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The Boltzmann Equation

I. Uniqueness and Local Existence*

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Abstract. An abstract form of the spatially non-homogeneous Boltzmann equation is derived which includes the usual, more concrete form for any kind of potential, hard or soft, with finite cutoff. It is assumed that the corresponding "gas" is confined to a bounded domain by some sort of reflection law. The problem then considered is the corresponding initial-boundary value problem, locally in time.

Two proofs of existence are given. Both are constructive, and the first, at least, provides two sequences, one converging to the solution from above, the other from below, thus producing, at the same time as existence, approximations to the solution and error bounds for the approximation.

The solution is found within a space of functions bounded by a multiple of a Maxwellian, and, in this space, uniqueness is also proved.

1. Introduction

This is the first in a projected sequence of papers on solutions of the Boltzmann equation in a domain V. Here, we restrict our attention to the local matters of uniqueness and local existence, needed in our later papers. In these subsequent papers, we hope to address questions of global existence, approximate solutions, numerical computation of solutions, and other matters.

The existence theorem we prove here is of interest for a number of reasons. First, with a single exception, it is the only such result we know of that applies to the spatially inhomogeneous Boltzmann equation. The exception is the theorem of Grad [1, §20] which, first of all, is limited to what we call *soft* interactions, and, more important, does not treat a physical domain with a boundary. Our result, on the other hand, applies to a non-homogeneous gas that is confined to a domain V by means of a reflection law and in which either soft or hard interactions may take place.

Second, we find the solution by strictly constructive means. We show that it is a limit of a sequence of functions that are themselves solutions of easily solved, first

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order, linear differential equations. At no point do we even use a compactness theorem to derive the main result.

Third, the method we use provides approximations, both upper and lower, that squeeze down on the solution and produce error bounds as part of the computation. This fact manifests itself in a feature of the method. It automatically assures us that the solution is non-negative. This has been a stumbling block for a number of proposed methods of solution, one of the major difficulties of which is that the purported approximations to the solution may be negative. (See, e.g., [1, §28].)

We also prove quite a general uniqueness theorem that says that the solution is unique within the class in which existence is proved. We believe this result includes all previous ones.

The plan of the paper is this. We begin with a section on the Boltzmann equation itself, in which we abstract certain properties of this equation which are precisely what we need later on. Any equation with these properties is then called an abstract Boltzmann equation. Then, in §3, we discuss the boundary conditions that must be imposed on a solution. After this, we solve a linear problem which is needed in later sections since we find the solution of the nonlinear abstract Boltzmann equation as a limit of solutions of linear problems.

This introductory material disposed of, we turn, in §5, to a method for solving the complete problem. We show that a solution exists and that it is a limit of two *monotone* sequences, one converging to the solution from above, the other from below.

The method of §5, while elegant and very suitable for computation, is unlikely to produce a solution in the large. Indeed, the iterates may not even be defined for all t. In §6, therefore, we present a second method of solution which, although it is restricted to what we call soft interactions and has the unfortunate feature that it requires the solution of a nonlinear problem at each step, nevertheless has the property that certain iterates are defined for all $t \geqq 0$. If t is small enough, we show that these iterates are the repeated images of a contraction mapping and so converge to a solution. We do not know at present whether they converge to a solution for all $t \geqq 0$, but they may do so.

Finally, in §7, we show the solution of §5 to be unique.

In all that follows, the letter c is reserved for a constant, independent of all the relevant variables. If the letter c, with or without subscripts, appears in a formula, it means that the formula is valid for some constant c.

2. The Abstract Boltzmann Equation

The Boltzmann equation has the form

$$f_t + \xi f_q = J(f, f). \tag{2.1}$$

Here, $f:(t, q, \xi) \rightarrow f(t, q, \xi)$ is a non-negative, real valued function defined on $I \times V \times R^3$, where I is an interval and V is a domain in R^3. t is the time, while $q \in V$ and $\xi \in R^3$ are the position and velocity of a molecule in a gas confined to the domain V. f_t is the derivative of f with respect to t, f_q the gradient of f with respect to the spatial variables q. ξf_q is the scalar product of the 3-vectors ξ and f_q. $J(f, g)$ is called the collision operator and is a certain bilinear map from two copies of some function space into another.

The physical interpretation of (2.1) is this. First, we imagine a collection of identical point masses, called molecules, interacting according to some law. Averages (which are everything in a molecular theory) are to be taken by multiplying by a certain probability density f and then integrating. (2.1) expresses a hypothesis about the development of this density in time. The left side of (2.1) is the rate of change of f along the characteristics $\frac{dq}{dt}=\xi$, while the right side is supposed to be the rate of change of f due to interactions between molecules. (2.1) then says that f changes in the interior of $V\times R^3$ only because of molecular interactions.

Our purpose in this section is to abstract certain properties of the collision operator, properties that are always available in the physically interesting cases. For this, we look at collisions more closely. An interaction between molecules is supposed to be an instantaneous event involving only two molecules at a point q and a time t. Let the velocities of the molecules before the interaction be ξ and η, and, after it, let them be ξ' and η'. If the collision conserves momentum and energy, then (ξ,η) and (ξ',η') are related by the equations[1]

$$\xi'+\eta'=\xi+\eta,\ |\xi'|^2+|\eta'|^2=|\xi|^2+|\eta|^2. \tag{2.2}$$

It is easy to verify from (2.2) that ξ' and η' have the form

$$\xi'=\xi+\zeta(\zeta\cdot v),\ \eta'=\eta-\zeta(\zeta\cdot v),\quad \text{where}\quad v=\eta-\xi, \tag{2.3}$$

and where ζ is a unit vector that determines the geometry of the collision. Notice that it follows from (2.3) that the inverse of the map $(\xi,\eta)\mapsto(\xi',\eta')$ is obtained by replacing ζ by $-\zeta$. Therefore, since the map is linear, its Jacobian (for fixed ζ) is unity [1].

With the notation just introduced, we can write down the collision operator J. It acts on functions keeping t and q fixed, transforming them only in the velocity variables. Thus, any dependence on t and q can be suppressed in displaying the formula for J, and, with this understanding, we have[2]

$$J(f,g)=\int_S\int_{R^3} k(\zeta\cdot v,|\xi-\eta|)\left\{\frac{f(\xi')g(\eta')+f(\eta')g(\xi')}{2}-f(\xi)g(\eta)\right\}d\eta d\zeta. \tag{2.4}$$

Here, S is the unit sphere $\{\zeta:|\zeta|=1\}$. The kernel k is determined by the details of the collision.

Of course, k is measurable, and, in all applications we know of [1], it satisfies $k(\theta,|\xi|)=k(-\theta,|\xi|)$, and

$$0\leqq k(\theta,|\xi|)\leqq c(|\xi|^{\lambda_1}+|\xi|^{\lambda_2}), \tag{2.5}$$

1 It is interesting to note, and apparently not widely understood, that (2.2) need be the *only* place where the science of mechanics enters into the Boltzmann equation. Schnute [4] has given an axiomatic derivation of the Boltzmann theory in which, among other things, he displays clearly the precise place of the conservation laws (2.2) in the theory

2 The object that appears in the Boltzmann equation is not (2.4), but the corresponding quadratic form $J(f,f)$. It is usual to introduce the corresponding symmetric bilinear form, rather than (2.4). We have not done this here, since we need the asymmetric form given

where $c>0$, λ_1 and λ_2 are constants. Either of the λs may be negative and, in most cases of physical interest, they do not exceed unity. What we need here is the hypothesis

$$-3<\lambda_1, \lambda_2<2, \tag{2.6}$$

the left hand inequality being needed in order for k to be locally integrable.

The right side of (2.4) is, apparently, the difference between two integrals. However, both these integrals diverge for all non-zero f and g unless one assumes there is no interaction between molecules that are far enough apart [1, p. 237]. Since this assumption of a *finite cutoff* is the only one consistent with the earlier hypotheses that molecular interactions are instantaneous and take place at a point, we make it freely in all that follows. The details of this hypothesis are not important to us. What matters is that it allows us to write J as a difference.

$$J(f,g)=Q(f,g)-P(f,g),$$

where

$$P(f,g)=f(\xi)\int_S\int_{R^3} k(\zeta\cdot v,|\xi-\eta|)\,g(\eta)\,d\eta d\zeta, \tag{2.7}$$

and

$$Q(f,g)=\int_S\int_{R^3} k(\zeta\cdot v,|\xi-\eta|)\frac{f(\xi')\,g(\eta')+f(\eta')\,g(\xi')}{2}\,d\eta d\zeta. \tag{2.8}$$

Notice that P is a product:

$$P(f,g)=f\cdot R(g), \tag{2.9}$$

where R is the linear operator defined by

$$R(g)(\xi)=\int_{R^3} g(\eta)\int_S k(\zeta\cdot v,|\xi-\eta|)\,d\zeta d\eta. \tag{2.10}$$

Inequalities play a large role in our argument. If f and g are two measurable functions, we always write $f\leqq g$ to mean $f(q,\xi)\leqq g(q,\xi)$ for almost all $(q,\xi)\in V\times R^3$.

