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On Symmetric Gauge Fields

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Abstract. The subgroups of the symmetry group of the gauge invariant Lagrangian are studied. For given subgroup G the G-invariant gauge fields are listed.

Let $F(\varphi)$ be a G-invariant functional and let H be a subgroup of the symmetry group G. It is easy to prove under certain conditions that every extremal of the functional $F(\varphi)$ considered only in the H-invariant fields is an extremal of this functional on all fields (see for instance [1]). This assertion can be used to search solutions of classical field equations especially in gauge theories. In these theories the functionals under consideration are invariant with respect to the group R generated by local gauge transformations and spatial symmetries. To apply the assertion above one must find the subgroups of the group R and for given subgroup $G \subset R$ one must find all G-invariant fields. In present paper we solve these two problems. Some results in this direction were obtained earlier by Burlankov [2] and used in [9].

To facilitate the reading to physicists we have divided the paper in two sections. The considerations of Section 1 used only notions familiar to physicists but in Section 2 we use the geometrical language of fibre space theory (see for instance [3]).

All manifolds and all maps under consideration will be supposed smooth.

Section 1

We denote by O the group of spatial symmetries. (This group acts on a manifold M; in physical applications usually M is three-dimensional or four-dimensional euclidean space.) The group of local gauge transformations will be denoted by K_∞ and the group generated by K_∞ and O will be denoted by R. The group K_∞ can be identified with the group of smooth functions on M taking values in the gauge group K. The group R can be considered as the group of pairs $(k(x), g)$ where $k(x) \in K_\infty$, $g \in G$ and the product of pairs $(k_1(x), g_1) \in R$, $(k_2(x), g_2) \in R$ is a pair $(k(x), g)$

given by formulae

$$k(x)=k_1(x)k_2(g_1^{-1}x)$$
$$g=g_1g_2\,. \tag{1}$$

The groups K_∞ and O can be considered as subgroups of R consisting of pairs having the form $(k(x),1)$ and $(1,g)$ respectively.

We shall study homomorphisms of fixed group G into the group R. For simplicity we assume that G is a compact connected Lie group. Let τ be such a homomorphism. This homomorphism transforms an element $g\in G$ into the pair $(\tau_g(x), \alpha(g))\in R$. It follows from (1) that α is a homomorphism of G into O and $\tau_g(x)$ satisfies

$$\tau_{g_1g_2}(x)=\tau_{g_1}(x)\tau_{g_2}(g_1^{-1}x)\,. \tag{2}$$

Two homomorphisms τ_1 and τ_2 are called gauge equivalent if $\tau_2=k\tau_1k^{-1}$ where $k(x)\in K_\infty$. It is easy to see that corresponding homomorphisms of G into O coincide: $\alpha_1=\alpha_2=\alpha$. The functions $\tau_g^{(1)}(x)$ and $\tau_g^{(2)}(x)$ are related by formula

$$\tau_g^{(2)}(x)=k(x)\tau_g^{(1)}(x)k^{-1}(\alpha(g^{-1})x)\,. \tag{3}$$

To list the homomorphisms of the group G into R we must solve the Equation (2). Let us fixe the homomorphism α of G into O and the point $x_0\in M$. It is convenient to study (2) at first assuming that x in (2) run over the set $N(x_0)$ of points having the form $\alpha(g)x_0$ where $g\in G$ (in other words we consider this equation on each orbit of the group $\alpha(G)$ separately). It is evident that (2) permits to express $\tau_g(x)$ for all $x\in N(x_0)$ through $\tau_g(x_0)$; namely

$$\tau_g(x)=\tau_{g_1}^{-1}(x_0)\tau_{g_1g}(x_0) \tag{4}$$

where g_1 satisfies $x=\alpha(g_1^{-1})x_0$.

