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On the Pairing of Polarizations

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Abstract. If F is a positive Lagrangian sub-bundle of a symplectic vector bundle (E, ω) we show by elementary means that the Chern classes of F are determined by ω. The notions of metaplectic structure for (E, ω), metalinear structure for and square root of K^F, the canonical bundle of F are shown to be essentially the same. If F and G are two positive Lagrangian sub-bundles with $F \cap \bar{G} = D^{\mathbb{C}}$, we define a pairing of K^F and K^G into the bundle $\mathscr{D}^{-2}(D)$ of densities of order -2 on D. This is the square of Blattner's half-form pairing and so characterizes the latter up to a sign.

Introduction

In order to construct a Hilbert space in the theory of geometric quantization [4, 6, 7], Kostant [3] introduced the notion of half-form normal to a positive polarization. If two positive polarizations F and G are such that $F \cap \bar{G} = D^{\mathbb{C}}$ is smooth, Blattner [1] showed the existence of a pairing of the half forms normal to F and G into the densities of order -1 on D.

If $F \cap \bar{G} = 0$, $\beta \in \Gamma K^F$, $\gamma \in \Gamma K^G$, K^F, K^G the canonical bundles of F and G, then $\beta \wedge \bar{\gamma}$ is a non-singular pairing of K^F and K^G into the volumes on X. Dividing by the Liouville volume gives a function $\langle \beta, \gamma \rangle_0$. In the general case where $F \cap \bar{G} = D^{\mathbb{C}}$, we observe that F and G project into $D^{\perp}/D$ to give Lagrangian sub-bundles F/D, G/D satisfying $F/D \cap \overline{G/D} = 0$. Thus by dividing out the intersection we can reduce to the case where $F \cap \bar{G} = 0$ and use the exterior product to define a pairing. This pairing is shown to be the square of Blattner's half-form pairing. It is often easier to compute this pairing of the canonical bundles and use continuity arguments to deduce properties of the half-form pairing.

Notation. Let V be a vector space over a field $\mathfrak{k}$, $b = (v_1, \ldots, v_r)$ an r-tuple of elements of V and $A = (A_{ij})$ an $r \times s$ matrix over $\mathfrak{k}$ then $b \cdot A$ will denote the s-tuple with j-th entry $\sum_{i=1}^{r} A_{ij} v_i$. If b_1, b_2 are r- and s-tuples, (b_1, b_2) will denote the $r+s$-tuple obtained in the obvious way. If T is an endomorphism of V and b an r-tuple, Tb will denote the r-tuple obtained by letting T act componentwise.

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If $\Omega: V \times V \to \mathfrak{k}$ is a function, b_1, b_2 r- and s-tuples then $\Omega(b_1, b_2)$ will denote the $r \times s$ matrix with ij-th entry $\Omega(v_i, w_j)$, $b_1 = (v_1, \ldots, v_r)$, $b_2 = (w_1, \ldots, w_s)$. If Ω is bilinear, A an $r \times p$ matrix, B an $s \times q$ matrix then

$$\Omega(b_1 \cdot A, b_2 \cdot B) = A^T \Omega(b_1, b_2) B$$

where A^T denotes the transpose of A. There is an obvious modification if $\mathfrak{k} = \mathbb{C}$ and Ω is anti-linear in one of its arguments.

Let J_n denote the $2n \times 2n$ matrix whose only non-zero entries are $(J_n)_{i,n+i} = 1$, $(J_n)_{n+i,i} = -1$, $i = 1, \ldots, n$. The real symplectic group Sp_n consists of all real $2n \times 2n$ matrices g with $gJ_ng^T = J_n$.

Let U_n be the subgroup of all g in Sp_n with $gJ_n = J_ng$, then U_n is maximal compact in Sp_n and isomorphic to $U(n)$. We shall identify U_n and $U(n)$. Explicitly, if $U + iV \in U(n)$ then

$$\begin{bmatrix} U & -V \\ V & U \end{bmatrix}$$

will be the corresponding element of U_n.

