm{Im}\,\theta \neq 0$) do appear as poles in the scattering amplitude (see [32] for a list of references). In the Stark case, perhaps the first question which arises is "which scattering amplitude?". A scattering operator can be constructed from the wave operators introduced in [5, 17]. Is there a physically observable object associated with this operator which has the complex eigenvalues of $H(\varepsilon, \theta)$ as poles?

Our final question concerns the calculation of the imaginary parts of the eigenvalues of $H(\varepsilon, \theta)$. Their calculation seems difficult especially for a potential falling off as slowly as the Coulomb potential. Can Oppenheimer's [26] original calculation of the hydrogen Stark lifetimes be duplicated using dilation analytic machinery?

Acknowledgement. The author would like to thank Barry Simon for some useful comments relevant to the present work, and Bill Reinhardt for his insistence on the viability of complex scaling in Stark problems.

References

1. Abramowitz, M., Stegun, I.: Handbook of mathematical functions, pp. 448–450. Washington, D.C.: National Bureau of Standards 1964
2. Agmon, S.: Proceedings of the Tokyo Int. Conf. on Functional Analysis and Related Topics, 1969
3. Aguilar, J., Combes, J.M.: Commun. math. Phys. **22**, 269–279 (1971)
4. Apostol, C., Foias, C., Voiculescu, D.: Rev. Roum. Math. Pures Appl. **19**, 549–577 (1974)
5. Avron, J., Herbst, I.: Commun. math. Phys. **52**, 239–254 (1977)
6. Avron, J., Herbst, I., Simon, B.: Phys. Lett. **62**A, 214–216 (1977)
7. Avron, J., Herbst, I., Simon, B.: Phys. Rev. Letters **39**, 1068–1070 (1977)
8. Avron, J., Herbst, I., Simon, B.: Schrödinger operators with magnetic fields. I. General interactions. Duke Math. J. (to appear)
9. Avron, J., Herbst, I., Simon, J.: Separation of the center of mass in homogeneous magnetic fields (Schrödinger operators with magnetic fields. II). Ann. Phys. **114**, 431–451 (1978)
10. Avron, J., Herbst, I., Simon, J.: Schrödinger operators with magnetic fields. III. Atoms in magnetic fields. (in preparation)
11. Balslev, E., Combes, J.M.: Commun. math. Phys. **22**, 280–294 (1971)
12. Brändas, E., Froelich, P.: Phys. Rev. A **16**, 2207–2210 (1977)
13. Cerjan, C., Reinhardt, W., Avron, J.: Spectra of atomic Hamiltonians in D.C. fields: Use of the numerical range to investigate the effect of a dilatation transformation. Preprint, University of Colorado 1977
14. Cerjan, C., Hedges, R., Holt, C., Reinhardt, W., Scheiber, K., Wendoloski, J.: Complex coordinates and the Stark effect. Int. J. Quant. Chem. (to be published)
15. Combes, J.M., Thomas, L.: Commun. math. Phys. **34**, 251–270 (1973)
16. Graffi, S., Grecchi, V.: Commun. math. Phys. **62**, 83–96 (1978)
17. Herbst, I.: Math. Z. **155**, 55–70 (1977)
18. Herbst, I., Simon, B.: The Stark effect revisited. Phys. Rev. Letters **41**, 67–69 (1978)
19. Herbst, I., Simon, B.: Dilation analyticity in constant electric field. II. The N-body problem, Borel summability. (in preparation)
20. Herrero, D.: Normal limits of nilpotent operators. Preprint (unpublished) Universidade Esadual de Campinas, Brasil
21. Howland, J.: Bull. Am. Math. Soc. **78**, 380–383 (1972)
22. Howland, J.: Pac. J. Math. **55**, 157–176 (1974)
23. Kato, T.: Perturbation theory for linear operators. Berlin, Heidelberg, New York: Springer 1976
24. Lavine, R., O'Caroll, M.: J. Math. Phys. **18**, 1908–1912 (1977)
25. Nelson, E.: Ann. Math. **70**, 572–614 (1959)
26. Oppenheimer, J.R.: Phys. Rev. **31**, 66–81 (1928)
27. Reed, M., Simon, B.: Methods of modern mathematical physics. I, II. New York, London: Academic Press 1972/1973
28. Reinhardt, W.: Int. J. Quant. Chem. Symp. **10**, 359–367 (1976)
29. Riddell, R.C.: Pac. J. Math. **23**, 377–401 (1967)
30. Simon, B.: Ann. Math. **97**, 247–274 (1973)
31. Simon, B.: Phys. Rev. Letters 1145–1146 (1972)
32. Simon, B.: Resonances and complex scaling: A rigorous overview. Int. J. Quant. Chem. (to appear)

33. Titchmarsh, E.C.: Eigenfunction expansions associated with second order differential equations. Oxford: Oxford Press 1958
34. Veselić, K., Weidmann, J.: Math. Z. **156**, 93 (1977)
35. Yosida, K.: Functional analysis. Berlin, Heidelberg, New York: Springer 1971

Communicated by J. Ginibre

Received June 7, 1978