Commun. math. Phys. 62, 43—53 (1978)

Communications in
Mathematical
Physics

Construction of the Affine Lie Algebra $A_1^{(1)}$

James Lepowsky[1* **] and Robert Lee Wilson[2***]

[1] Department of Mathematics, Yale University, New Haven, Connecticut 06520, USA

[2] Department of Mathematics, Rutgers University, New Brunswick, New Jersey 08903, USA

Abstract. We give an explicit construction of the affine Lie algebra $A_1^{(1)}$ as an algebra of differential operators on $\mathbb{C}[x_1, x_3, x_5, \ldots]$. This algebra is spanned by the creation and annihilation operators and by the homogeneous components of a certain "exponential generating function" which is strikingly similar to the vertex operator in the string model.

1. Introduction

For every complex simple Lie algebra $\mathfrak{a}$ there is an infinite-dimensional Lie algebra $\mathfrak{a}\hat{}$ called the associated affine algebra. The affine algebras are among the generalized Cartan matrix (GCM) Lie algebras (or Kac-Moody Lie algebras), which were introduced and studied by Kac [3a] and Moody [7], and which have recently received a great deal of attention. The simplest non-trivial GCM Lie algebra is the affine Lie algebra $\mathfrak{sl}(S, \mathbb{C})\hat{}$. [Here $\mathfrak{sl}(2, \mathbb{C})$ denotes the Lie algebra of traceless 2 by 2 complex matrices. The Kac-Moody definition of $\mathfrak{sl}(2, \mathbb{C})\hat{}$ is given in §2 below.] This algebra is denoted $A_1^{(1)}$ by Kac. For convenience we will henceforth write $\mathfrak{g}$ for $\mathfrak{sl}(2, \mathbb{C})$ and $\mathfrak{g}\hat{}$ for $\mathfrak{sl}(2, \mathbb{C})\hat{}$.

The main purpose of this paper is to construct $\mathfrak{g}\hat{}$ as a concrete Lie algebra of differential operators on the space $\mathbb{C}[x_1, x_3, x_5, \ldots]$ of polynomials in infinitely many variables. (This space is naturally graded by setting $\deg x_k = -k$.) In this construction (described in detail in §5) $\mathfrak{g}\hat{}$ is spanned by the identity, the creation and annihilation operators [$L(x_k)$ and $\partial/\partial x_k$], and the homogeneous components of

$$\exp(\sum 4L(x_k)/k)\exp(-\sum \partial/\partial x_k)\,.$$

---

* Partially supported by a Sloan Foundation Fellowship and NSF grant MCS 76-10435

** Current address: Department of Mathematics, Rutgers University, New Brunswick, New Jersey 08903, USA

*** Most of this work was done while the author was a Visiting Fellow at Yale University, supported in part by NSF grant MCS 77-03608 and in part by a Faculty Academic Study Plan grant from Rutgers University

0010-3616/78/0062/0043/$02.20

The last expression may be thought of as an "exponential generating function" for $\hat{\mathfrak{g}}$. It is strikingly similar (see the remark following Theorem 5.7) to an expression for the vertex operator in the string model; see for example [8, p. 285, Formula (1.71)]. (We are indebted to Howard Garland for calling our attention to this similarity.)

Our construction depends on two main results. The first of these (§ 3) is that $\hat{\mathfrak{g}}$ contains a subalgebra $\mathfrak{s}$ consisting of elements of odd degree which is isomorphic to a Heisenberg algebra on infinitely many variables. The second (§ 4) is that $\hat{\mathfrak{g}}$ has an irreducible module V (one of the standard $\hat{\mathfrak{g}}$-modules introduced and studied by Kac [3b]) which remains irreducible when considered as an $\mathfrak{s}$-module. We call V the fundamental module for $\hat{\mathfrak{g}}$. The uniqueness of the Heisenberg commutation relations then allows us to identify V with $\mathbb{C}[x_1, x_3, \ldots]$ and $\mathfrak{s}$ with the algebra of creation and annihilation operators. The construction of the rest of $\hat{\mathfrak{g}}$ is then accomplished using commutator relations between elements of $\mathfrak{s}$ and elements of $\hat{\mathfrak{g}}$.