Let $H(x_0)$ denote the set of elements $h\in G$ satisfying $\alpha(h)x_0=x_0$ (the isotropy subgroup at point $x_0\in M$). One can check easily that $\tau_g(x)$ defined by (4) satisfies (2) if and only if the function $\mu(g)=\tau_g(x_0)$ satisfies

$$\mu(hg)=\mu(h)\mu(g) \tag{5}$$

for every $g\in G$, $h\in H(x_0)$. It is sufficient therefore to study the Equation (5). Let $\mathscr{G}$ denote the Lie algebra of the Lie group G, and $\mathscr{H}$ denote the subalgebra of $\mathscr{G}$ corresponding to the subgroup $H=H(x_0)$. The orthogonal complement of $\mathscr{H}$ in $\mathscr{G}$ will be denoted by $\mathscr{V}$ and the set of elements of G having the form $\exp(v)$ where $v\in\mathscr{V}$ will be denoted by V. There exists such a neighbourhood U of unity in G that all elements $g\in U$ have unique representation in the form $g=h(g)v(g)$ where $h(g)\in H$, $v(g)\in V$. It is easy to find the general solution of Equation (5) in U; namely

$$\mu(g)=\lambda(h(g))\sigma(v(g)) \tag{6}$$

where $\lambda(h)$ is an arbitrary homomorphism of H into K and $\sigma(v)$ is an arbitrary function. [It follows from (5) that μ considered on H is a homomorphism and $\mu(g)=\mu(h(g))\mu(v(g))$.] The proof that (6) satisfies (5) is straightforward. The appearance of an arbitrary function $\sigma(v)$ is a consequence of gauge invariance; it follows from

(3) that the solutions of (5) are gauge equivalent if and only if corresponding homomorphisms λ_1, λ_2 are conjugate, i.e. $\lambda_2 = k\lambda_1 k^{-1}$, $k \in K$ (see Section 2 for details).

We have solved the Equation (5) only locally. In Section 2 we shall see that in the case when $\mu(g)$ is a solution of (5) which is continuous on G corresponding homomorphism $\lambda(h)$ of H into K must satisfy certain topological conditions. Here we consider only the case $G = SO(3)$, $H = SO(2)$ we prove that in this case the homomorphisms λ and λ^{-1} must be homotopic (two maps are called homotopic if one can connect they by continuous family of maps). To check this assertion one can use the parametrization of $SO(3)$ by means of Euler angles φ_1, θ, φ_2; in this parametrization $\mu(\varphi_1, \theta, \varphi_2) = \lambda(\varphi_1)\sigma(\theta, \varphi_2)$ and the function μ is continuous on $SO(3)$ if and only if λ and σ are continuous, $\sigma(0, \varphi) = \lambda(\varphi)$, $\sigma(\pi, \varphi) = \lambda(-\varphi)$. Hence the function $\sigma(\theta, \varphi)$ can be considered as a continuous deformation connecting $\lambda(\varphi)$ and $\lambda(-\varphi)$.

We consider now the G'-invariant Yang-Mills fields $A_\mu(x)$ where $G' = \tau(G)$ is the image of G by the homomorphism τ of G into R. (The description of G'-invariant fields of matter is easier and we shall omit it.) It is evident that the G'-invariant field $A_\mu(x)$ in all points of the orbit $N(x_0)$ can be expressed through $A_\mu(x_0)$ and $A_\mu(x_0)$ must satisfy the condition

$$A_\mu(x_0) = \tilde{h}_\mu^\nu(\tau_h^{-1}(x_0)A_\nu(x_0)\tau_h(x_0) - \tau_h^{-1}(x_0)\partial_\mu\tau_h(x_0)) \tag{7}$$

where $\tilde{h}_\mu^\nu$ is the Jacobian matrix of transformation $\alpha(h)$ at point x_0, $h \in H$. To describe the G'-invariant fields we must find all $A_\mu(x_0)$ satisfying (7). At first we shall eliminate the last term in (7) by means of the gauge transformation (this is possible because this term vanishes in the case when $d\sigma(v) = 0$ for $v = 1$). Let us define two representations of the group H by means of formulae

$$\Lambda_h a = \lambda(h) a \lambda^{-1}(h)$$

$$\Gamma_h b^\mu = \tilde{h}_\nu^\mu b^\nu$$

[here $\lambda(h) = \tau_h(x_0)$, the representation Λ_h acts on the Lie algebra $\mathscr{K}$ of the group K and Γ_h acts on the tangent space $\mathscr{M}_{x_0}$ to manifold M at the point x_0]. One can consider $A_\mu(x_0)$ as a linear operator $\hat{A}$ mapping $\mathscr{M}(x_0)$ into $\mathscr{K}$; the equality (7) can be written in the form $\Lambda_h \hat{A} = \hat{A}\Gamma_h$ (i.e. the operator $\hat{A}$ intertwine the representations Λ_h and Γ_h). Now all Yang-Mills fields satisfying (7) can be listed by means of Schur's lemma.

