Commun. math. Phys. 63, 265—267 (1978)

Communications in
Mathematical
Physics

Cantoni's Generalized Transition Probability

Stanley P. Gudder

Department of Mathematics, University of Denver, Denver, Colorado 80208, USA

Abstract. It is observed that Cantoni's generalized transition probability can be derived from certain physically motivated axioms.

1. Introduction

In a recent paper [2], V. Cantoni has defined a generalized transition probability and has shown that his definition reduces to the usual transition probability for pure states in the Hilbert space formulation of quantum mechanics. In a later paper [3], Cantoni has constructed a Riemannian structure on the state space using this generalized transition probability. The present note observes that Cantoni's generalized transition probability can be derived from physically motivated axioms.

Let $(\mathcal{O}, \mathcal{S})$ be a pair of nonempty sets the elements of which we call *observables* and *states*, respectively; and let $\mathcal{P}$ be the set of probability measures on $\mathcal{B}(R)$. We assume the existence of a map $p: \mathcal{O} \times \mathcal{S} \to \mathcal{P}$ satisfying Mackey's Axioms I and II [2, 11] and call $(\mathcal{O}, \mathcal{S})$ a *Mackey system*. For $x \in \mathcal{O}, s \in \mathcal{S}$, the measure $s_x = p(x, s) \in \mathcal{P}$ gives the distribution of the observable x in the state s. For $x \in \mathcal{O}$, $s, t \in \mathcal{S}$, let $\tau \in \mathcal{P}$ satisfy $s_x, t_x \ll \tau$ (i.e., s_x and t_x are absolutely continuous relative to τ). Following Cantoni [2], we define

$$T_x^{1/2}(s,t) = \int_R \left[\frac{ds_x}{d\tau} \frac{dt_x}{d\tau} \right]^{1/2} d\tau$$

and call $T(s,t) = \inf_{x \in \mathcal{O}} T_x(s,t)$ the *generalized transition probability* of s to t.

2. Transition Measures

For $\alpha, \beta \in \mathcal{P}, \Delta \in \mathcal{B}(R)$, the (α, β) *transition measure* on Δ is a real number $m_{\alpha,\beta}(\Delta)$ satisfying:

(1) $m_{\alpha,\beta}(\cdot)$ is a nonnegative measure on $\mathcal{B}(R)$;
(2) $m_{\alpha,\beta}(\cdot) = m_{\beta,\alpha}(\cdot)$ for all $\alpha, \beta \in \mathcal{P}$;
(3) if $\beta(\Delta) = 0$, then $m_{\alpha,\beta}(\Delta) = 0$;
(4) if $\alpha, \beta \ll \tau$, then

0010-3616/78/0063/0265/$01.00

$$\frac{dm_{\alpha,\beta}}{d\tau} = \frac{dm_{\alpha,\tau}}{d\tau}\frac{dm_{\tau,\beta}}{d\tau} \quad \text{a.e. } [\tau].$$

Physically, we think of α and β as the distributions of some observable x in two different states s and t, respectively; and $m_{\alpha,\beta}(\Delta)$ as a likelyhood of a transition from s to t given that an observation of x yields a result in Δ. Condition (1) follows from the reasonable physical assumption that

$$m_{\alpha,\beta}(\Delta_1 \cup \Delta_2) = m_{\alpha,\beta}(\Delta_1) + m_{\alpha,\beta}(\Delta_2) \text{ if } \Delta_1 \cap \Delta_2 = \emptyset.$$

Condition (2) is a symmetry assumption that is usually made for transition probabilities. This condition holds on all sufficiently regular quantum systems [1, 7–9, 13]. Although Mielnik [12] has raised the possibility that such a condition may not hold in certain "quantum worlds", the usefulness of such nonregular systems has not been thoroughly demonstrated. Condition (3) says that if there is zero probability that x has a value in Δ in the state t, then an x-observation resulting in Δ gives no contribution to the likelihood of an s to t transition. It follows from (3) that if $\alpha, \beta \ll \tau$, then the Radon-Nikodym derivatives in (4) exist. Condition (4) states that the "transition density" from α to β equals the product of the "transition densities" from α to τ and from τ to β. In integrated form, (4) becomes

$$m_{\alpha,\beta}(\Delta) = \int_\Delta \frac{dm_{\alpha,\tau}}{d\tau}(\lambda)\, m_{\tau,\beta}(d\lambda) \tag{2.1}$$

for every $\Delta \in \mathscr{B}(R)$. Equation (2.1) is a kind of Chapman-Kolmogorov equation which holds for state transitions in Markov processes [6]. Equation (2.1) is also reminiscent of the familiar Hilbert space expression $\langle \phi, \psi \rangle = \sum \langle \phi, \phi_i \rangle \langle \phi_i, \psi \rangle$ where $\{\phi_i\}$ is an orthonormal basis, although the analogy does not seem to be particularly accurate [1].

Theorem. *For any $\alpha, \beta \in \mathscr{P}$, $m_{\alpha,\beta}$ exists, is unique and satisfies*

$$m_{\alpha,\beta}(\Delta) = \int_\Delta \left[\frac{d\alpha}{d\tau}\frac{d\beta}{d\tau}\right]^{1/2} d\tau$$

for all $\Delta \in \mathscr{B}(R)$ and all $\tau \in \mathscr{P}$ with $\alpha, \beta \ll \tau$.