The first of these results is proved using results of Kac [3a] and Moody [7] on the structure of $\hat{\mathfrak{g}}$. Namely, $\hat{\mathfrak{g}}$ has a one-dimensional center $\mathbb{C}z$ and the quotient $\tilde{\mathfrak{g}} = \hat{\mathfrak{g}}/\mathbb{C}z$ is easily described:

$$\tilde{\mathfrak{g}} \cong \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}],$$

where $\mathbb{C}[t, t^{-1}]$ is the algebra of Laurent polynomials in the indeterminate t. The Lie bracket in $\tilde{\mathfrak{g}}$ is given by the formula

$$[x \otimes t^m, y \otimes t^n] = [x, y] \otimes t^{m+n}$$

for all $x, y \in \mathfrak{g}$ and $m, n \in \mathbb{Z}$ (the set of integers).

If we view the variable t as the function $e^{i\theta}$ (θ a real number between 0 and 2π), then we see that $\tilde{\mathfrak{g}}$ is simply the Lie algebra of trigonometric polynomials with values in $\mathfrak{g}$, and hence is a Lie algebra of functions from the circle into $\mathfrak{g}$. Philosophically, our construction of $\hat{\mathfrak{g}}$ amounts to "quantizing" these functions, by making them act as certain operators on a Fock space.

The second of the above-mentioned results is proved by using results of [3b] and [6] to show that V and $\mathbb{C}[x_1, x_3, \ldots]$ have the same "character". It was the unexpected discovery [1] that the classical partition function $p(n)$ occurs in the character formula for the fundamental module which led to this result and to the discovery of $\mathfrak{s}$.

In addition to the papers cited above, the reader may wish to consult [2, 3c] and [5] for further information and bibliography on GCM Lie algebra theory and its connections with other parts of mathematics. For the convenience of the reader, however, the treatment below is almost entirely self-contained.

Our construction of $A_1^{(1)}$ is generalized in [4].

2. Definition of $\mathfrak{sl}(2, \mathbb{C})\hat{}$

The 3-dimensional Lie algebra $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ has basis

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

satisfying the bracket relations

$$[h,e]=2e\,, \quad [h,f]=-2f\,, \quad [e,f]=h\,.$$

The infinite-dimensional Lie algebra $\mathfrak{g}^{\sim}$ is defined to be $\mathfrak{g}\otimes\mathbb{C}[t,t^{-1}]$, where $\mathbb{C}[t,t^{-1}]$ is the algebra of polynomials in the indeterminate t and its inverse t^{-1}. Hence $\mathfrak{g}^{\sim}$ has basis

$$\{h\otimes t^m, e\otimes t^n, f\otimes t^p \mid m,n,p\in\mathbb{Z}\}$$

with brackets

$$[h\otimes t^m, e\otimes t^n]=2e\otimes t^{m+n}\,,$$

$$[h\otimes t^m, f\otimes t^p]=-2f\otimes t^{m+p}\,,$$

$$[e\otimes t^n, f\otimes t^p]=h\otimes t^{n+p}\,,$$

and the remaining brackets among basis elements are zero. Note that $\mathfrak{g}^{\sim}$ may be viewed as the Lie algebra of 2 by 2 matrices whose coefficients are finite Laurent series in t.