Now, if k satisfies (2.5) and (2.6), then, if g decreases rapidly enough at infinity, (2.10) shows that

$$|R(g)(\xi)|\leqq c(1+|\xi|^\lambda),$$

where $\lambda=\max(0,\lambda_1,\lambda_2)$, and c depends on g. In particular, if g is the (non-normalized) Maxwellian distribution m_α, defined by

$$m_\alpha(\xi)=e^{-\alpha|\xi|^2}, \tag{2.11}$$

then, because of (2.5) and (2.6)

$$0\leqq R(m_\alpha)\leqq c(1+|\xi|^\lambda), \tag{2.12}$$

where $0\leqq\lambda<2$ and c depends only on α.

We need a number of function spaces. Let L^1_{loc} be the set of all measurable, locally integrable functions defined on $V\times R^3$, L^∞ the essentially bounded functions in L^1_{loc}, L^1 the integrable functions there. We denote the norm in L^1 by $\|\cdot\|$:

$$\|f\| = \int_V \int_{R^3} |f(q,\xi)| d\xi dq .$$

The Maxwellians (2.11) have a special role to play, for we always work in the space of functions bounded by a Maxwellian. We call these spaces M^α. Precisely:

$$M^\alpha = \{f \in L^\infty : |f| \leqq c m_\alpha \quad \text{for some constant } c\}; \tag{2.13}$$

the constant may depend on f. We define a convenient notion of convergence on M^α and a corresponding idea of continuity. We say that a sequence $\{f_n\}$ *converges* in M^α if $\{f_n(q,\xi)\}$ converges for almost all $(q,\xi) \in V \times R^3$, and if $|f_n| \leqq c m_\alpha$, where the constant c is independent of n. We say that a map $L: M^\alpha \times M^\alpha \to L^1$ is *sequentially continuous* if, whenever $\{f_n\}$, $\{g_n\}$ converge in M^α to f and g, respectively, then $L(f_n, g_n)$ converges to $L(f,g)$ in L^1.

We show that P is a sequentially continuous map of $M^\alpha \times M^\alpha$ into L^1. Let $\{f_n\}$ and $\{g_n\}$ be two sequences, converging in M^α to f and g, respectively. Consider

$$\|P(f,g) - P(f_n, g_n)\|$$

$$= \int_V \int_{R^3} \int_{R^3} |f(q,\xi) g(q,\eta) - f_n(q,\xi) g_n(q,\eta)| \int_S k(\zeta\cdot v, |\xi-\eta|)\, d\zeta d\eta d\xi dq .$$

The integrand of the outer triple integral goes to zero almost everywhere in $V \times R^3 \times R^3$. Also, this integrand is bounded by a multiple of $m_\alpha(\xi) m_\alpha(\eta) \int_S k (\zeta\cdot v, |\xi-\eta|)\, d\zeta$, which is integrable over $V \times R^3 \times R^3$ if V has finite volume. The Lebesgue convergence theorem then shows that $\{P(f_n, g_n)\}$ converges to $P(f,g)$ in L^1. Thus, *$P: M^\alpha \times M^\alpha \to L^1$ is sequentially continuous.*

When $f, g \in M^\alpha$, then, since $k \geqq 0$, $P(f,g) \leqq cP(m_\alpha, m_\alpha) \in L^1$, if V has finite volume. Thus, it makes sense to define

$$(f,g)_P = \int_S \int_{R^3} P(f,g)\, d\xi dq$$

for $f, g \in M^\alpha$. Notice that $(f,g)_P = (g,f)_P$, since $k(\zeta\cdot v, |\xi-\eta|) = k(\zeta\cdot(-v), |\xi-\eta|)$.

We need the positive cones in L^1 and M^α. We denote these by L^1_+ and M^α_+:

$$L^1_+ = \{f \in L^1 : f \geqq 0\}, M^\alpha_+ = \{f \in M^\alpha : f \geqq 0\} .$$

$Q(f,g)$ is defined, and in L^1, whenever $f, g \in M^\alpha$. For, (2.2) implies that $|\xi-\eta| = |\xi'-\eta'|$, that $\zeta\cdot(\eta'-\xi') = -\zeta\cdot v$, and, finally, that the transformation $(\xi,\eta) \mapsto (\xi',\eta')$ (for fixed ζ) has unit Jacobian. Therefore, making the transformation $(\xi,\eta) \mapsto (\xi',\eta')$, we find, since $k \geqq 0$ and since, by hypothesis, $k(-\theta, |\xi|) = k(\theta, |\xi|)$, $\|Q(f,g)\| \leqq \|Q(|f|,|g|)\| \leqq \frac{1}{2}\{\|P(|f|,|g|)\| + \|P(|g|,|f|)\|\} = \|P(|f|,|g|)\|$. In the rest of our argument, we need a little less than this, namely,

$$\|Q(f,g)\| \leqq c\|P(f,g)\| \quad \text{for} \quad f, g \in M^\alpha_+ . \tag{2.14}$$

Notice that (2.14) and the bilinearity of Q imply that, as a map from $M^\alpha \times M^\alpha$ into L^1, *Q is sequentially continuous.*

It is also *symmetric*, by its definition, (2.8).

One fact about the mapping R of (2.10) that we need is that, since $k \geqq 0$, R is *monotone* on M_+^α. That is, if $f, g \in M_+^\alpha$ and $f \leqq g$, then $0 \leqq R(f) \leqq R(g)$.

Because of (2.2), it is also easy to show from (2.8) and (2.7) that $Q(m_\alpha, m_\alpha) = P(m_\alpha, m_\alpha)$, where m_α is defined by (2.11). Again, we don't need this much. In agreement with (2.14), we assume only

$$0 \leqq Q(m_\alpha, m_\alpha) \leqq cP(m_\alpha, m_\alpha).$$

Finally, notice that, since $k \geqq 0$, Q, like R, is *monotone* on M_+^α : if $f, g \in M_+^\alpha$ and $f \leqq g$, then $0 \leqq Q(f, f) \leqq Q(f, g) \leqq Q(g, g)$.

The properties just derived are all we need to solve the Boltzmann equation. For easy reference later, we list them here. Recall that we define the Maxwellian m_α by (2.11) and the corresponding space M^α by (2.13) with the notions of convergence and sequential continuity defined there. It is also convenient to write p_λ for the function defined by

$$p_\lambda(\xi) = 1 + |\xi|^\lambda. \tag{2.15}$$

Then, we assume:

(P_1) $P: M^\alpha \times M^\alpha \to L^1$ is a sequentially continuous, bilinear map, having the form $P(f, g) = f \cdot R(g)$;

(P_2) $R: M^\alpha \to L^1_{\text{loc}}$ is monotone on M_+^α: if $f, g \in M_+^\alpha$ and $f \leqq g$, then $0 \leqq R(f) \leqq R(g)$;

(P_3) $(f, g)_P = \int\limits_V \int\limits_{R^3} P(f, g)\, d\xi\, dq = (g, f)_P$;

(P_4) $R(m_\alpha) \leqq cp_\lambda$ for some constants $c > 0$ and λ, $0 \leqq \lambda < 2$ (c may depend on α);

(Q_1) $Q: M^\alpha \times M^\alpha \to L^1$ is a symmetric, sequentially continuous, bilinear map;

(Q_2) if $f, g \in M_+^\alpha$ and $f \leqq g$, then $0 \leqq Q(f, f) \leqq Q(f, g) \leqq Q(g, g)$;

(Q_3) if $f, g \in M_+^\alpha$, then $\|Q(f, g)\| \leqq c \|P(f, g)\|$;

(Q_4) $Q(m_\alpha, m_\alpha) \leqq cP(m_\alpha, m_\alpha)$.

We refer to the hypotheses (P_1)–(P_4) collectively as (P) and the hypotheses (Q_1)–(Q_4) as (Q).