Positive Almost-Complex Structures and Positive Lagrangian Sub-Bundles

Let X be a smooth manifold and (E, ω) a symplectic vector bundle over X. E is a real vector bundle of fibre dimension $2n$ and ω_x a non-singular skew-symmetric bilinear form on E_x for each $x \in X$. A $2n$-tuple b of elements of E_x is called a symplectic frame at x if $\omega_x(b, b) = J_n$. If $g \in \mathrm{Sp}_n$, $b \cdot g$ is again a symplectic frame at x, and the space $B(E, \omega)$ of all symplectic frames of E is a principal Sp_n bundle over X.

Since $U(n)$ is maximal compact in Sp_n, $B(E, \omega)$ can be reduced to a $U(n)$ bundle and in this way Chern classes $c_i(E, \omega) \in H^{2i}(X, \mathbb{Z})$, $i = 1, \ldots, n$ are associated to (E, ω). They are independent of the reduction. A smooth section J of End E will be called a positive, compatible almost-complex structure (PCACS) in (E, ω) if

(i) $J_x^2 = -1$, $\quad \forall x \in X$; and

(ii) $S_x^J(v, w) = \omega_x(v, J_xw)$, $\quad v, w \in E_x$

defines a symmetric, positive definite bilinear form on E_x for each $x \in X$. Then we may define

$$B(E, \omega, J) = \{b \in B(E, \omega) \mid b = (b_1, b_2) \quad \text{with} \quad b_2 = Jb_1\}.$$

$B(E, \omega, J)$ is a reduction of $B(E, \omega)$ to a $U(n)$ bundle and every reduction arises this way. Let E^J denote E regarded as a complex n-dimensional vector bundle by means of the PCACS J. E^J has a Hermitian structure H^J given by

$$H_x^J(v, w) = S_x^J(v, w) - i\omega_x(v, w), \qquad v, w \in E_x^J.$$

If b_1 is an H_x^J-orthonormal frame for E_x^J then $(b_1, Jb_1) \in B(E, \omega, J)_x$ and conversely. Thus $B(E, \omega, J)$ can be identified with the bundle of orthonormal frames for E^J and the $c_i(E, \omega)$ are the Chern classes of the complex vector bundle E^J.

Let $E^{\mathbb{C}}$ denote the complexification of E, extend ω by linearity and let $\bar{v}$ denote complex conjugation in $E^{\mathbb{C}}$. A sub-bundle F of $E^{\mathbb{C}}$ is called a positive Lagrangian sub-bundle (PLS) if

(i) $\dim_{\mathbb{C}} F_x = n, \quad \forall x \in X;$

(ii) $\omega_x(v, w) = 0, \quad \forall v, w \in F_x, \quad x \in X;$

(iii) $-i\omega_x(v, \bar{v}) \geqq 0, \quad \forall v \in F_x, \quad x \in X.$

In addition we say F is positive definite if $-i\omega_x(v, \bar{v}) > 0$ for all non-zero v in F_x, x in X. According to the proof of Lemma 3.11 in [1], if F is a PLS, F is positive definite if and only if $F \cap \bar{F} = 0$, in which case $E^{\mathbb{C}} = F \oplus \bar{F}$ is a direct sum. Further, if F and G are PLS's then $F \cap \bar{G} = \bar{F} \cap G = (F \cap \bar{F}) \cap (G \cap \bar{G})$ and in particular $F \cap \bar{G} = 0$ for all PLS's F if G is positive definite.

If J is a PCACS, extend it to $E^{\mathbb{C}}$ by linearity, then $P^J = \frac{1}{2}(1 - iJ)$ is a field of projections; let $F^J = P^J(E^{\mathbb{C}}) = P^J(E)$. It is easily checked that F^J is a PLS and is positive definite. $P^J : E^J \to F^J$ is a complex linear isomorphism and the kernel of P^J on $E^{\mathbb{C}}$ is $\overline{F^J}$. But $F \cap \overline{F^J} = 0$ for any PLS F by the remarks in the previous paragraph so P^J regarded as a map from F into F^J is also an isomorphism. Thus as complex vector bundles $F \cong F^J \cong E^J$ for any PLS F and PCACS J. In particular the Chern classes of a PLS F are $c_i(E, \omega)$, $i = 1, \dots, n$. Further the Hermitian structure H^J on E^J transports, via the above isomorphisms, to F. It follows that fixing a PCACS J gives an isomorphism of a $U(n)$ bundle of orthonormal frames of F with a $U(n)$-reduction of $B(E, \omega)$.