The GCM Lie algebra $\mathfrak{g}^{\wedge}$ is defined as the Lie algebra generated by the six elements h_0, h_1, e_0, e_1, f_0, f_1 subject to the relations

$$\begin{aligned}
&[h_0,h_1]=0\\
&[e_i,f_j] = \delta_{ij}h_i\\
&[h_i,e_j] = A_{ij}e_j\\
&[h_i,f_j] = -A_{ij}f_j\\
&[e_i,[e_i[e_i,e_j]]]=0=[f_i,[f_i,[f_i,f_j]]] \quad \text{if} \quad i\neq j\,,
\end{aligned}$$

where A is the 2 by 2 "generalized Cartan matrix" $\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$. (The original definition of GCM Lie algebras appears in [3a] and [7]. For $\mathfrak{g}^{\wedge}$ the definition may be simplified to this form by Proposition 13 of [3a].) Note that the last relations may be rewritten

$$(\operatorname{ad}e_i)^{-A_{ij}+1}e_j=0=(\operatorname{ad}f_i)^{-A_{ij}+1}f_j \quad \text{if} \quad i\neq j\,.$$

Define $z=h_0+h_1$. Then z is central, i.e., $[z,\mathfrak{g}^{\wedge}]=0$, since $\operatorname{ad}z$ annihilates all six generators of $\mathfrak{g}^{\wedge}$.

The "abstract" Lie algebra $\mathfrak{g}^{\wedge}$ may be described using the more "concrete" algebra $\mathfrak{g}^{\sim}$ as follows: We may identify $\mathfrak{g}^{\wedge}$ with the direct sum of vector spaces

$$\text{(2.1)}\quad \mathfrak{g}^{\wedge}=\mathfrak{g}^{\sim}\oplus\mathbb{C}z\,.$$

In this description of $\mathfrak{g}^{\wedge}$, $\mathfrak{g}^{\sim}$ is not a Lie subalgebra of $\mathfrak{g}^{\wedge}$, but instead, for all $x,y\in\mathfrak{g}$ and $m,n\in\mathbb{Z}$,

$$\text{(2.2)}\quad [x\otimes t^m, y\otimes t^n]=[x,y]\otimes t^{m+n}+m\delta_{m,-n}\operatorname{tr}(xy)z$$

in $\mathfrak{g}\hat{}$, where $\operatorname{tr}(xy)$ is the ordinary trace of matrices. The elements e_i, f_i, h_i in the first description of $\mathfrak{g}\hat{}$ correspond to the following elements in the second description:

$$\begin{aligned}
&e_0 \leftrightarrow f \otimes t\\
&e_1 \leftrightarrow e \otimes 1\\
&f_0 \leftrightarrow e \otimes t^{-1}\\
&f_1 \leftrightarrow f \otimes 1\\
&h_0 \leftrightarrow -h \otimes 1 + z\\
&h_1 \leftrightarrow h \otimes 1\ .
\end{aligned}$$

(The second description of $\mathfrak{g}\hat{}$ may be deduced easily from [3a] and [7].)

Our main goal in this paper is to construct $\mathfrak{g}\hat{}$ as an algebra of differential operators.

3. Heisenberg Subalgebras

We use the description (2.1), (2.2) of $\mathfrak{g}\hat{}$. For all odd integers j, define the following elements of $\mathfrak{g}\hat{}$:

$$B_j = e \otimes t^{(j-1)/2} + f \otimes t^{(j+1)/2}$$

and

$$C_j = -e \otimes t^{(j-1)/2} + f \otimes t^{(j+1)/2}\ .$$

For all nonzero even integers j, define

$$H_j = h \otimes t^{j/2}\ ,$$

and finally, define

$$H_0 = h \otimes 1 - \tfrac{1}{2} z = \tfrac{1}{2}(h_1 - h_0)\ .$$

Then the B_j, C_j, H_j, and z form a new basis of $\mathfrak{g}\hat{}$, and (2.2) easily gives us the following bracket relations, for all appropriate j and k:

$$(3.1)\qquad \begin{aligned}
[B_j, B_k] &= j\delta_{j,-k} z\\
[C_j, C_k] &= -j\delta_{j,-k} z\\
[H_j, H_k] &= j\delta_{j,-k} z\\
[B_j, H_k] &= 2C_{j+k}\\
[C_j, H_k] &= 2B_{j+k}\\
[B_j, C_k] &= 2H_{j+k}\ .
\end{aligned}$$

Let

$$\begin{aligned}
\mathfrak{s} &= \operatorname{span}\{z\} \cup \{B_j \mid j \text{ odd}\}\ ,\\
\mathfrak{s}' &= \operatorname{span}\{z\} \cup \{C_j \mid j \text{ odd}\}\ ,\\
H &= \operatorname{span}\{z\} \cup \{H_j \mid 0 \neq j \text{ even}\}\ .
\end{aligned}$$

Then $\mathfrak{s}$, $\mathfrak{s}'$, and H are clearly infinite-dimensional Heisenberg subalgebras of $\mathfrak{g}\hat{}$.