We have considered above the Equation (2) on the fixed orbit $N(x_0)$. Our considerations can be immediately generalized to study this equation on a $\alpha(G)$-invariant set M_1 if the isotropy group $H(b)$ depends continuously on $b \in M_1$ and the set of orbits $M_1/\alpha(G)$ is homeomorphic to the convex subset T of euclidean space. In this case one can select continuously one point of each orbit in such a way that isotropy groups in all these points coincide; in other words there exists such a map q of T into M_1 that $H(q(t))$ does not depend on $t \in T$ and $pq(t) = t$ where p is the natural projection of M_1 onto $M_1/\alpha(G) = T$ (see [4]). The solution of (2) on M_1 can be written in the form

$$\tau_g(x) = \mu^{-1}(g_1(x))\mu(g_1(x)g)$$

where $g_1(x)$ satisfies $\alpha(g_1(x))x \in q(T)$ and $\mu(g)$ is a solution of (5). The G'-invariant Yang-Mills field can be expressed through $A_\mu(q(t))$, $t \in T$ and the possible values of $A_\mu(q(t))$ can be listed by means of Schur's lemma. It is important to note that all gauge invariant quantities can be expressed through $A_\mu(q(t))$, $t \in T$ if M_1 is an open subset of M having full measure (i.e. $M \setminus M_1$ is a set of measure zero). Such a choice of M_1 is always possible (see [4]). It is convenient therefore to consider the fields only on M_1 where the invariant fields can be described completely (the fields on M_1 are in general discontinuous on M but this is not essential if the basic physical quantities are finite).

Let us indicate now how our considerations can be used to study spherically symmetric fields in three-dimensional euclidean space E^3. In this case $G = SO(3)$ or $G = SU(2)$ and α is an identity map $SO(3) \to SO(3)$ or a covering map $SU(2) \to SO(3)$. If we delete the origin of coordinates from E^3 we obtain a manifold $M_1 = E^3 \setminus \{0\}$. The set $q(T)$ where T is a ray $0 < t < \infty$, $q(t) = (0, 0, t)$ intersects each orbit of $\alpha(G) = SO(3)$ in a unique point and $h = H(q(t))$ does not depend on t [namely $H = SO(2)$ or $H = U(1)$]. The type of spherical symmetry is characterized up to gauge equivalence by a homomorphism λ of H into K. The possible values of spherically symmetric field on positive z-axis can be described by means of Schur's lemma and gauge invariant quantities can be expressed through these values.

Let us consider for example the case $K = SU(n)$; then every homomorphism of H into K is conjugate to the homomorphism having the form $\lambda(\varphi) = \exp(im_a\varphi)\delta^{ab}$ where in the case $G = SO(3)$, $H = SO(2)$ the Euler angle φ runs over interval $[0; 2\pi]$ and m_a are integers [in the case $G = SU(2)$, $H = U(1)$ the angle φ satisfies $0 \leqq \varphi \leqq 4\pi$ and $2m_a$ are integers].

The possible values of spherically symmetric Yang-Mills field A_μ on the z-axis are antihermitian matrices satisfying $A_1^{ab} = 0$ if $|m_a - m_b| \neq 1$, $A_2^{ab} = i\,\mathrm{sgn}(m_a - m_b)A_1^{ab}$, $A_3^{ab} = 0$ if $m_a \neq m_b$. If the scalar fields $\phi = (\phi^{ab})$ transform according adjoint representation of $SU(n)$ then on the z-axis the spherically symmetric fields obey $\phi^{ab} = 0$ if $m_a \neq m_b$. [In general if the scalar fields $\phi(\phi^1, \ldots, \phi^n)$ transform according the representation ϱ of the group K the values of spherically symmetric fields on the z-axis satisfy $\varrho(\lambda(h))\phi = \phi$ for every $h \in H$.]

One can prove that the energy of spherically symmetric Yang-Mills field in euclidean space with usual metrics can be finite only in the case when the homomorphism λ can be extended to the homomorphism of G into K but every spherically symmetric Yang-Mills field has finite energy in some spherically symmetric metrics in E^3. The proof of these assertions will be given in a separate paper [5]. This paper contains also an analysis of spherically symmetric solutions of field equations; in particular the solutions having magnetic charge are studied.