Let F be a PLS of $E^{\mathbb{C}}$ and F^0 the sub-bundle of the dual $(E^{\mathbb{C}})^*$ of all linear forms vanishing on F. Let $v \mapsto v^\omega = \omega(v, \cdot)$ be the isomorphism of $E^{\mathbb{C}}$ with $(E^{\mathbb{C}})^*$ determined by ω. F is Lagrangian when $F^\omega = F^0$, in particular for a PLS, F, and F^0 are isomorphic. Thus F^0 has dimension n and $K^F = \Lambda^n F^0$ is a line bundle, which we call the canonical bundle of F (denoted N^F by some authors). It follows K^F has $c_1(E, \omega)$ as its Chern class. If $b = (v_1, \dots, v_n)$ is a frame for F_x, set $b^\omega = v_1^\omega \wedge \dots \wedge v_n^\omega$, then b^ω is a frame for K_x^F and for all $g \in \mathrm{GL}(n, \mathbb{C})$, $(b \cdot g)^\omega = \mathrm{Det}[g] b^\omega$.

Square Roots, Metalinear and Metaplectic Structures

The groups Sp_n and $\mathrm{GL}(n, \mathbb{C})$ have the same fundamental group as $U(n)$ which is $\mathbb{Z}$. All three groups have thus unique (up to isomorphism) connected double covering groups Mp_n, $ML(n, \mathbb{C})$, and $MU(n)$ respectively, and $MU(n)$ may be regarded as a maximal compact subgroup of both Mp_n and $ML(n, \mathbb{C})$. $ML(n, \mathbb{C})$ has a unique character $\mathrm{Det}^{1/2}$ such that if $\sigma : ML(n, \mathbb{C}) \to \mathrm{GL}(n, \mathbb{C})$ denotes the covering map,

$$(\mathrm{Det}^{1/2}[g])^2 = \mathrm{Det}[\sigma(g)], \quad g \in ML(n, \mathbb{C}).$$

The reason for introducing these groups is the existence of this square root; see [3].

Let (E, ω) be a symplectic vector bundle of dimension $2n$, $B(E, \omega)$ the Sp_n bundle of symplectic frames and $\pi : B(E, \omega) \to X$ the projection. A metaplectic structure on (E, ω) is an isomorphism class of double coverings $\sigma : \tilde{B} \to B(E, \omega)$ by principal Mp_n-bundles $\tilde{\pi} : \tilde{B} \to X$ such that

$$\begin{array}{ccccc} \tilde{B} \times Mp_n & \longrightarrow & \tilde{B} & \overset{\tilde{\pi}}{\searrow} & \\ {\scriptstyle \sigma\times\sigma}\downarrow & & {\scriptstyle \sigma}\downarrow & & X \\ B(E, \omega) \times \mathrm{Sp}_n & \longrightarrow & B(E, \omega) & \underset{\pi}{\nearrow} & \end{array}$$

commutes, horizontal arrows being given by group actions and where two such coverings $\tilde{B}_1$, $\tilde{B}_2$ are isomorphic if there is a diffeomorphism $\tau:\tilde{B}_1\to\tilde{B}_2$ such that

$$\begin{array}{ccccc} \tilde{B}_1\times Mp_n & \to & \tilde{B}_1 & \xrightarrow{\tilde{\pi}_1} & \\ \tau\times\mathrm{id}\downarrow & & \tau\downarrow & \searrow^{\sigma_1} B(E,\omega)\xrightarrow{\pi} X & \\ \tilde{B}_2\times Mp_n & \to & \tilde{B}_2 & \nearrow_{\sigma_2} \xrightarrow{\tilde{\pi}_2} & \end{array}$$

commutes.

The notion of metalinear structure in a PLS F is defined analogously in terms of double coverings of its frame bundle $B(F)$ by principal $ML(n,\mathbb{C})$ bundles.