4. The Fundamental $\hat{\mathfrak{g}}$-Module as a Fock Space

In [3b], Kac introduced a family of $\hat{\mathfrak{g}}$-modules called standard modules. The standard modules cannot be viewed as $\tilde{\mathfrak{g}}$-modules. The "smallest" of these is called the *fundamental* module V. V is an infinite-dimensional irreducible $\hat{\mathfrak{g}}$-module generated by a nonzero vector v_0 with the following properties:

$$\begin{aligned} &e_0 \cdot v_0 = 0 = e_1 \cdot v_0 , \\ (4.1) \quad &h_0 \cdot v_0 = v_0 , \\ &h_1 \cdot v_0 = 0 . \end{aligned}$$

The vector v_0 is called a *highest weight vector* of V. V is spanned by elements of the form

$$(4.2) \quad f_{i_1} f_{i_2} \cdots f_{i_j} \cdot v_0 ,$$

where each i_m is either 0 or 1. We assign the element (4.2) *principal degree* equal to $-j$. In particular, v_0 itself has principal degree zero. (The useful concepts of principal specialization and by implication, principal degree, for standard $\hat{\mathfrak{g}}$-modules were introduced in [6].) For every integer j, let V_j be the span of all elements (4.2) of principal degree j. Note that $V_j = (0)$ if $j > 0$. Every element of V_j will be said to have principal degree j.

We next recall a basic result from [6] concerning the generating function of the dimensions of the spaces V_j. (This generating function is a kind of "character".) The proof (in [6]) uses the Weyl-Kac character formula [3b] and the method of principal specialization.

Proposition 4.3. *For an indeterminate q we have*

$$\sum_{j \geqq 0} (\dim V_{-j}) q^j = \prod_{j \geqq 1} (1 - q^{2j-1})^{-1} .$$

In particular, $\dim V_{-j}$ *is the number of partitions of j into odd parts (i.e., the number of ways of writing j as a sum of nonincreasing positive odd integers).*

Definition. Let $T: V \to V$ be a linear operator. Then T is said to have *degree* $j \in \mathbb{Z}$ if $T(V_k) \subseteqq V_{k+j}$ for all $k \in \mathbb{Z}$.

The following result, concerning the elements introduced in §3, is straightforward.

Proposition 4.4. *For every odd integer j and every even integer k, the Lie algebra elements B_j and C_j have degree j as operators on V, and the element H_k has degree k. Moreover z has degree zero, and in fact z acts as the identity operator on V.*

It is an important fact that V actually remains irreducible under $\mathfrak{s}$. The proof of this fact will use degree considerations. The uniqueness of the Heisenberg commutation relations will then give us a concrete model for V. (This approach would work just as well with $\mathfrak{s}'$ in place of $\mathfrak{s}$. However, it would *not* work with H because V is not irreducible under H.)

Define a *multi-index* α to be a sequence $\alpha(1)$, $\alpha(3)$, $\alpha(5), \ldots$ of nonnegative integers such that only finitely many of them are nonzero. For a multi-index α, define B_-^α to be the linear operator

$$B_{-1}^{\alpha(1)} B_{-3}^{\alpha(3)} B_{-5}^{\alpha(5)} \dots$$

on V, and define B_+^α to be the linear operator

$$B_1^{\alpha(1)} B_3^{\alpha(3)} B_5^{\alpha(5)} \dots$$

on V. Also, for multi-indices α and β define

$$\alpha! = \prod \alpha(j)!\,,$$
$$|\alpha| = \sum \alpha(j)\,,$$
$$\|\alpha\| = \sum j\alpha(j)\,,$$

and

$$\binom{\alpha}{\beta} = \prod \binom{\alpha(j)}{\beta(j)}\,,$$

where all sums and products are over all positive odd integers and we use the convention that the binomial coefficient $\binom{a}{b}$ is zero if a and b are nonnegative integers with $a<b$. Note that by Proposition 4.4, B_+^α has degree $\|\alpha\|$, and B_-^α has degree $-\|\alpha\|$.