Section 2

Let us consider a manifold E and a compact connected Lie group K acting on E on the right. If $ek \neq e$ for every $e \in E$, $k \in K$, $k \neq 1$ this action determines a principal fibration $\xi(E, M, K, p)$ with the space $E = E^\xi$, the group K, the base $M = E/K$ and the projection p. It is well known that Yang-Mills fields can be considered as connections in principal fibrations. (It is sufficient usually to regard Yang-Mills fields as connections in trivial fibration but in some questions non-trivial fibrations occur; see for instance [6, 7].) A map φ of E onto E will be called an automorphism of principal fibration if $\varphi(e)k=\varphi(ek)$ for every $e\in E$, $k\in K$. Each automorphism φ determines a transformation of the base $M=E/K$; we shall denote this transformation by $\pi(\varphi)$. The group of automorphisms of the principal fibration ξ determining an identity transformation of the base will be denoted by K_∞^ξ; the group of automorphisms satisfying $\pi(\varphi)\in O$ where O is a fixed group acting on M will be denoted by R^ξ. It is easy to verify that in the case of trivial fibration the groups K_∞^ξ and R^ξ coincide with the groups K_∞ and R defined in Section 1.

We shall fixe a compact connected Lie group G and a homomorphism α of G into O. The set of homomorphisms $\tau: G\to R$ satisfying $\pi(\tau(g))=\alpha(g)$ for every $g\in G$ will be denoted by $\mathscr{A}(\xi,\alpha)$. A homomorphism $\tau\in\mathscr{A}(\xi,\alpha)$ determines an action of the group $G\times K$ onto E^ξ; namely the transformation corresponding to a pair (g,k) maps $e\in E$ into $\tau(g^{-1})ek$. The isotropy group of this action at $e\in E$ [i.e., the set of pairs $(g,k)\in G\times K$ satisfying $\tau(g^{-1})ek=e$] can be described as set of pairs $(h,\lambda(h))$ where $h\in H(p(e))$ and $\lambda=\lambda_{\tau,e}$ is a homomorphism of $H(p(e))$ into K depending on τ and e [here $H(b)$ is the isotropy group at $b\in M$ i.e. the set of elements $h\in G$ satisfying $\alpha(h)b=b$].

If $p(e)=p(e_1)$ then the homomorphisms $\lambda=\lambda_{\tau,e}$ and $\lambda_1=\lambda_{\tau,e_1}$ are conjugate; really $\lambda_1=k^{-1}\lambda k$ where $k\in K$ satisfies $e_1=ek$. If the homomorphism $\tau'\in\mathscr{A}(\xi,\alpha)$ is gauge equivalent to the homomorphism $\tau\in\mathscr{A}(\xi,\alpha)$ where $\gamma\in K_\infty^\xi$, then the homomorphisms $\lambda=\lambda_{\tau',e}$ and $\lambda=\lambda_{\tau,e}$ are conjugate. [To check this assertion one must note that $\lambda_{\tau',e}=\lambda_{\tau,\gamma e}$ and $p(e)=p(\gamma e)$.]

We consider firstly the simplest case when the group $\alpha(G)$ acts on M transitively.

Theorem 1. *If $b\in M$ and λ is a homomorphism of $H(b)$ into K then there exist a principal fibration $\xi(\lambda)$ $(E(\lambda), M, K, p(\lambda))$ and a homomorphism $\tau_\lambda\in\mathscr{A}(\xi(\lambda),\alpha)$ satisfying $\lambda=\lambda_{\tau_\lambda,e}$ for certain $e\in E(\lambda)$.*

To construct $\xi(\lambda)$ we define the action of $H(b)$ on $G\times K$ assuming that $h\in H(b)$ transforms $(g,k)\in G\times K$ into $(hg,\lambda(h)k)$. The coset space of $G\times K$ with respect to this action will be denoted by $E(\lambda)$ and the identification map of $G\times K$ onto $E(\lambda)$ will be denoted by ψ. The right action of K on $G\times K$ induces the right action of K on $E(\lambda)$ which determines a principal fibration $\xi(\lambda)$ $(E(\lambda)$, M, K, $p(\lambda))$. For every $g\in G$ we define $\tau_\lambda(g)$ as an automorphism of $\xi(\lambda)$ satisfying $\tau_\lambda(g)\psi(\gamma,k)=\psi(\gamma g^{-1},k)$ for all $(\gamma,k)\in G\times K$. It is evident that $\tau_\lambda\in\mathscr{A}(\xi(\lambda),\alpha)$ and $\lambda=\lambda_{\tau_\lambda,e}$ where $e=\psi(1,1)$.