Let F have a metalinear structure $\sigma:\tilde{B}\to B(F)$ (it is convenient to work with representatives rather than isomorphism classes). Pick a PCACS J, transfer H^J from E^J to F as before and reduce $B(F)$ to a $U(n)$ bundle $B(F,J)$. Then $\sigma^{-1}(B(F,J))=\tilde{B}^J$ is an $MU(n)$-bundle and hence $\tilde{B}^J\times_{MU(n)}Mp_n$ a principal Mp_n bundle. Using the isomorphisms $B(E,\omega)\cong B(E,\omega,J)\times_{U(n)}\mathrm{Sp}_n$ and $B(F,J)\cong B(E,\omega,J)$ it is easy to exhibit $\tilde{B}^J\times_{MU(n)}Mp_n$ as a double covering of $B(E,\omega)$ giving (E,ω) a metaplectic structure. This sets up a bijection between the metaplectic structures on (E,ω) and metalinear structures on F. The condition for these to exist is $c_1(E,\omega)\equiv 0 \bmod 2$ and $H^1(X,\mathbb{Z}_2)$ acts simply-transitively on both sets of structures. The bijection we have outlined intertwines this action.

If $\pi:K\to X$ is a complex line bundle over X, a square root of K is an isomorphism class of pairs (Q,i) where Q is a line bundle over X and i an isomorphism of $Q\otimes Q$ with K. Two pairs $(Q_r,i_r)r=1,2$ being isomorphic if there is an isomorphism $\tau:Q_1\to Q_2$ of line bundles with $i_2\circ\tau\otimes\tau=i_1$. K has a square root if and only if its Chern class is zero modulo 2, and $H^1(X,\mathbb{Z}_2)$ parametrizes the set of square roots.

Let (E,ω) be a symplectic vector bundle, $F\subset E^{\mathbb{C}}$ a PLS then F has a metalinear structure if and only if K^F has a square root, since both conditions are equivalent to $c_1(E,\omega)\equiv 0 \bmod 2$. In fact there is a bijection of metalinear structures and square roots as follows: We have $K^F\cong B(F)\times_{\mathrm{GL}(n,\mathbb{C})}\mathbb{C}$, $\mathrm{GL}(n,\mathbb{C})$ acting on $\mathbb{C}$ by the character Det, so that if $\sigma:\tilde{B}\to B(F)$ is a metalinear structure, $\tilde{B}\times_{ML(n,\mathbb{C})}\mathbb{C}$ is a square root where $ML(n,\mathbb{C})$ acts on $\mathbb{C}$ by $\mathrm{Det}^{1/2}$. Conversely, let (Q,i) be a square root for K^F and set

$$\tilde{B}=\{(b,q)\in B(F)\times Q|\pi b=\pi q, \quad b^{\omega}=i(q\otimes q)\}.$$

Let $ML(n,\mathbb{C})$ act on $\tilde{B}$ on the right by

$$((b,q),g)\mapsto(b\cdot\sigma(g),\mathrm{Det}^{1/2}[g]q)$$

and define $\sigma:\tilde{B}\to B(F)$ by $\sigma(b,q)=b$, then $\sigma:\tilde{B}\to B(F)$ is a metalinear structure on F. At the level of isomorphism classes this sets up the required bijection.

In [1] Blattner constructs specific metalinear structures and square roots for PLS's F and their canonical bundles K^F starting from a metaplectic structure on (E,ω). This construction is compatible with the bijections described above. We denote this particular square root for K^F by (Q^F,i^F). Corresponding with $F\cong G$ for two PLS's F and G we have $Q^F\cong Q^G$ as line bundles if they arise from the same metaplectic structure on (E,ω). Indeed, one may consider a metaplectic structure on (E,ω) as a consistent assignment of square roots of the canonical bundles of all the

PLS's of $E^{\mathbb{C}}$. This consistency is necessary for the pairing. Sections of Q^F are called half-forms normal to F.