Using the commutation relations in the Heisenberg Lie algebra $\mathfrak{s}$, together with the fact that $B_j \cdot v_0 = 0$ for positive odd j and the fact that $z \cdot v_0 = v_0$, we easily obtain:

Lemma 4.5. *For all multi-indices α and β we have*

$$B_+^\alpha B_-^\beta \cdot v_0/\alpha! = \binom{\beta}{\alpha} \left(\prod j^{\alpha(j)}\right) B_-^{\beta-\alpha} \cdot v_0\,.$$

In particular,

$$B_+^\alpha B_-^\alpha \cdot v_0/\alpha! = \left(\prod j^{\alpha(j)}\right) v_0 \neq 0$$

while if $|\alpha| \geqq |\beta|$ and $\beta \neq \alpha$, then

$$B_+^\alpha B_-^\beta \cdot v_0 = 0\,.$$

Corollary 4.6. *Let V' be the $\mathfrak{s}$-submodule of V generated by v_0. Then V' is irreducible under $\mathfrak{s}$, and V' has as a basis $\{B_-^\alpha \cdot v_0\}$, where α ranges through the set of multi-indices.*

Now the element $B_-^\alpha \cdot v_0$ of V has degree $-\|\alpha\|$. Thus $V' \cap V_{-j}$ (j a nonnegative integer) has a basis consisting of those $B_-^\alpha \cdot v_0$ such that $\|\alpha\| = j$, and the number of such α is exactly the number of partitions of j into odd parts. By Proposition 4.3, it follows that

$$\dim(V' \cap V_{-j}) = \dim V_{-j}\,,$$

and so we must have $V' = V$. Thus we have proved:

Theorem 4.7. *The fundamental $\hat{\mathfrak{g}}$-module V is irreducible under the Heisenberg subalgebra $\mathfrak{s}$.*

Now let Z be the polynomial algebra $\mathbb{C}[x_1, x_3, x_5, \ldots]$ on the infinitely many variables $x_1, x_3, x_5, \ldots$. For a multi-index α, write

$$x^\alpha = \prod x_j^{\alpha(j)} \in Z$$

and

$$D^\alpha = \prod (\partial/\partial x_j)^{\alpha(j)} .$$

For $f \in Z$, write $L(f)$ for the operator on Z which multiplies a given polynomial by f.

It is clear (from the standard realization of the Heisenberg commutation relations) that the linear map τ defined by (for all odd $j > 0$)

$$\begin{aligned} &\tau : B_j \mapsto \tfrac{1}{2} j (\partial/\partial x_j) \\ (4.8) \qquad &\tau : B_{-j} \mapsto 2L(x_j) \\ &\tau : z \mapsto L(1) = \text{the identity operator} \end{aligned}$$

is an isomorphism of $\mathfrak{s}$ onto the span of

$$\{L(1)\} \cup \{L(x_j) | j \text{ odd}, j > 0\} \cup \{\partial/\partial x_j | j \text{ odd}, j > 0\} .$$

Then Lemma 4.5, Corollary 4.6, and Theorem 4.7 imply the "uniqueness of the Heisenberg commutation relations" in the present setting:

Theorem 4.9. *The linear map τ' defined by*

$$\tau' : B_-^\alpha \cdot v_0 \mapsto 2^{|\alpha|} x^\alpha$$

(α an arbitrary multi-index) is an isomorphism of V onto Z. If τ is as in (4.8) then

$$\tau(u) \cdot \tau'(v) = \tau'(u \cdot v)$$

for all $u \in \mathfrak{s}$ and $v \in V$. Hence V and Z are isomorphic as $\mathfrak{s}$-modules.