The structure inherent in the formulas (2.7–10) is irrelevant to us, except insofar as it is manifested in the hypotheses (P) and (Q). We focus attention on what we actually need by calling an equation of the form

$$f_t + \xi f_q + P(f, f) = Q(f, f)$$

an *abstract Boltzmann equation* if (P) and (Q) are satisfied. To distinguish it from the abstract equation, we call (2.1) with $J = Q - P$, and P and Q defined by (2.7) and (2.8), the *concrete Boltzmann equation.* Our efforts in this section can then be summarized as follows; *the concrete Boltzmann equation is an abstract Boltzmann equation when k satisfies* (2.5) *and* (2.6) *and the volume of V is finite.*

In all that follows, we assume V has finite volume, although we do not always mention this. See, however, the remark on domains with infinite volume in §5.

3. Boundary Conditions and Trajectories

We want to solve the abstract Boltzmann equation with an initial condition, but first we must say a word about boundary conditions. We suppose that a molecule, upon coming in contact with the boundary of V, has its velocity instantaneously changed according to a definite law so that it returns to V. Such a law is called a reflection law. More precisely, let ∂V be a C^1-manifold, and let $n=n(q)$ be the inner normal to ∂V at q. A map $A:\partial V\times R^3\to R^3$ is called a *reflection law* [3] if, for every $q\in\partial V$, $n(q)\cdot\xi<0$ implies $n(q)\cdot A(q,\xi)>0$. This condition says only that a molecule trying to escape from V has its velocity changed to drive it back into V.

If f is a solution of the Boltzmann equation and A the *reflection law* that determines the motion of the molecules when they encounter the boundary, there is an associated boundary condition that f must satisfy. It is this:

$$f(t+0,q,A(q,\xi))=\left|\frac{n(q)\cdot\xi}{n(q)\cdot A(q,\xi)}\right|\frac{f(t-0,q,\xi)}{|A'(q,\xi)|}, \tag{3.1}$$

whenever $q\in\partial V$ and $n(q)\cdot\xi<0$. Here, $A'(q,\xi)$ denotes the Jacobian of A with respect to ξ, with q fixed. (3.1) was first derived in [3, Lemma 3.3], using considerations of conservation of probability. It was first proposed and justified directly as a boundary condition for the Boltzmann equation in [4].

Having stated (3.1), we can now also define the complete problem we want to solve. We are given an abstract Boltzmann equation

$$f_t+\xi f_q+P(f,f)=Q(f,f) \tag{3.2}$$

and a reflection law A for a domain V. We want to solve (3.2) subject to the boundary condition (3.1) and an initial condition

$$f|_{t=0}=\phi\,. \tag{3.3}$$

We refer to this problem as the initial-boundary value problem for the abstract Boltzmann equation.

Of course, this problem is one of the development of f in time, so it is appropriate to speak of spaces of functions of t. We need the spaces $C^0(0,T;L^1)$ and $L^p(0,T;L^1)$ of functions, defined on $[0,T]$, taking values in L^1, that are continuous and in L^p, respectively, as functions of t. We also need the space $L^\infty(0,T;M^\alpha)$ of functions $f:[0,T]\to M^\alpha$ such that f/m_α is a bounded function of all its variables. Finally, we write $f\in AC(0,T;L^1)$ if $f\in C^0(0,T;L^1)$, if the limit

$$\dot f(t)=\lim_{h\to 0}\frac{f(t+h)-f(t)}{h}$$

exists (in L^1) for almost all t, $0\leqq t\leqq T$, and if $\dot f\in L^1(0,T;L^1)$.

The problem (3.1–3) can be posed in a different way, using the idea of a trajectory. We think of a trajectory as a union of characteristics of the equation (3.2), two characteristics being identified as belonging to the same trajectory if one is obtained from the other by means of reflection. That is, let $(q_0,\xi_0)\in V\times R^3$. A trajectory is defined as the straight line

$$q=q_0+\xi t,\ \xi=\xi_0 \tag{3.4}$$

when t is so small that $q_0+\xi_0 t\in V$. If t_0 is the first value of t for which $q_0+\xi_0 t_0\in\partial V$, the trajectory continues for $t>t_0$ as the line

$$q=q_1+A(q_1,\xi_0)(t-t_0),\ \xi=A(q_1,\xi_0), \tag{3.5}$$

where $q_1=q_0+\xi_0 t_0$. If this line intersects the boundary when $t=t_1$, then, for $t>t_1$,

$$q=q_2+A(q_2, A(q_1,\xi_0))(t-t_1),\ \xi=A(q_2, A(q_1,\xi_0)),$$

where $q_2=q_1+A(q_1,\xi_0)(t_1-t_0)$, and so on. These equations also define a family $\{\psi^t\}$ of maps, the value of ψ^t at (q_0,ξ_0) being the point (q,ξ) on the trajectory at which (q_0,ξ_0) has arrived by time t. Thus, $\psi^t(q_0,\xi_0)=(q,\xi)$, where (q,ξ) is defined by (3.4) for $0\leqq t<t_0$, by (3.5) for $t_0<t<t_1$, etc. ψ^t can be defined at $t=t_0,t_1,\dots$ by continuity from either the right or the left. Following [3,4], we define ψ^t to be continuous from the right, and we call it *the trajectory map.* Notice that ψ^t depends only on V and the reflection law A, and not on P or Q. We always assume that V and A are such that the trajectories intersect the boundary only finitely often in finite time. This is the case, for example, for reverse reflection [see (3.8)] no matter what V may be, and it is the case for specular reflection whenever V has bounded curvature.

A certain class of reflection laws occurs frequently and is of particular importance. We say that a reflection law A is *regular* if energy is conserved at each reflection, if the associated trajectory map ψ^t takes $\bar{V}\times R^3$ onto itself for every t, and if the Jacobian of ψ^t is always unity. This last condition can be written directly in terms of A as follows [3, Theorem 2.2]:

$$|n(q)\cdot A(q,\xi)|\cdot|A'(q,\xi)|=|n(q)\cdot\xi|\,. \tag{3.6}$$

Most of the usual reflection laws are regular. For example, the traditional *specular reflection*:

$$A(q,\xi)=\xi-2(n(q)\cdot\xi)\,n(q) \tag{3.7}$$

is regular, as is the far more realistic *reverse reflection*[3].

$$A(q,\xi)=-\xi\,. \tag{3.8}$$

In the rest of this paper, we consider only regular reflection laws. For such laws, the boundary condition (3.1) takes the simpler form

$$f(t+0,q,A(q,\xi))=f(t-0,q,\xi)\quad\text{for}\quad q\in\partial V\quad\text{and}\quad n(q)\cdot\xi<0\,. \tag{3.9}$$

We need a piece of notation associated with the trajectory map. If $f:(t,q,\xi)\to f(t,q,\xi)$ is any function, we define a function $f^\#$ as the function f considered along trajectories. Precisely,

$$f^\#(t,q,\xi)=f(t,\psi^t(q,\xi))\,.$$

The boundary condition (3.9) for a regular reflection law takes a particularly simple form when it is restated in terms of $f^\#$. To see this, notice that, if $\psi^{t-0}(q_0,\xi_0)=(q,\xi)\in\partial V\times R^3$, then $\psi^t(q_0,\xi_0)=\psi^{t+0}(q_0,\xi_0)=(q,A(q,\xi))$. Thus, (3.9) reads

$$f^\#(t+0,q,\xi)=f^\#(t-0,q,\xi) \tag{3.10}$$

i.e., $f^\#$ is continuous in t.

[3] We call (3.8) more realistic than (3.7), and it is, but, even more, (3.8) should be called the most realistic reflection law. This is so because (3.8) implies the associated gas does not slip on ∂V [3]. Even more, it is shown in [3] that a simple generalization of (3.8) is essentially the only reflection law that implies the no-slip boundary condition; the generalization is $A(q,\xi)=-\lambda(q,\xi)\xi$, where $\lambda(q,\xi)>0$. (3.8) is the only law of this form that conserves energy

(3.2) also takes a simpler form when written in terms of $f^{\#}$. Indeed, at least for smooth functions, (3.2) is equivalent to

$$\dot{f}^{\#}+P^{\#}(f,f)=Q^{\#}(f,f). \tag{3.11}$$

$\Big($Notice here that $P^{\#}(f,f)=[P(f,f)]^{\#}$. $P^{\#}(f,f)$ is *not* $P(f^{\#},f^{\#})$, although again, it is $f^{\#}R^{\#}(f)=f^{\#}[R(f)]^{\#}$, because of (P_1). Also notice that $\dot{f}^{\#}=\frac{d}{dt}(f^{\#})$.$\Big)$

We end this section with

Lemma 3.1. *Let A be a regular reflection law. Then, if $f\in C^0(0,T;L^1)$, we have* $\|f(t)\| = \|f^{\#}(t)\|$.