Densities and Pairings

Let GL_n denote $\mathrm{GL}(n,\mathbb{R})$ or $\mathrm{GL}(n,\mathbb{C})$ according to context. By complexifying one need really only consider the complex case. Let D be a real or complex n-dimensional vector bundle and $B(D)$ the principal GL_n bundle of frames of D. Let $\alpha\in\mathbb{R}$, then $\mu: B(D)\to\mathbb{C}$ is an α-density or density of order α on D if

$$\mu(b\cdot g)=|\mathrm{Det}[g]|^{\alpha}\mu(b)\ \forall b\in B(D)\,,\qquad g\in\mathrm{GL}_n\,.$$

Let $\mathscr{D}^{\alpha}(D)$ denote the complex line bundle $B(D)\times_{\mathrm{GL}_n}\mathbb{C}$ where GL_n acts on $\mathbb{C}$ by the character $|\mathrm{Det}[\cdot]|^{\alpha}$. Then the α-densities on D can be identified with the sections of $\mathscr{D}^{\alpha}(D)$. We identify $\mathscr{D}^{\alpha}(D)$ with $\mathscr{D}^{\alpha}(D^{\mathbb{C}})$ in the obvious way for any real bundle D.

If μ is an α-density, v a β-density on D then the pointwise product μv is an $(\alpha+\beta)$-density. If $D\subset E$ is a sub-bundle and $\lambda\in\Gamma\mathscr{D}^1(E)$ a nowhere vanishing section, there is an isomorphism of $\mathscr{D}^{\alpha}(D)$ with $\mathscr{D}^{-\alpha}(E/D)$ given by

$$\tilde{\mu}(\tilde{b})=\mu(e)/\lambda(e,b)$$

where $\mu\in\mathscr{D}^{\alpha}(D)$, $\tilde{b}\in B(E/D)$, $(e,b)\in B(E)$ such that b projects onto $\tilde{b}$. In particular, if (E,ω) is symplectic one can choose the Liouville density λ defined by $\lambda(b)=1$ for b in $B(E,\omega)$. This is consistent since $\mathrm{Det}[g]=1$ for all g in Sp_n.

Let (E,ω) be a sympletic vector bundle over X, F, G PLS's. We shall suppose $F\cap\bar{G}$ has constant dimension and then $F\cap\bar{G}=D^{\mathbb{C}}$ for some real sub-bundle D of E. D is isotropic:

$$\omega_x(v,w)=0\ \forall v,w\in D_x,\, x\in X\,,$$

If $D\subset E$ is any isotropic sub-bundle and

$$D^{\perp}=\{v\in E\,|\,\omega(v,w)=0\ \forall w\in D\}$$

then $D\subset D^{\perp}$ and D is the kernel of the restriction of ω to $D^{\perp}$. Hence there is an induced symplectic structure ω/D on $D^{\perp}/D$ by setting

$$(\omega/D)_x(\tilde{v},\tilde{w})=\omega_x(v,w)$$

where $\tilde{v},\tilde{w}\in(D^{\perp}/D)_x$ and $v,w\in D_x^{\perp}$ project to $\tilde{v},\tilde{w}$ respectively. It may easily be checked that if $F\cap\bar{G}=D^{\mathbb{C}}$, then $F\subset(D^{\perp})^{\mathbb{C}}$, as is G, and F and G project to PLS's F/D, G/D of $(D^{\perp}/D)^{\mathbb{C}}$ where $F/D\cap\overline{G/D}=0$.

If F,G are PLS's of $E^{\mathbb{C}}$, we say F and G are transverse if $F\cap\bar{G}=0$ (this is not consistent with the usual notion of transverse, $E^{\mathbb{C}}=F+G$, but should not cause confusion). In the general case where $F\cap\bar{G}=D^{\mathbb{C}}$, we reduce to the transverse case by passing to the quotient $(D^{\perp}/D,\omega/D)$. Let $F\cap\bar{G}=0$ and $\beta\in K_x^F$, $\gamma\in K_x^G$ then $\beta\wedge\bar{\gamma}\in\Lambda^{2n}(E^{\mathbb{C}})^*$ and is non-zero if β and γ both are, so K^F and K^G are non-singularly paired by $(\beta,\gamma)\mapsto\beta\wedge\bar{\gamma}$.