Remark. For each odd positive j, let c_j be any nonzero complex number. Then the images of B_j and B_{-j} may be replaced by $jc_j(\partial/\partial x_j)$ and $c_j^{-1} L(x_j)$, respectively.

The fundamental module V has now been identified with the Fock space Z in such a way that $\mathfrak{s}$ acts as the creation and annihilation operators. In § 5, we shall determine how the rest of $\hat{\mathfrak{g}}$ acts on Z.

5. The Exponential Generating Function for $\hat{\mathfrak{g}}$

The notions of degree for elements of V and for operators on V of course carry over to the corresponding notions of degree for elements of Z and for operators on Z. In particular, for a multi-index α, $x^\alpha \in Z$ has degree $-\|\alpha\|$. (Note that our conventions force us to assign negative degrees to monomials in Z.) Also, the operator $L(x^\alpha)$ has degree $-\|\alpha\|$, and the operator D^α has degree $\|\alpha\|$. We easily see:

Proposition 5.1. *Every homogeneous operator of degree j from Z to itself can be written as a sum*

$$\sum_{\|\nu\| - \|\mu\| = j} d(\mu, \nu) \frac{L(x^\mu)}{\mu!} \frac{D^\nu}{\nu!}$$

for some scalars $d(\mu, \nu)$, where the sum ranges over all multi-indices μ and ν such that $\|\nu\| - \|\mu\| = j$.

(Note that even if the above sum is infinite, it is still a well-defined operator of degree j from Z to itself.)

For all $j \in \mathbb{Z}$, we now define

$$X_j = \begin{cases} C_j & \text{if } j \text{ is odd} \\ H_j & \text{if } j \text{ is even} . \end{cases}$$

By Proposition 4.4, the operator X_j on V is homogeneous of degree j, and so

$$X_j = \sum_{\|\nu\| - \|\mu\| = j} d_j(\mu, \nu) \frac{L(x^\mu)}{\mu!} \frac{D^\nu}{\nu!}$$

for some scalars $d_j(\mu, \nu)$. We shall use the bracket relations (3.1) to compute the $d_j(\mu, \nu)$.

Let α and β be multi-indices such that $\|\beta\| - \|\alpha\| = j$. Below k will always range through the positive odd integers and, unless indicated otherwise, all sums and products are over k. We have

$$\prod (\operatorname{ad} B_k)^{\alpha(k)} \prod (\operatorname{ad} B_{-k})^{\beta(k)} X_j = 2^{|\alpha| + |\beta|} X_0 \tag{5.2}$$

by (3.1). On the other hand, in view of Theorem 4.9, this operator also equals

$$2^{-|\alpha| + |\beta|} \prod k^{\alpha(k)} \prod (\operatorname{ad}(\partial/\partial x_k))^{\alpha(k)} \prod (\operatorname{ad}(L(x_k)))^{\beta(k)}$$

$$\left(\sum_{\|\nu\| - \|\mu\| = j} d_j(\mu, \nu) \frac{L(x^\mu)}{\mu!} \frac{D^\nu}{\nu!} \right) .$$

It is easily checked that

$$\prod (\operatorname{ad}(\partial/\partial x_k))^{\alpha(k)} \left(\frac{L(x^\mu)}{\mu!} \right) = \frac{L(x^{\mu - \alpha})}{(\mu - \alpha)!}$$

and

$$\prod (\operatorname{ad}(L(x_k)))^{\beta(k)} \left(\frac{D^\nu}{\nu!} \right) = (-1)^{|\beta|} \frac{D^{\nu - \beta}}{(\nu - \beta)!} ,$$

where we adopt the convention that $x^\gamma = 0$ and $D^\gamma = 0$ if $\gamma(k) < 0$ for some k. Using these remarks the above expression simplifies to

$$2^{-|\alpha| + |\beta|} \prod k^{\alpha(k)} (-1)^{|\beta|} \sum_{\|\nu\| - \|\mu\| = j} d_j(\mu, \nu) \frac{L(x^{\mu - \alpha})}{(\mu - \alpha)!} \frac{D^{\nu - \beta}}{(\nu - \beta)!} .$$