Proof. Let $f\in C^0(0,T;L^1)$. Since A is regular, ψ^t maps onto, and its Jacobian is unity. Therefore,

$$\begin{aligned}\|f(t)\| &= \int_V \int_{R^3} |f(t,q,\xi)|\,d\xi\,dq\\ &= \int_V \int_{R^3} |f(t,\psi^t(q,\xi)|\,d\xi\,dq\\ &= \|f^{\#}(t)\|.\end{aligned}$$

4. An Associated Linear Problem

In the next section, we find the solution of the initial-boundary value problem (3.2), (3.3), (3.9) as a limit of solutions of a sequence of *linear* problems. In this section, we show that these linear problems have solutions, derive some of their properties, and, most important, display an explicit formula for them. The problem we want to solve is this. Let V be a domain in R^3, and let A be a regular reflection law. Then, we have an equation

$$f_t+\xi f_q+P(f,g)=h, \tag{4.1}$$

where P satisfies (P_1) and (P_2), and g and h are given. We want to solve (4.1) with the boundary condition

$$f(t+0,q,A(q,\xi))=f(t-0,q,\xi) \quad \text{for} \quad q\in\partial V \quad \text{and} \quad n(q)\cdot\xi<0 \tag{4.2}$$

[cf. (3.9)], and the initial condition

$$f(0,q,\xi)=\phi(q,\xi). \tag{4.3}$$

The solution of this problem is straightforward when ϕ, g, and h are smooth and go to zero quickly enough at infinity; we present an explicit formula for the solution presently. But first, to cover the general situation, we define the concept of a mild solution. We say a function f is a *mild solution* of (4.1)–(4.3) in an interval $[0,T]$ if $f\in L^\infty(0,T;M^\alpha)$, $f^{\#}\in AC(0,T;L^1)$, and if

$$\dot{f}^{\#}+P^{\#}(f,g)=h^{\#}, \tag{4.4}$$

and

$$f(0)=\phi. \tag{4.5}$$

As we saw in § 3, if f is a mild solution, it satisfies (4.2) since $f^{\#}$ is continuous, while (4.4) is equivalent to (4.1) if f is smooth, and (4.5), of course, is the same as (4.3).

When ϕ, g, and h are nice enough, the mild solution of (4.1)–(4.3) is immediate. It is defined by

$$f^{\#}(t)=\phi\exp\left(-\int_0^t R^{\#}(g)(\sigma)d\sigma\right)+\int_0^t h^{\#}(\tau)\exp\left(-\int_\tau^t R^{\#}(g)(\sigma)d\sigma\right)d\tau. \tag{4.6}$$

In what follows, we write $j\in L^\infty(0,T;M_+^\alpha)$ if $j\in L^\infty(0,T;M^\alpha)$ and $j(t)\in M_+^\alpha$ for almost all t in $(0,T)$. With this definition and (4.6) before us, we can simply read off most of

Lemma 4.1. *Let P satisfy* (P_1), (P_2), *and* (P_4). *Let $\phi\in M_+^\alpha$, $g,h\in L^\infty(0,T;M_+^\alpha)$ for some T, $0<T<\infty$. Then,* (4.1)–(4.3) *has a unique mild solution $f\in L^\infty(0,T;M_+^\alpha)$ satisfying*

$$\|f(t)\|\leqq\|\phi\|+\int_0^t\|h(\tau)\|\,d\tau. \tag{4.7}$$

Also, $\|f(\cdot)\|$ is absolutely continuous, and

$$\frac{d}{dt}\|f(t)\|+(f(t),g(t))_P=\|h(t)\|. \tag{4.8}$$

Proof. Formally, we define the mild solution by (4.6). This definition makes sense almost everywhere since (P_4) and the hypothesis on g give that $R(g)$ is an integrable function of t for almost all (q,ξ). We prove $f^{\#}\in C^0(0,T;L^1)$. Let $0\leqq s\leqq t\leqq T$, and consider

$$\begin{aligned} f^{\#}(s)-f^{\#}(t)=&\left[\phi\exp\left(-\int_0^s R^{\#}(g)(\sigma)\,d\sigma\right)+\int_0^s h^{\#}(\tau)\exp\left(-\int_\tau^s R^{\#}(g)(\sigma)\,d\sigma\right)d\tau\right]\\ &\cdot\left[1-\exp\left(-\int_s^t R^{\#}(g)(\sigma)\,d\sigma\right)\right]\\ &-\int_s^t h^{\#}(\tau)\exp\left(-\int_\tau^t R^{\#}(g)(\sigma)\,d\sigma\right)d\tau. \end{aligned}$$

Since $g(t)\geqq 0$ (a.e.), (P_2) gives $R(g)(t)\geqq 0$ (a.e.), and then we find, for some constants c,

$$|f^{\#}(s)-f^{\#}(t)|\leqq cm_\alpha(1-e^{-c(t-s)R(m_\alpha)})+c(t-s)m_\alpha$$

since $T<\infty$. It now follows easily that

$$|f^{\#}(s)-f^{\#}(t)|\leqq c(t-s)m_\alpha p_\lambda.$$

Since V has finite volume, $m_\alpha p_\lambda\in L^1$, and it follows that $f^{\#}\in C^0(0,T;L^1)$.

Next, since ϕ, $h(t)\geqq 0$ (a.e.), (4.6) shows that $f^{\#}(t)\geqq 0$. Since $R(g)(t)\geqq 0$ (a.e.), we have

$$0\leqq f^{\#}(t)\leqq\phi+\int_0^t h^{\#}(\tau)\,d\tau. \tag{4.9}$$

Therefore, $f^{\#}(t)\leqq cm_\alpha$ so that $f^{\#}\in C^0(0,T;M^\alpha)$. Also, integrating (4.9) and using Lemma 3.1, we obtain (4.7).

$f^{\#}$ is differentiable, and we have, clearly,

$$\dot f^{\#}(t)+P^{\#}(f,g)(t)=h^{\#}(t), \tag{4.10}$$

while, setting $t=0$ in (4.6) gives $f(0)=f^{\#}(0)=\phi$. Also, (4.10) implies easily that $f^{\#}\in AC(0,T;L^1)$, and f is a mild solution of (4.1)–(4.3).

Now, integrating (4.10) and using the positivity of $f^{\#}(t)$, we find

$$\frac{d}{dt}\|f^{\#}(t)\|+\|P^{\#}(f,g)(t)\|=\|h^{\#}(t)\|.$$

(4.8) follows from this and Lemma 3.1.

To prove the uniqueness, we must show that, when ϕ and h are zero, then f is zero. We cannot use (4.9) directly, since that depends on the representation (4.6). However, because of (P_4), we see that, when $h=0$, (4.1) can be written in the form $\frac{d}{dt}\left[f^{\#}(t)\exp\left(\int\limits_0^t R^{\#}(g)(\sigma)\,d\sigma\right)\right]=0$. The uniqueness follows from this and the fact that $f(0)=0$.

The next result follows easily from (4.6) and (P_2).

Lemma 4.2. *Let $\phi_i, g_i, h_i (i=1,2)$ satisfy the hypotheses imposed in Lemma 4.1 on ϕ, g, and h, respectively. Let $f_i\ (i=1,2)$ denote the corresponding mild solutions. If $\phi_1\leqq\phi_2$, while $g_1(t)\geqq g_2(t)$, $h_1(t)\leqq h_2(t)$* (a.e.), *then $f_1(t)\leqq f_2(t)$ for all t, $0\leqq t\leqq T$.*

Briefly, increasing ϕ or h, or decreasing g, has the effect of increasing f.

In § 5, we need the solution for a wider class of functions ϕ and h. For this, let $g\in L^\infty(0,T;M^\alpha)$. Then, of course, $R(g)(t)$ is defined (a.e.) and lies in L^1_{loc} [cf. (P_2)]. It may happen that, for a given function $f\in L^\infty(0,T;L^1)$, $f\cdot R(g)\in L^1(0,T;L^1)$. For such functions f, we can define $P(f,g)$ in the obvious way, as $f\cdot R(g)$. Given a $g\in L^\infty(0,T;M^\alpha)$, we write $\boldsymbol{D}_g$ for the class of f's in $L^\infty(0,T;L^1)$ for which $f\cdot R(g)\in L^1(0,T;L^1)$. We denote the extension of $P(\cdot,g)$ to $\boldsymbol{D}_g$ by $P(\cdot,g)$ again. This will cause no confusion. We extend the notion of a mild solution to this case by calling a function f a *weak solution* of (4.1)–(4.3) if $f\in\boldsymbol{D}_g$, $f^{\#}\in AC(0,T;L^1)$, and (4.4) and (4.5) are satisfied.