If $F\cap\bar{G}=D^{\mathbb{C}}$ we need some way of passing from K^F to $K^{F/D}$. Observe that if b is a frame for F_x and $\beta\in K_x^F$ then $\beta=fb^{\omega}$ for some $f\in\mathbb{C}$. If e is a frame for D_x it can always be extended to a frame $(e, b_1)=b$ for F, and $b^\omega = e^\omega \wedge b_1^\omega$. Thus $\beta = e^\omega \wedge \beta_1$ where $\beta_1 = f b_1^\omega$. Moreover b_1 projects to a frame $\tilde{b}_1$ for $(F/D)_x$. We set

$$\tilde{\beta}_e = f \tilde{b}_1^{\omega/D}.$$

One may easily check that $\tilde{\beta}_e$ depends only on e and not the choice of b_1 extending e to a frame of F_x. Clearly $\tilde{\beta}_e \in K_x^{F/D}$ and

$$\tilde{\beta}_{e\cdot g} = \mathrm{Det}[g]^{-1} \tilde{\beta}_e, \quad g \in \mathrm{GL}_k$$

where $k = \dim D$. Since $K^{F/D}$ and $K^{G/D}$ are non-singularly paired we obtain, for each $\beta \in \Gamma K^F$, $\gamma \in \Gamma K^G$ a function $\langle \beta, \gamma \rangle_0$ on $B(D)$ from

$$\langle \beta, \gamma \rangle_0(e) \lambda^{D^\perp/D} = (-1)^{n-k} \tilde{\beta}_e \wedge \bar{\tilde{\gamma}}_e, \quad e \in B(D)$$

where $\lambda^{D^\perp/D}$ denotes the Liouville volume on $D^\perp/D$ ($\lambda^{D^\perp/D}(b)=1$ if $b \in B(D^\perp/D, \omega/D)$). $\langle \cdot, \cdot \rangle_0$ is a nonsingular sesquilinear pairing of $K^F \times K^G$ into $\mathscr{D}^{-2}(D)$. Let $\langle \cdot, \cdot \rangle$ be the corresponding pairing into $\mathscr{D}^2(E/D)$ obtained from $\langle \cdot, \cdot \rangle_0$ using the Liouville density to identify $\mathscr{D}^2(E/D)$ and $\mathscr{D}^{-2}(D)$.

To obtain an explicit formula for $\langle \cdot, \cdot \rangle_0$, choose a frame e for D, an extension (e, b_1) to a frame for F, (e, b_2) to a frame for G with $\omega(b_1, \bar{b}_2)=1$, then $(\tilde{b}_1, \bar{\tilde{b}}_2)$ is a (complex) symplectic frame for $D^\perp/D^{\mathbb{C}}$. Thus

$$\langle \beta, \gamma \rangle_0(e) = (-1)^{n-k} \tilde{\beta}_e \wedge \bar{\tilde{\gamma}}_e(\tilde{b}_1, \bar{\tilde{b}}_2) = \beta(f, \bar{b}_2) \bar{\gamma}(f, b_1) \qquad (*)$$

where $(e, b_1, f, \bar{b}_2)$ is an extension to a (complex) symplectic frame of $E^{\mathbb{C}}$. In [1] Blattner constructs pairings

$$\langle \cdot, \cdot \rangle_0 : Q^F \times Q^G \to \mathscr{D}^{-1}(D), \quad \langle \cdot, \cdot \rangle : Q^F \times Q^G \to \mathscr{D}^1(E/D)$$

and Theorem 3.20 of [1], together with the formula (*) shows

$$\langle i^F(\varphi \otimes \varphi), i^G(\psi \otimes \psi) \rangle_0 = \langle \varphi, \psi \rangle_0^2$$

for $\varphi \in \Gamma Q^F$, $\psi \in \Gamma Q^G$. Thus the pairing of canonical bundles determines the half-form pairing up to a global sign. This may be sufficient in many applications.