Now this operator has degree zero, and when we apply it to the constant polynomial $1 \in Z$, only one term in the sum is nonzero. We get the constant polynomial

$$2^{-|\alpha| + |\beta|} \prod k^{\alpha(k)} (-1)^{|\beta|} d_j(\alpha, \beta) .$$

Similarly, when we apply (5.2) to 1, the result is

$$2^{|\alpha|+|\beta|}d_0(0,0)\,.$$

Equating these two constants, we find that

$$d_j(\alpha,\beta)=\prod(4/k)^{\alpha(k)}(-1)^{|\beta|}d_0(0,0)\,.$$

Hence

$$(5.3)\quad X_j=d_0(0,0)\sum_{\|\nu\|-\|\mu\|=j}\cdot\left\{\prod(4/k)^{\mu(k)}\left(\frac{L(x^\mu)}{\mu!}\right)\right\}\left\{(-1)^{|\nu|}\left(\frac{D^\nu}{\nu!}\right)\right\}.$$

In order to compute $d_0(0,0)$, we simply apply $X_0=H_0=\frac{1}{2}(h_1-h_0)$ to the highest weight vector $v_0\in V$, which corresponds to $1\in Z$ under the identification of V with Z (Theorem 4.9). From (4.1), we find that

$$(5.4)\quad d_0(0,0)=-\tfrac{1}{2}\,.$$

From (5.3) and (5.4) we get:

Theorem 5.5. *Let $j\in\mathbb{Z}$. As an operator on $Z=\mathbb{C}[x_1,x_3,x_5,\ldots]$,*

$$X_j=-\tfrac{1}{2}\sum_{\|\nu\|-\|\mu\|=j}\{\prod L(4x_k/k)^{\mu(k)}/\mu(k)!\}\{\prod(-\partial/\partial x_k)^{\nu(k)}/\nu(k)!\}\,,$$

where k ranges through the positive odd integers.

Now note that as an operator on Z,

$$\begin{aligned}\exp(-\textstyle\sum(\partial/\partial x_k)) &= \sum_{m\geqq 0}(-\textstyle\sum(\partial/\partial x_k))^m/m!\\ &= \sum_{m\geqq 0}\sum_{|\nu|=m}\prod(-(\partial/\partial x_k))^{\nu(k)}/\nu(k)!\\ &= \sum_{\nu}\prod(-(\partial/\partial x_k))^{\nu(k)}/\nu(k)!\,,\end{aligned}$$

where ν ranges through the set of multi-indices.

Also, let $\hat{Z}$ be the algebra $\mathbb{C}[[x_1,x_3,x_5,\ldots]]$ of formal power series in the variables $x_1,x_3,x_5,\ldots$. Then $\exp(\sum L(4x_k/k))$ is a well-defined operator from Z to $\hat{Z}$. We have

$$\begin{aligned}\exp(\textstyle\sum L(4x_k/k)) &= \sum_{n\geqq 0}(\textstyle\sum L(4x_k/k))^n/n!\\ &= \sum_{n\geqq 0}\sum_{|\mu|=n}\prod L(4x_k/k)^{\mu(k)}/\mu(k)!\\ &= \sum_{\mu}\prod L(4x_k/k)^{\mu(k)}/\mu(k)!\,,\end{aligned}$$

where μ ranges through the set of multi-indices.