We can now prove

Lemma 4.3. *Let P satisfy* (P_1) and (P_2). *Let $\phi\in L^1_+$, $g\in L^\infty(0,T;M^\alpha_+)$, $h\in L^\infty(0,T;L^1_+)$ for some T, $0<T<\infty$. Then,* (4.1)–(4.3) *has a unique, non-negative, weak solution f satisfying* (4.8).

Proof. Given any $j\in L^1_+$, we write

$$T_n j=\begin{cases} j, & \text{whenever } \ j\leqq nm_\alpha\\ nm_\alpha & \text{otherwise}.\end{cases}$$

By Lemma 4.1, there is a sequence $\{f_n\}$ of mild solutions of (4.1)–(4.3) with ϕ replaced by $T_n\phi$ and h by $T_n h$. Each f_n satisfies

$$\dot{f}_n^{\#}(t)+P^{\#}(f_n,g)(t)=(T_n h)^{\#}(t) \tag{4.11}$$

and

$$f_n(0)=T_n\phi, \tag{4.12}$$

as well as

$$\|f_n(t)\|\leqq\|T_n\phi\|+\int\limits_0^t\|(T_n h)(\tau)\|\,d\tau, \tag{4.13}$$

and

$$\frac{d}{dt}\|f_n(t)\| + (f_n(t), g(t))_P = \|(T_n h)(t)\|. \tag{4.14}$$

by (4.7) and (4.8).

The sequences $\{T_n\phi\}$ and $\{T_n h(t)\}$ are non-decreasing. By Lemma 4.2, $\{f_n(t)\}$ is non-decreasing also, and (4.13) shows that $\{f_n(t)\}$ is bounded in L^1. Since L^1 has the Levi property [2, § 18], $\{f_n(t)\}$ converges to a function $f(t)$ in L^1 for every $t\in[0, T]$.

Integrate (4.14). A simple estimation gives

$$\begin{aligned}\int_0^T (f_n(t), g(t))_P\,dt &\leqq \|T_n\phi\| + \int_0^T \|(T_n h)(t)\|\,dt \\ &\leqq \|\phi\| + \int_0^T \|h(t)\|\,dt.\end{aligned} \tag{4.15}$$

Since $\{f_n(t)\}$ is non-decreasing, the same is true of $\{P(f_n, g)\} = \{f_n R(g)\}$. Since $g(t)$ and $f_n(t)$ are non-negative (a.e.), (4.15) shows that $\{P(f_n, g)\}$ is bounded in $L^1(0, T; L^1)$. Thus, the Levi property of $L^1(0, T; L^1)$ shows that $\{P(f_n, g)\}$ converges there. Since $P(f_n, g) = f_n R(g)$ and $f_n \uparrow f$, the limit of $\{P(f_n, g)\}$ must be $P(f, g)$. Now, we can let n tend to infinity in (4.11) and (4.12) to show that f is the desired weak solution.

Uniqueness follows as it did for Lemma 4.1.

We also have the following extension of Lemma 4.2.

Lemma 4.4. *Let P satisfy* (P_1), (P_2), *and* (P_4). *For $i=1, 2$, let $\phi_i\in L^1_+$, $g_i\in L^\infty(0, T; M^\alpha_+)$, $h_i\in L^\infty(0, T; L^1_+)$, and let f_i be the corresponding weak solution of Lemma 4.3. If $\phi_1 \leqq \phi_2$, while $g_1(t)\geqq g_2(t)$, $h_1(t)\leqq h_2(t)$* (a.e.), *then $f_1(t)\leqq f_2(t)$ for $0\leqq t\leqq T$.*

For, the representation (4.6) is still valid, even for weak solutions, when $g(t)\geqq 0$.

5. Local Existence

Next, we want to solve the nonlinear initial-boundary value problem (3.1)–(3.3) for the abstract Boltzmann equation. Naturally, we cannot expect the solution to be any better than the solution of a similar linear problem, which means that we must look for some sort of mild solution, a concept that we now define. Let A be a regular reflection law. We say a function f is a *mild solution* of the initial-boundary value problem for the abstract Boltzmann Equation (3.2) in the interval $[0, T]$ if $f\in L^\infty(0, T; M^\alpha)$, $f^\#\in AC(0, T; L^1)$, and if

$$\dot{f}^\# + P^\#(f, f) = Q^\#(f, f) \tag{5.1}$$

while

$$f(0) = \phi. \tag{5.2}$$

As before, (5.1) comes from (3.2), while the boundary condition (3.9) follows from the continuity of $f^\#$. Notice that, since we require $f(t)\in M^\alpha_+$, we have $f(t)\geqq 0$ as part of the definition of a mild solution.

In this section, we show a mild solution of (5.1)–(5.2) exists and is the limit of two sequences, $\{l_n\}$ and $\{u_n\}$, the one monotone increasing, the other monotone decreasing, and both squeezing down on f. Moreover, we show that each function l_n and u_n for $n \geqq 1$ is a solution of a simple *linear* problem of the sort we solved explicitly in (4.6). Specifically, $\{l_n\}$ and $\{u_n\}$ are defined recursively as follows. If $l_0, \ldots, l_{n-1}, u_0, \ldots, u_{n-1}$ are known, then l_n and u_n are the weak solutions of

$$\dot{l}_n^{\#} + P^{\#}(l_n, u_{n-1}) = Q^{\#}(l_{n-1}, l_{n-1}), \tag{5.3}$$

$$\dot{u}_n^{\#} + P^{\#}(u_n, l_{n-1}) = Q^{\#}(u_{n-1}, u_{n-1}), \tag{5.4}$$

$$l_n(0) = u_n(0) = \phi . \tag{5.5}$$

Naturally, we must begin with a pair of functions (l_0, u_0). We say that such a pair satisfies the *beginning condition* in $[0, T]$ if $u_0 \in L^\infty(0, T; M^\alpha)$ and

$$0 \leqq l_0(t) \leqq l_1(t) \leqq u_1(t) \leqq u_0(t) \quad \text{for} \quad 0 \leqq t \leqq T. \tag{5.6}$$

We discuss the beginning condition later, but, first, we prove

Lemma 5.1. *Let the hypotheses* (P_1), (P_2), (P_4), (Q_2), *and* (Q_3) *be satisfied. Let* $\phi \in M_+^\alpha$. *Let* (l_0, u_0) *satisfy the beginning condition in* $[0, T]$. *Then, the sequences* $\{l_n\}$ *and* $\{u_n\}$, *defined by* (5.3)–(5.5), *exist for all n and belong to* $L^\infty(0, T; M_+^\alpha)$. *Moreover, these sequences satisfy*

$$0 \leqq l_0(t) \leqq l_1(t) \leqq \ldots \leqq l_n(t) \leqq \ldots \leqq u_n(t) \leqq \ldots \leqq u_1(t) \leqq u_0(t) \quad \textit{for} \quad 0 \leqq t \leqq T. \tag{5.7}$$

Consequently, $\{l_n(t)\}$ *and* $\{u_n(t)\}$ *converge in* M^α *for every t with* $0 \leqq t \leqq T$.

Proof. The proof is inductive. Suppose $l_1, \ldots, l_{k-1}, u_1 \ldots, u_{k-1}$ all exist, belong to $L^\infty(0, T; M^\alpha)$ and satisfy

$$0 \leqq l_0(t) \leqq \ldots \leqq l_{k-1}(t) \leqq u_{k-1}(t) \leqq \ldots \leqq u_0(t) \quad \text{for} \quad 0 \leqq t \leqq T. \tag{5.8}$$

Notice that the beginning condition is the case $k=2$ of this hypothesis. l_0, l_1 and u_1 all lie in $L^\infty(0, T; M_+^\alpha)$ because of (5.6) and the fact that u_0 lies there.

Because of (Q_2) and (5.8), we have $0 \leqq Q(l_{k-1}(t), l_{k-1}(t)) \leqq Q(u_{k-1}(t), u_{k-1}(t)) \leqq Q(u_0(t), u_0(t)) \leqq Q(m_\alpha, m_\alpha)$. Because of (Q_3), (P_1), and (P_4), then, the right sides of (5.3) and (5.4) lie in $L^\infty(0, T; L_+^1)$. Also, and again because of (5.8), l_{k-1} and u_{k-1} lie in $L^\infty(0, T; M_+^\alpha)$.

