Commun. math. Phys. 56, 277—279 (1977)

Communications in
Mathematical
Physics

A Comment on the Symmetries of Kerr Black Holes

C. D. Collinson and P. N. Smith

Department of Applied Mathematics, The University, Hull HU6 7RX, England

Abstract. Conditions are given for the linear dependence of the two Killing vectors, found by Hughston and Sommers to exist in a class of Einstein-Maxwell fields of Petrov type D. The Killing tensors associated with these fields are shown to be contracted products of Killing Yano tensors.

1. Introduction

In an elegant paper Hughston and Sommers [1] have shown how the symmetries of the Kerr black hole can be inferred by an argument centering around properties of the curvature tensor. They also show that an inference of this sort holds for the class of Petrov type D Einstein-Maxwell fields for which the electromagnetic field is non null with principal rays aligned with those of the Weyl tensor. If spinors o_A, i_A with $o_A i^A = 1$ are chosen so that the null vectors $o_A \bar{o}_{A'}$ and $i_A \bar{i}_{A'}$ are in the direction of these principal rays then the Maxwell spinor ϕ_{AB} can be written in terms of a single scalar field γ as

$$\phi_{AB} = \gamma^{2/3} o_{(A} i_{B)} . \tag{1.1}$$

Two further spinor fields X_{AB} and γ_{ABCD} are defined by

$$X_{AB} = \gamma^{-1/3} o_{(A} i_{B)} \tag{1.2}$$

and

$$\gamma_{ABCD} = \gamma o_{(A} o_B i_C i_{D)} . \tag{1.3}$$

These three fields satisfy the Maxwell equation, twistor equation and spin two rest mass field equation respectively, i.e.

$$\nabla^A{}_{A'} \phi_{AB} = 0 \tag{1.4}$$

$$\nabla^{A'}{}_{(A} X_{BC)} = 0 \tag{1.5}$$

and

$$\nabla^A{}_{A'} \gamma_{ABCD} = 0 . \tag{1.6}$$

Using only the twistor Equation (1.5) and the Ricci identity Hughston and Sommers show that the spinor $\xi_{A'A}=\nabla^{B}{}_{A'}X_{BA}$ corresponds to a complex Killing vector ξ_a. In general two real Killing vectors can be constructed from ξ_a and these are linearly dependent when ξ_a is degenerate, that is when X_{AB} can be renormalized with a constant phase factor to give $\xi_a=\bar{\xi}_a$. Whenever degeneracy occurs then a Killing tensor K_{ab} is found to exist and then Hughston and Sommers show that the vector

$$\eta_a=K_a{}^b\xi^b$$

is also a Killing vector. The two vectors ξ_a and η_a generate the axial symmetry and stationary nature of the charged Kerr black hole. The question of the possible linear dependence of the two vectors, not discussed by Hughston and Sommers, is discussed in Section 2. In Section 3 it is shown that the degeneracy of ξ_a implies the existence of a Killing-Yano tensor from which the above Killing tensor can be constructed.

2. Conditions for the Linear Dependence of the Two Killing Vectors

Written in terms of a null tetrad [2] constructed from the spinors o_A and i_A

$$\xi_a=\gamma^{-1/3}[-\mu l_a+\varrho n_a+\tau\bar{m}_a+\pi m_a]\,. \tag{2.1}$$

It follows that ξ_a is degenerate if and only if

$$\gamma^{-1/3}\mu=e^{2ic}\bar{\gamma}^{-1/3}\bar{\mu},\ \gamma^{-1/3}\varrho=e^{2ic}\bar{\gamma}^{-1/3}\varrho,\ \gamma^{-1/3}\tau=e^{2ic}\bar{\gamma}^{-1/3}\bar{\pi}\,, \tag{2.2}$$

for some real function c. When these conditions are satisfied

$$K_{ab}=(\gamma\bar{\gamma})^{-1/3}l_{(a}n_{b)}-\tfrac{1}{8}(e^{-ic}\gamma^{-1/3}+e^{ic}\bar{\gamma}^{-1/3})^2 g_{ab} \tag{2.3}$$

and

$$\begin{aligned}\eta_a=K_{ab}\xi^b= & -\tfrac{1}{8}(e^{-ic}\gamma^{-1/3}-e^{ic}\bar{\gamma}^{-1/3})^2\bar{\gamma}^{1/3}(-\mu l_a+\varrho n_a)\\ & -\tfrac{1}{8}(e^{-ic}\gamma^{-1/3}+e^{ic}\bar{\gamma}^{-1/3})^2\gamma^{-1/3}(\tau\bar{m}_a+\pi m_a)\,.\end{aligned} \tag{2.4}$$