Flat Partial Connections

A partial connection is a covariant derivative ∇_ξ defined only for ξ in a sub-bundle F of the tangent bundle. Let F be a sub-bundle of TX or $TX^{\mathbb{C}}$. F is called involutive if $\xi, \eta \in \Gamma F \Rightarrow [\xi, \eta] \in \Gamma F$. If $f \in C^\infty(X)$ let $d^F f$ denote the restriction of df to F. It is a section of F^*, where F^* is the dual bundle of F. Let E be a real or complex (it must be complex if F is) vector bundle. An F-connection in E is a linear map $\nabla : \Gamma E \to \Gamma F^* \otimes E$ with

$$\nabla f s = f \nabla s + d^F f \otimes s$$

for all $f \in C^\infty(X)$, $s \in \Gamma E$. Then for $\xi \in \Gamma F$ one may define ∇_ξ by

$$\nabla_\xi s = (\nabla s)(\xi)$$

regarding $F^* \otimes E$ as $\mathrm{Hom}(F, E)$. The F-connection ∇ is said to be flat if

$$[\nabla_\xi, \nabla_\eta]s = \nabla_{[\xi,\eta]}s$$

for all $\xi, \eta \in \Gamma F$, $s \in \Gamma E$. Some properties of flat F-connections in line bundles are studied in [5].

The Lie derivative in any bundle associated to the frame bundle of the normal bundle of F defines a flat F-connection. In the case F is real and integrable this is the flat connection along the leaves due to Bott. Two special cases in the symplectic situation are $D \subset TX$, isotropic and $F \subset TX^{\mathbb{C}}$, a PLS on a symplectic manifold (X, ω). Then $\mathscr{D}^\alpha(TX/D)$ has a flat D-connection. Since $F^0 \cong (TX^{\mathbb{C}}/F)^*$, K^F has a flat F-connection. But ΓK^F consists of differential forms so the Lie derivative is given by

$$\theta(\xi) = i(\xi) \cdot d + d \circ i(\xi), \quad \xi \in \Gamma F .$$

However for $\beta \in \Gamma K^F$, $i(\xi)\beta = 0$ so that

$$\nabla_\xi \beta = i(\xi) d\beta, \quad \xi \in \Gamma F$$

defines the natural flat F-connection in K^F.

Let (X, ω) be a symplectic manifold, then a positive polarization is an involutive PLS $F \subset TX^{\mathbb{C}}$. If (TX, ω) admits a metaplectic structure, we have the square root (Q^F, i^F) of K^F. As observed by Gawedzki in [2], a flat F-connection ∇ in K^F induces a unique flat F-connection $\nabla^{1/2}$ in Q^F such that

$$\nabla_\xi i^F(\varphi \otimes \psi) = i^F(\nabla_\xi^{1/2} \varphi \otimes \psi + \varphi \otimes \nabla_\xi^{1/2} \psi), \quad \xi \in \Gamma F , \quad \varphi, \psi \in \Gamma Q^F .$$

Let $D \subset TX$ be isotropic, and $D^\perp \subset TX$ its orthogonal complement with respect to ω as before, so that $D \subset D^\perp$. If $\dim D = k$, $\dim X = 2n$ then $\dim D^\perp/D = 2(n-k)$. For $x \in X$ choose any neighbourhood U with a $2(n-k)$-tuple b of vector fields in $D^\perp$ on U with $\omega(b, b) = J_{n-k}$ at each point of U. Then b spans a complement of D in $D^\perp$ on U and projects to a symplectic frame field for $(D^\perp/D, \omega/D)$ on U. Writing $b = (v_1, \ldots, v_{n-k}, w_1, \ldots, w_{n-k})$ we define

$$\theta_x^D(v) = \textstyle\sum_{j=1}^{n-k} \omega_x([v_j, w_j], v) , \qquad v \in D_x .$$

Then θ_x^D is independent of the choice of frame-field b with the above properties and defines a smooth section of D^*. This is Blattner's obstruction to projecting the half-form pairing to X/D. One may compute

$$\nabla_\xi \langle \beta, \gamma \rangle = \langle \nabla_\xi \beta, \gamma \rangle + \langle \beta, \nabla_\xi \gamma \rangle - \theta^D(\xi) \langle \beta, \gamma \rangle$$

where $\beta \in \Gamma K^F$, $\gamma \in \Gamma K^G$, $F \cap \bar{G} = D^{\mathbb{C}}$, $\xi \in \Gamma D$. We thus obtain the obstruction at the canonical bundle level.

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