Therefore, we can apply Lemma 4.3 to (5.3)–(5.5) with $n=k$ to find l_k and u_k as weak solutions of these problems.

Next, (5.8) in (Q_2) gives $Q(l_{k-2}(t), l_{k-2}(t)) \leqq Q(l_{k-1}(t), l_{k-1}(t)) \leqq Q(u_{k-1}(t), u_{k-1}(t)) \leqq Q(u_{k-2}(t), u_{k-2}(t))$. Therefore, comparing the Equations (5.3) when $n=k$ and $n=k-1$ and using Lemma 4.4, we find $l_{k-1}(t) \leqq l_k(t)$. Similarly, comparing Equations (5.4) when $n=k$ and $n=k-1$ gives $u_k(t) \leqq u_{k-1}(t)$. Finally, comparing (5.3) and (5.4) with $n=k$, we find $l_k(t) \leqq u_k(t)$. Thus, we have

$$l_{k-1}(t) \leqq l_k(t) \leqq u_k(t) \leqq u_{k-1}(t) . \tag{5.9}$$

To complete the induction, it remains to prove l_k and u_k belong to $L^\infty(0, T; M_+^\alpha)$. But this is immediate in view of (5.9) and (5.8).

Finally, we must prove the convergence of $\{l_n(t)\}$ and $\{u_n(t)\}$ in M^α. But, once again, this is immediate because of (5.7) and the hypothesis $u_0 \in L^\infty(0, T; M^\alpha)$. This completes the proof of Lemma 5.1.

Next, we prove

Lemma 5.2. *Let the hypotheses* (P) *and* (Q) *be satisfied. Let* $\phi \in M_+^\alpha$. *Let* (l_0, u_0) *satisfy the beginning condition in* $[0, T]$. *Denote the limits of* $\{l_n(t)\}$ *and* $\{u_n(t)\}$ *by* $l(t)$ *and* $u(t)$. *Then,* $l(t) = u(t)$ *for all* t, $0 \leqq t \leqq T$. *This common limit is a mild solution of the abstract Boltzmann problem.*

Proof. We saw in Lemma 5.1 that $\{l_n(t)\} \uparrow l(t)$, $\{u_n(t)\} \downarrow u(t)$ in M^α for all $t \in [0, T]$. Moreover,

$$0 \leqq l_n(t) \leqq u_n(t) \leqq c m_\alpha, \qquad n = 0, 1, \ldots, \tag{5.10}$$

because of (5.7) and the hypothesis $u_0 \in L^\infty(0, T; M^\alpha)$. Because of the Levi property, it follows from (5.10) that $\{l_n(t)\}$ and $\{u_n(t)\}$ also converge to $l(t)$ and $u(t)$ in L^1.

By (4.8)

$$\|l_n(t)\| + \int_0^t (l_n(\tau), u_{n-1}(\tau))_P \, d\tau = \|\phi\| + \int_0^t \|Q(l_{n-1}(\tau), l_{n-1}(\tau))\| \, d\tau$$

and

$$\|u_n(t)\| + \int_0^t (u_n(\tau), l_{n-1}(\tau))_P \, d\tau = \|\phi\| + \int_0^t \|Q(u_{n-1}(\tau), u_{n-1}(\tau))\| \, d\tau .$$

Because of (P_1), (Q_1), and the bound (5.10), we can let n tend to infinity in these equations to find

$$\|l(t)\| + \int_0^t (l(\tau), u(\tau))_P \, d\tau = \|\phi\| + \int_0^t \|Q(l(\tau), l(\tau))\| \, d\tau \tag{5.11}$$

and

$$\|u(t)\| + \int_0^t (u(\tau), l(\tau))_P \, d\tau = \|\phi\| + \int_0^t \|Q(u(\tau), u(\tau))\| \, d\tau . \tag{5.12}$$

Notice that, because of (5.7), $u(t) \geqq l(t)$. Therefore, $\|u(t)\| - \|l(t)\| = \|u(t) - l(t)\|$. Similarly, because of (5.7), (Q_2) and (Q_1), $\|Q(u(\tau), u(\tau))\| - \|Q(l(\tau), l(\tau))\| = \|Q(u(\tau), u(\tau)) - Q(l(\tau), l(\tau)\| = \|Q(u(\tau) - l(\tau), u(\tau) + l(\tau))\|$. Therefore, subtracting (5.11) from (5.12) and using (P_3) as well as (Q_3), we find

$$\begin{aligned} \|u(t) - l(t)\| &= \int_0^t \|Q(u(\tau) - l(\tau), u(\tau) + l(\tau))\| \, d\tau \\ &\leqq c \int_0^t \|P(u(\tau) - l(\tau), u(\tau) + l(\tau))\| \, d\tau . \end{aligned} \tag{5.13}$$

Now, $0 \leqq l(t) \leqq u(t) \leqq c m_\alpha$ by (5.10). Because of this, (P_2), and (P_4), we have $R(u(t) + l(t)) \leqq c p_\lambda$. Therefore,

$$\|P(u(t) - l(t), u(t) + l(t))\| \leqq c \|p_\lambda (u(t) - l(t))\| .$$

We evaluate the norm appearing here in two parts. First, we integrate over $S_n = V \times \{\xi \in R^3 : |\xi| < n\}$, then we integrate over $\tilde{S}_n = V \times \{\xi \in R^3 : |\xi| \geqq n\}$. In the integration over S_n, we estimate p_λ by

$$|p_\lambda(\xi)| \leqq c(1 + n^\lambda) \quad \text{in} \quad S_n .$$

This estimate gives

$$\|p_\lambda(u(t)-l(t))\|_{L^1(S_n)} \leqq c(1+n^\lambda)\,\|u(t)-l(t)\|\,.$$

In the integration over $\tilde{S}_n$, we use the fact that l and u lie in $L^\infty(0,T;M^\alpha)$. This entails the fact that $0 \leqq u(t)-l(t) \leqq cm_\alpha$. Therefore,

$$\begin{aligned}\|p_\lambda(u(t)-l(t))\|_{L^1(\tilde{S}_n)} &\leqq c \int_{|\xi|>n} (1+|\xi|^\lambda) m_\alpha(\xi)\,d\xi \\ &\leqq ce^{-\frac{\alpha}{2}n^2}\,.\end{aligned}$$

Using these facts in (5.13), we obtain

$$\|u(t)-l(t)\| \leqq c(1+n^\lambda)\int_0^t \|u(\tau)-l(\tau)\|\,d\tau + ce^{-\frac{\alpha}{2}n^2}\,. \tag{5.14}$$

Because of (5.5), $\|u(0)-l(0)\|=0$. It follows from this and (5.14) that $\|u(t)-l(t)\| \leqq ce^{-\frac{\alpha}{2}n^2+ct(1+n^\lambda)}$. But the hypothesis ($P_4$) requires $\lambda<2$. Therefore, letting $n\to\infty$, we find $u(t)=l(t)$.

Let $u(t)=f(t)=l(t)$. It remains to show that f is a mild solution of the Boltzmann problem. $f\in L^\infty(0,T;M_+^\alpha)$, by (5.10). Also, by (5.3)–(5.5),

$$l_n^\#(t)+\int_0^t P^\#(l_n,u_{n-1})(\tau)\,d\tau=\phi+\int_0^t Q^\#(l_{n-1},l_{n-1})(\tau)\,d\tau\,. \tag{5.15}$$

$\{l_n\}$ and $\{u_n\}$ converge to f in $L^\infty(0,T;M_+^\alpha)$, by Lemma 5.1 and (5.10). Sending n to infinity in (5.15), then, we obtain the integrated version of (5.1). It follows that $f^\#\in AC(0,T;L^1)$, and the rest of Lemma 5.2 is easily proved.

Notice that it is the beginning condition that makes the solution local. According to Lemma 5.2, in any interval in which the beginning condition can be satisfied, a solution exists. If the beginning condition can be satisfied globally, then a solution exists globally. It remains, then, to study the beginning condition.