For every operator $A:Z\to Z^{\hat{}}$ and every $j\in\mathbb{Z}$, it is clear what we mean by the *homogeneous component* A_j (using our usual notion of degree). Then A is the formal infinite sum $\sum_{j\in\mathbb{Z}} A_j$. Also, for each j, $A_j(Z)$ is contained in the subspace Z of $Z^{\hat{}}$, i.e., A_j is a linear operator taking Z to itself. Theorem 5.5 thus has the following reformulation:

Theorem 5.6. *Let Y be the well-defined operator*

$$Y=-\tfrac{1}{2}\exp(\textstyle\sum L(4x_k/k))\exp(-\sum(\partial/\partial x_k))$$

from $Z=\mathbb{C}[x_1,x_3,x_5,\ldots]$ to $Z^{\hat{}}=\mathbb{C}[[x_1,x_3,x_5,\ldots]]$, where k ranges through the positive odd integers. Then

$$Y=\sum_{j\in\mathbb{Z}} X_j\,,$$

i.e., the j-th homogeneous component Y_j of Y is X_j, for all $j\in\mathbb{Z}$.

Remark. Y may be thought of as an "exponential generating function" for the Lie algebra $\mathfrak{g}^{\hat{}}$.

Summarizing and recalling Theorem 4.9, we have:

Theorem 5.7. *The Lie algebra $\mathfrak{g}^{\hat{}}$ has as a basis the elements $B_{\pm k}$ (k odd positive), $X_j=C_j$ (j odd), $X_j=H_j$ (j even) and z (as defined in §3), and these elements can be realized as the following differential operators on $\mathbb{C}[x_1,x_3,x_5\ldots]$:*

$$\begin{aligned} B_k &\mapsto \tfrac{1}{2}k(\partial/\partial x_k)\\ B_{-k} &\mapsto 2L(x_k)\\ z &\mapsto L(1)\\ X_j &\mapsto Y_j\,, \end{aligned}$$

where Y_j is as described in Theorem 5.6.

Remark. If we let

$$B_k\mapsto kc_k(\partial/\partial x_k)$$

and

$$B_{-k}\mapsto c_k^{-1}L(x_k)$$

for positive odd k (see the remark following Theorem 4.9), then Y is replaced by

$$-\tfrac{1}{2}\exp(\textstyle\sum L(2x_k/kc_k))\exp(-\sum 2c_k(\partial/\partial x_k))\,.$$

In particular, we can let

$$B_k\mapsto \sqrt{k}(\partial/\partial x_k)$$

and

$$B_{-k}\mapsto \sqrt{k}L(x_k)\,,$$

and then the exponential generating function becomes

$$-\tfrac{1}{2}\exp\left(\sum L(2x_k/\sqrt{k})\right)\exp\left(-\sum 2(\partial/\partial x_k)/\sqrt{k}\right),$$

where the summations are over all positive odd integers k.

References

1. Feingold, A., Lepowsky, J.: The Weyl-Kac character formula and power series identities (to appear)
2. Garland, H., Lepowsky, J.: Lie algebra homology and the Macdonald-Kac formulas. Inventiones math. **34**, 37—76 (1976)

3a. Kac, V.: Simple irreducible graded Lie algebras of finite growth (in Russian). Izv. Akad. Nauk SSSR **32**, 1323—1367 (1968); English translation: Math. USSR-Izv. **2**, 1271—1311 (1968)

3b. Kac, V.: Infinite-dimensional Lie algebras and Dedekind's η-function (in Russian). Funkt. Anal. Ego Prilozheniya **8**, 77—78 (1974); English translation: Funct. Anal. Appl. **8**, 68—70 (1974)

3c. Kac, V.: Infinite-dimensional algebras, Dedekind's η-function, classical Möbius function, and the very strange formula. Advan. Math. (to appear)

4. Kac, V., Kazhdan, D., Lepowsky, J., Wilson, R.: (manuscript in preparation)
5. Lepowsky, J.: Generalized Verma modules, loop space cohomology, and Macdonald-type identities (to appear)
6. Lepowsky, J., Milne, S.: Lie algebraic approaches to classical partition identities. Advan. Math. (to appear)
7. Moody, R.: A new class of Lie algebras. J. Algebra **10**, 211—230 (1968)
8. Schwartz, J.: Dual resonance theory. Phys. Rept. **8** C, 269—335 (1973)

Communicated by A. Jaffe

Received March 9, 1978