Proof. Cantoni [2] has already noted (see also [11]) that the expression

$$F_{\alpha,\beta}(\Delta) = \int_\Delta \left[\frac{d\alpha}{d\tau}\frac{d\beta}{d\tau}\right]^{1/2} d\tau$$

if finite and independent of τ, where $\alpha, \beta \ll \tau$. It is clear that $F_{\alpha,\beta}$ satisfies (1) and (2). For (3), if $\beta(\Delta) = 0$, then $d\beta/d\tau = 0$ on Δ a.e. $[\tau]$ and hence $F_{\alpha,\beta}(\Delta) = 0$. For (4), suppose $\alpha, \beta \ll \tau$. Then $F_{\alpha,\tau}(\Delta) = \int_\Delta (d\alpha/d\tau)^{1/2} d\tau$ so $dF_{\alpha,\tau}/d\tau = (d\alpha/d\tau)^{1/2}$ a.e. $[\tau]$. Hence,

$$\frac{dF_{\alpha,\tau}}{d\tau}\frac{dF_{\tau,\beta}}{d\tau} = \left[\frac{d\alpha}{d\tau}\frac{d\beta}{d\tau}\right]^{1/2} = \frac{dF_{\alpha,\beta}}{d\tau} \quad \text{a.e. } [\tau].$$

We now show that $m_{\alpha,\beta}$ is unique. Letting $\alpha = \beta = \tau$ in (4) gives $dm_{\alpha,\alpha}/d\alpha =$

$(dm_{\alpha,\alpha}/d\alpha)^2$. Hence, $dm_{\alpha,\alpha}/d\alpha = 1$ a.e. $[\alpha]$ so $m_{\alpha,\alpha} = \alpha$. Now suppose that $\alpha \ll \tau$. From (2) and (4) we have

$$\frac{d\alpha}{d\tau} = \frac{dm_{\alpha,\alpha}}{d\tau} = \frac{dm_{\alpha,\tau}}{d\tau}\frac{dm_{\tau,\alpha}}{d\tau} = \left(\frac{dm_{\alpha,\tau}}{d\tau}\right)^2 \quad \text{a.e. } [\tau] .$$

Hence, $dm_{\alpha,\tau}/d\tau = (d\alpha/d\tau)^{1/2}$ a.e. $[\tau]$. If $\alpha, \beta \ll \tau$, we have by (2) and (2.1)

$$m_{\alpha,\beta}(\Delta) = \int_\Delta \frac{dm_{\alpha,\tau}}{d\tau}\frac{dm_{\beta,\tau}}{d\tau} d\tau = \int_\Delta \left[\frac{d\alpha}{d\tau}\frac{d\beta}{d\tau}\right]^{1/2} d\tau .$$

3. Generalized Transition Probability

Let $(\mathcal{O}, \mathcal{S})$ be a Mackey system. For $x \in \mathcal{O}, s, t \in \mathcal{S}$, we call $F_x(s,t) = m_{s_x,t_x}(R)$ the *transition amplitude* of s to t given x. Mielnik [12] defines the "detection ratio" $s:t$ as the minimal fraction of s particles detected by an instrument that detects all t particles, and shows that this reduces to the usual transition probability in the Hilbert space framework. Following this line of reasoning, we define the *transition amplitude* $F(s,t)$ of s to t by $F(s,t) = \inf_{x \in \mathcal{O}} F_x(s,t)$.

To treat more general states and particle beams, it is important to consider unnormalized states $\lambda s, \lambda > 0, s \in \mathcal{S}$, where $p(x, \lambda s)(\Delta) = \lambda p(x,s)(\Delta)$ [4, 5, 10, 12]. We extend F to such states by using the same definition as before but allowing α and β to be finite nonnegative measures instead of just probability measures. For physical reasons it would be desirable to have $F(\lambda s, t) = \lambda F(s,t), s, t \in \mathcal{S}, \lambda > 0$, but this does not hold. It is clear that the only function of F that satisfies the above homogeneity condition is F^2. We thus define the *generalized transition probability* to be $T(s,t) = F^2(s,t)$. This is precisely Cantoni's definition.

References

1. Ax, J.: Found. Phys. **6**, 371 (1976)
2. Cantoni, V.: Commun. math. Phys. **44**, 125 (1975)
3. Cantoni, V.: Commun. math. Phys. **56**, 189 (1977)
4. Davies, E. B.: Commun. math. Phys. **15**, 277 (1969); **19**, 83 (1970); **22**, 51 (1971)
5. Davies, E. B., Lewis, J. T.: Commun. math. Phys. **17**, 239 (1970)
6. Doob, J.: Stochastic processes. New York: Wiley 1953
7. Landé, A.: Phys. Rev. **108**, 891 (1957)
8. Landé, A.: New foundations of quantum mechanics. London: Cambridge University Press 1960
9. Landé, A.: Phys. Today **20**, 55 (1967)
10. Ludwig, G.: Commun. math. Phys. **4**, 331 (1967); **9**, 1 (1968); **26**, 78 (1972)
11. Mackey, G. W.: Mathematical foundations of quantum mechanics. New York: Benjamin 1963
12. Mielnik, B.: Commun. math. Phys. **9**, 55 (1968); **15**, 1 (1969); **37**, 221 (1974)
13. Uhlmann, A.: Rept. Math. Phys. **9**, 273 (1976)

Communicated by R. Haag

Received April 5, 1978