But before this, we want to make a remark about the condition that the volume of V is finite. This hypothesis is needed mainly because m_α, being independent of q, is in L^1 only if V has finite volume. However, this difficulty can be circumvented by replacing m_α by any strictly positive, L^1-function $\mu:(q,\xi)\to\mu(q,\xi)$ defined on $V\times R^3$. Given such a function, we can define a space

$$\Lambda^\mu=\{f\in L^1:|f|\leqq c\mu \qquad \text{for some constant } c\}\,, \tag{5.16}$$

and a sense of convergence like that of M^α on it: $\{f_n\}$ converges in Λ^μ if $\{f_n(q,\xi)\}$ converges almost everywhere and $\{f_n/\mu\}$ is bounded. If, now, one replaces m_α by μ and M^α by Λ^μ throughout the argument, Lemma 5.1 (with M^α replaced by Λ^μ) can still be proved. Moreover, if μ is taken to have the form $\mu(q,\xi)=\beta(q)m_\alpha(\xi)$, where $\beta\in L^1(V)\cap L^\infty(V)$, then Lemma 5.2 can be proved also. Thus, it is entirely because of the beginning condition that the hypothesis on the volume of V is needed.

We now turn to the beginning condition. There are two cases, according to whether the exponent λ appearing in (P_4) is zero or not. If $\lambda=0$, we call the molecular interactions *soft*; otherwise, we call them *hard*. The case of soft interactions is easier. In that case, we take $l_0=0$ and $u_0=c_0 m_\alpha$, where c_0 is a constant. Then, (5.3)–(5.5) give

$$l_1^\#(t)=\phi\exp\left(-\int_0^t R^\#(cm)(\tau)\,d\tau\right),\qquad u_1^\#(t)=\phi+c_0^2 t Q^\#(m_\alpha,m_\alpha). \tag{5.17}$$

It is obvious that $0\leqq l_1(t)\leqq u_1(t)$. All that remains of the beginning condition, therefore, is the requirement $u_1(t)\leqq u_0(t)$. Let ψ^t be the trajectory map, and write $\psi^t(q,\xi)=(q^t(q,\xi),\xi^t(q,\xi))$. If the reflection law is regular, it conserves energy, and this means $|\xi^t(q,\xi)|=|\xi|$. Therefore, $m_\alpha(\xi^t)=m_\alpha(\xi)$. Also, by ($Q_4$), $Q^\#(m_\alpha,m_\alpha)(t)\leqq cP^\#(m_\alpha,m_\alpha)(t)=cm_\alpha(\xi^t)R^\#(m_\alpha)(t)\leqq cm_\alpha(\xi)$ for soft interactions. If $\phi\in M_+^\alpha$, it now follows immediately from (5.17) that $u_1^\#(t)\leqq u_0^\#(t)$ if c_0 is large and t is small enough. Thus, the beginning condition is satisfied. This completes the proof of the following theorem in the case of soft interactions.

Theorem 5.3. *Let $V\subset R^3$ be a domain with C^1-boundary and finite volume. Let A be a regular reflection law. Let the hypotheses* (P) *and* (Q) *be satisfied with $\alpha=\alpha_1>0$. Then, if $0\leqq\phi\leqq cm_{\alpha_0}$ where $0<\alpha_1<\alpha_0$, the initial-boundary value problem for the abstract Boltzmann equation has a mild solution in some interval $0\leqq t\leqq T$, with $T>0$.*

Proof. The theorem follows from Lemma 5.2 once we can show the beginning condition can be satisfied. For this, we take l_0 and u_0 to have the form

$$l_0(t)=0,\, u_0(t)=\beta(t)\,m_{\alpha(t)}.$$

We just saw that, for soft interactions, this choice of l_0 and u_0 leads to the beginning condition even when β and α are constant. In general, however, it is necessary to allow β and α to vary.

Now, however we choose β and α, it is true that $0\leqq l_0(t)\leqq l_1(t)\leqq u_1(t)$, as a trivial computation shows. As before, then, it remains to prove $u_1(t)\leqq u_0(t)$. For this, it suffices to have $u_1(0)\leqq u_0(0)$ and $\dot u_1^\#(t)\leqq \dot u_0^\#(t)$. The first of these conditions is satisfied by taking $\alpha(0)=\alpha_0$ and $\beta(0)$ large enough, since $\phi\leqq cm_{\alpha_0}$. As for the second, we have

$$\dot u_1^\#(t)=Q^\#(u_0,u_0)(t)\leqq cP^\#(u_0,u_0)(t),$$

by (Q_4). On the other hand, $P(u_0,u_0)=\beta^2 m_\alpha\cdot R(m_\alpha)\leqq c\beta^2 m_\alpha\cdot p_\lambda$, where $p_\lambda=1+|\xi|^\lambda$, by ($P_4$). The constant c depends on α, but it is bounded as long as $\alpha\geqq\alpha_1>0$.

Since energy is conserved at reflections, $[c\beta^2(t)\,m_{\alpha(t)}p_\lambda]^\#=c\beta^2(t)\,m_{\alpha(t)}p_\lambda$. For the same reason, $\dot u_0^\#(t)=\dot u_0(t)$. Consequently, $\dot u_1^\#(t)\leqq\dot u_0^\#(t)$ if

$$c\beta^2(t)\,m_{\alpha(t)}p_\lambda\leqq\frac{d}{dt}[\beta(t)\,m_{\alpha(t)}]. \tag{5.18}$$

Written out in full, (5.18) reads

$$c\beta^2(t)(1+|\xi|^\lambda)\leqq\dot\beta(t)-\beta(t)\,\dot\alpha(t)|\xi|^2. \tag{5.19}$$

Choose α and β to have the form

$$\alpha(t)=\alpha_0\left(1-\frac{t}{t_0}\right)^a,\beta(t)=\beta_0\left(1-\frac{t}{t_0}\right)^{-b}.$$

We have the inequality

$$|\xi|^\lambda \leqq \left(1-\frac{\lambda}{2}\right)\frac{1}{\varepsilon}+\frac{\lambda}{2}\varepsilon^{\frac{2}{\lambda}-1}|\xi|^2,$$

where $\varepsilon>0$ is arbitrary. Use this inequality on the left side of (5.19), and choose ε so that the terms in $|\xi|^2$ cancel from both sides of the result. Then, a simple computation shows that (5.19) is satisfied if

$$2(1-b)\geqq \lambda a>0 \tag{5.20}$$

and

$$c\beta_0^2+\left(\frac{c^2\beta_0^{4-\lambda}t_0^\lambda}{a^\lambda a_0^\lambda}\right)^{\frac{1}{2-\lambda}}\leqq\frac{b\beta_0}{t_0}. \tag{5.21}$$

Moreover, β_0 may be chosen freely, since, once β_0 is fixed, and a and b chosen so that (5.20) is satisfied, t_0 can still be chosen small enough that (5.21) holds. Finally, with all the constants chosen, T can be restricted to be so small that $\alpha(t)\geqq\alpha_1$ for $0\leqq t\leqq T$, α_1 being the value of α occurring in the statement of the theorem. The result follows from this.

Note: The referee has pointed out that the proof of Theorem 5.3 implies an inequality of the form

$$f(t,q,\xi)\leqq\beta(t)\exp\{-\alpha(t)|\xi|^2\}$$

and posed the question: what are the best possible functions α and β? We do not know the answer, but it is at least possible that it might provide a condition for a global solution.

6. Another Method

The method of proof of Theorem 5.3 is highly satisfactory, giving, as it does, two explicitly constructible sequences, one pushing down on the solution from above, the other up from below. Still, the beginning condition is awkward and, perhaps, unnatural, since, on the one hand, it appears nowhere in the problem itself, while, on the other, it is the beginning condition that gives Theorem 5.3 its local character. Indeed, at least as long as we satisfy the beginning condition by requiring (as we did in the proof of Theorem 5.3) that $l_0(t)=0$ and $\dot u_1^{\#}(t)\leqq\dot u_0^{\#}(t)$, it is impossible that $u_0(t)$ even exists globally.

In this section, we present another method in which the beginning condition does not appear. This second method seems to be limited to soft interactions, but, on the other hand, the successive approximations to the solution exist for all time. Of course, we only show they converge for a finite time; the question of global convergence remains open.

The method of §5 gives very easily the solution of the nonlinear problem

$$\dot f^{\#}(t)+P^{\#}(f,f)(t)=h^{\#}(t), \tag{6.1}$$

$$f(0)=\phi, \tag{6.2}$$

when $\phi \in M^\alpha_+$ and $h \in L^\infty(0, T; L^1)$. Indeed, one only has to define

$$\dot{l}^\#_n(t) + P^\#(l_n, u_{n-1})(t) = h^\#(t), \tag{6.3}$$

$$\dot{u}^\#_n(t) + P^\#(u_n, l_{n-1})(t) = h^\#(t), \tag{6.4}$$

$$l_n(0) = u_n(0) = \phi. \tag{6.5}$$

The argument of Lemma 5.1 goes through very easily, and we don't need the additional conditions of Lemma 5.2 to prove that $\{l_n(t)\}$ and $\{u_n(t)\}$ converge to the same limit since, in this case, the right sides of (5.11) and (5.12) do not depend on l and u. Therefore, subtraction of (5.11) from (5.12) gives immediately that $\|u(t) - l(t)\| = 0$, instead of (5.13). Moreover, the beginning condition is easily seen to be satisfied in $[0, T]$ when $l_0 = 0, u_0 = cm_\alpha$, if $h \in L^\infty(0, T; M^\alpha_+)$. Therefore, we have

Lemma 6.1. *Let the hypotheses* (P) *be satisfied. If $\phi \in M^\alpha_+$ and $h \in L^\infty(0, T; M^\alpha_+)$, then the initial-boundary value problem* (6.1)–(6.2) *has a mild solution in* $[0, T]$.