From (2.1) and (2.4) the following Theorem can be deduced.

Theorem. *The vectors η_a and ξ_a are linearly dependent (i.e. $\eta_a=\lambda\xi_a$ where because both vectors are Killing vectors the scalar λ is necessarily a constant) if and only if either*

(i) $\varrho=\mu=0$ or (ii) $\tau=\pi=0$.

If both Conditions (i) and (ii) hold then ξ_a vanishes. Furthermore using (2.2) it can be seen that η_a vanishes in Case (i) if and only if $\tau+\bar{\pi}=0$ and in Case (ii) if and only if $\varrho=\bar{\varrho}$ and $\mu=\bar{\mu}$. These results have a direct geometrical interpretation in terms of the geometry of the congruencies defined by the two principal rays.

Using the classification due to Kinnersley [3] it follows that, for empty space-times, ξ_a must be non-zero, η_a and ξ_a being linearly dependent if and only if the space-time corresponds to Case IV or Case I (the N.U.T. space-times [4]). In both cases further Killing vectors are known to exist. The vector η_a vanishes if and only if the space-time is of Kinnersley Case IV with $a=0$ or a N.U.T. space-time with $\varrho_0=0$. The last class includes the Schwarzchild space-time so that the result given here generalizes a footnote appearing in the paper of Hughston and Sommers.

3. The Existence of a Killing-Yano Tensor

One of the physically important properties of Killing vectors K_a and Killing tensors K_{ab} is that they generate scalar fields $K_a v^a$ and $K_{ab}v^a v^b$ which remain constant along a given congruence of geodesics with tangent vector v^a. The vector field $F_{ab}v^b$ remains covariantly constant along all geodesics if and only if

$$F_{a(b;c)}=0\,. \tag{3.1}$$

A skew symmetric tensor satisfying (3.1) is called a Killing-Yano tensor. Notice that the magnitude of a covariantly constant vector field is a constant scalar field so that the "square" of a Killing-Yano tensor is a Killing tensor. The relationship between such tensors is given by Collinson [5].

A Killing-Yano tensor F_{ab} corresponds to a symmetric spinor f_{AB} satisfying the two equations

$$\nabla_{(A}{}^{A'} f_{BC)}=0 \tag{3.2}$$

and

$$\nabla^B{}_{A'} f_{BA}+\nabla_A{}^{B'} \bar{f}_{B'A'}=0\,. \tag{3.3}$$

Notice that (3.2) is just the twistor equation and that if ξ_a is degenerate and X_{AB} is renormalized to give $\xi_a=\bar{\xi}_a$ then $f_{AB}=iX_{AB}$ satisfies both equations. Thus a Killing-Yano tensor exists and it is easy to show that the Killing tensor found by Hughston and Sommers is just the square of this Killing-Yano tensor.

Acknowledgements. One of the authors (P.N.S.) gratefully acknowledges the support of the S.R.C.

References

1. Hughston, L.P., Sommers, P.: Commun. math. Phys. **33**, 129—133 (1973)
2. Newman, E., Penrose, R.: J. Math. Phys. **3**, 565—578 (1962)
3. Kinnersley, W.: J. Math. Phys. **10**, 1195—1203 (1969)
4. Newman, E., Tanburino, L., Unti, T.: J. Math. Phys. **4**, 915—923 (1963)
5. Collinson, C.D.: Int. J. Theor. Phys. **15**, 311—314 (1976)

Communicated by R. Geroch

Received June 16, 1977