Lemma 6.1 allows us to define a map $F: j \to F(j)$ as follows. Take $j \in L^\infty(0, T, M^\alpha_+)$. We define $F(j) = f$ as the mild solution of

$$\dot{f}^\#(t) + P^\#(f, f)(t) = Q^\#(j, j)(t), \tag{6.6}$$

$$f(0) = \phi. \tag{6.7}$$

Notice that (P_4), (Q_2), and (Q_4) (with $\lambda = 0$) give $Q(j, j) \in L^\infty(0, T; M^\alpha)$, so that f exists.

We show first that F maps the convex set $\boldsymbol{K}_{c_1} = \{j \in L^\infty(0, T; M^\alpha_+): j(t) \leqq c_1 m_\alpha\}$ into itself, if T is small enough. For, let $j \in \boldsymbol{K}_{c_1}$. Then, $f(t) \geqq 0$ (since this is so for all the functions l_n). Therefore,

$$\begin{aligned} f^\#(t) &\leqq \phi + \int_0^t Q^\#(j, j)(\tau)\, d\tau \\ &\leqq \phi + c \int_0^t P^\#(m_\alpha, m_\alpha)(\tau)\, d\tau = \phi + ctm_\alpha, \end{aligned}$$

where (Q_2), (Q_4), (P_1), and (P_4) have been used. If $\phi \in M^\alpha_+$, it follows that $f \in \boldsymbol{K}_{c_1}$ if T is small enough.

Next, we show that F is a contraction on $\boldsymbol{K}_{c_1}$. Let $j_1, j_2 \in \boldsymbol{K}_{c_1}$, and let $f_1 = F(j_1)$, $f_2 = F(j_2)$. Write $j = j_1 - j_2$, $f = f_1 - f_2$. Then, using (P_1) and (Q_1), we find

$$\dot{f}^\# + P^\#(f, f_1) + P^\#(f_2, f) = Q^\#(j, j_1 + j_2).$$

Since $f(0) = 0$, it follows that

$$f^\#(t) = \int_0^t \exp\left(-\int_\tau^t R^\#(f_1)(\sigma)\, d\sigma\right) [Q^\#(j, j_1 + j_2)(\tau) - P^\#(f_2, f)(\tau)]\, d\tau.$$

Now, $f_1(t) \geqq 0$. Therefore, $R(f_1)(t) \geqq 0$, and we have

$$|f^\#(t)| \leqq \int_0^t [|Q^\#(j, j_1 + j_2)(\tau)| + |P^\#(f_2, f)(\tau)|]\, d\tau.$$

Thus, integrating and using Lemma 3.1, we find

$$\begin{aligned}\|f(t)\| &\leqq \int_0^t [\|Q(j,j_1+j_2)\| + \|P(f_2,f)\|]\,d\tau \\ &\leqq c\int_0^t [\|P(j,m_\alpha)\| + \|P(f,m_\alpha)\|]\,d\tau \\ &\leqq c\int_0^t [\|j(\tau)\| + \|f(\tau)\|]\,d\tau,\end{aligned}$$

where the hypotheses (Q_3), (P_3), and (P_4) have been used. It follows easily from this that F is a contraction if T is small enough. Thus, we have proved

Theorem 6.2. *Let $V\subset R^3$ be a domain with a C^1-boundary and finite volume. Let A be a regular reflection law. Let the hypotheses* (P) *and* (Q) *be satisfied with $\lambda=0$. If $0\leqq\phi\leqq cm_\alpha$, then the initial-boundary value problem for the abstract Boltzmann equation has a unique mild solution in an interval $0\leqq t\leqq T$, with $T>0$.*

Notice that, starting with any $j_0\in L^\infty(0,\infty;M^\alpha_+)$ (say, even, $j_0=0$), the iterates $j_n=F^n(j_0)$ exist for all $t>0$. They converge for small enough t. Whether they do so for all t remains open.

7. Uniqueness

The solution for soft interactions is automatically unique since, as we saw in §6, the map F is then a contraction. To study the uniqueness question for hard interactions, we return to the method of §5. This allows us to emphasize a feature of that method, which is that certain properties of the solution (in particular, uniqueness) follow directly from the method of proving existence.

In §5, we constructed two sequences, $\{l_n\}$ and $\{u_n\}$, satisfying the equations

$$\dot{l}_n^\# + P^\#(l_n,u_{n-1}) = Q^\#(l_{n-1},l_{n-1}), \tag{7.1}$$

$$\dot{u}_n^\# + P^\#(u_n,l_{n-1}) = Q^\#(u_{n-1},u_{n-1}), \tag{7.2}$$

$$l_n(0)=u_n(0)=\phi, \tag{7.3}$$

and converging to a mild solution f of the Boltzmann equation if $l_0=0$ and $u_0=\beta m_\alpha$, where β and α are two functions, the one increasing like a power of $1-t/t_0$, the other decreasing like a power of $(1-t/t_0)^{-1}$.

Let f_1 be a mild solution of the Boltzmann problem

$$\dot{f}_1^\# + P^\#(f_1,f_1) = Q^\#(f_1,f_1), \tag{7.4}$$

$$f_1(0)=\phi. \tag{7.5}$$

By Lemma 4.1, the linear problem

$$\dot{g}^\# + P^\#(g,f_1) = Q^\#(f_1,f_1), \tag{7.6}$$

$$g(0)=\phi.$$

has a unique mild solution. Since, obviously, f_1 is a solution, it follows that $g=f_1$.

Now, $f_1 \in L^\infty(0, T; M^\alpha_+)$ for some $T>0$. Therefore, there exist functions α_1, β_1, of the sort used in the proof of Theorem 5.3, such that $f_1(t) \leqq \beta_1(t) m_{\alpha_1(t)}$ in a small enough interval $[0, t_0]$. Write $l_0=0, u_0=\beta_1 m_{\alpha_1}$, and define the sequences $\{l_n\}$ and $\{u_n\}$ by (7.1)–(7.3). We have $l_0(t) \leqq f_1(t) \leqq u_0(t)$. Suppose $l_{n-1}(t) \leqq f_1(t) \leqq u_{n-1}(t)$. Comparing (7.1) with (7.6) and using Lemma 4.2, we find $l_n(t) \leqq g(t) = f_1(t)$. Similarly, comparing (7.2) with (7.6), we find $f_1(t) = g(t) \leqq u_n(t)$. Therefore,

$$l_n(t) \leqq f_1(t) \leqq u_n(t), n=0, 1, \ldots.$$

Now, let f_1 and f_2 be two mild solutions of the Boltzmann problem. If we choose α_1 and β_1 such that $0 \leqq f_i(t) \leqq \beta_1(t) m_{\alpha_1(t)}$, $i=1, 2$, the above argument than shows that

$$l_n(t) \leqq f_i(t) \leqq u_n(t), n=0, 1, \ldots.$$

But, according to the results of §5, the sequences $\{l_n(t)\}$ and $\{u_n(t)\}$ have a common limit. Therefore, each of the $f_i(t)$ is equal to this limit, and, therefore, the functions f_i are equal to each other in an interval $[0, t_0]$. The argument can now be repeated to conclude $f_1(t) = f_2(t)$ for $0 \leqq t \leqq T$. Thus, we have

Theorem 7.1. *Let the conditions of Theorem* 5.3 *be satisfied. Then, the mild solution constructed there is unique.*

Notice that Theorem 7.1 provides uniqueness in the space M^α of functions bounded by Maxwellians. It would be desirable, of course, to provide a proof of uniqueness assuming only that the solution has a certain number of moments that are finite, as was done in [5] in the homogeneous case when f is independent of q. At present, however, we are unable to do this. We believe it remains true, though, that Theorem 7.1 is the best yet available when the solution is non-homogeneous and the molecules are restricted to lie in a domain V.

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