Theorem 2. *The homomorphism $\tau\in\mathscr{A}(\xi,\alpha)$ is gauge equivalent to $\tau'\in\mathscr{A}(\xi,\alpha)$ if and only if the homomorphisms $\lambda=\lambda_{\tau,e}$ and $\lambda'=\lambda_{\tau',e}$ are conjugate for some $e\in E^\xi$.*

If $\tau\in\mathscr{A}(\xi,\alpha)$ and $e\in E^\xi$ we can construct a map $v=v_{\tau,e}$ of $G\times K$ onto E^ξ by means of formula $v(g,k)=\tau(g^{-1})ek$. It is evident that $v(hg,\lambda(h)k)=\tau(g^{-1})\tau(h^{-1})e\lambda(h)k$ $=v(g,k)$ and therefore v determines a map $\hat{v}$ of $E(\lambda)$ onto E^ξ. The map $\hat{v}$ commutes with action of K on $E(\lambda)$ and E^ξ and hence it can be considered as an isomorphism of principal fibrations $\xi(\lambda)$ and ξ. It is evident that $\tau(g)\hat{v}=\hat{v}\tau_\lambda(g)$.

Let us consider homomorphisms $\tau\in\mathscr{A}(\xi,\alpha)$ and $\tau'\in\mathscr{A}(\xi,\alpha)$ generating conjugate homomorphisms $\lambda=\lambda_{\tau,e}$ and $\lambda'=\lambda_{\tau',e}$. If $\lambda'=k^{-1}\lambda k$ then $\lambda_{\tau,e}=\lambda_{\tau',e'}$, where $e'=ek$. Now the element $\gamma\in K_\infty^\xi$ satisfying $\tau'=\gamma\tau\gamma^{-1}$ can be obtained by means of formula $\gamma=\hat{v}'\hat{v}^{-1}$ where $\hat{v}=\hat{v}_{\tau,e}$ and $\hat{v}'=\hat{v}_{\tau',e'}$ are isomorphisms of $\xi(\lambda)$ and ξ constructed above. (The isomorphisms $\hat{v}$ and $\hat{v}'$ induce the identity map of the base M hence

$\gamma \in K^{\xi}_{\infty}$.) We see that τ and τ' are gauge equivalent; this proves one of assertions of Theorem 2. The second assertion was proved earlier.

Theorem 3. *The homomorphism λ of $H(b)$ into K can be represented in the form $\lambda_{\tau,e}$ where τ is a homomorphism of G into R satisfying $\pi(\tau(g)) = \alpha(g)$ if and only if the composition $\varrho_\lambda \sigma$ of the maps $\sigma: M \to B_H$ and $\varrho_\lambda: B_H \to B_K$ is homotopic to zero [here $\sigma: M \to B_H$ is a classifying map of the principal fibration $(G, H, G/H)$ and the map ϱ_λ is induced by the homomorphism $\lambda: H \to K$].*

We have mentioned above that the group R is isometric to the group $R^{\xi(0)}$ where $\xi(0)$ is a trivial fibration. It follows immediately from Theorems 1 and 2 that $\lambda = \lambda_{\tau,e}$ where $\tau \in \mathscr{A}(\xi(0), \alpha)$ if and only if the fibration $\xi(\lambda)$ is trivial. One can verify that $\varrho_\lambda \sigma$ is a classifying map of the fibration $\xi(\lambda)$; this proves the theorem.

In the case when $G/H(b)$ is a topological sphere S^m the Theorem 3 can be reformulated as follows. Let $\varphi: S^{m-1} \to H(b)$ be a characteristic map of the principal fibration $(G, H, G/H)$. Then $\lambda\varphi: S^{m-1} \to K$ is a characteristic map of the fibration $\xi(\lambda)$ and therefore this fibration is trivial if and only if the map $\lambda\varphi$ is homotopic to zero. In the case $H = U(1)$ the degree of the characteristic map $\varphi: S^1 \to U(1)$ is equal to 1 hence $\xi(\lambda)$ is trivial if and only if the map $\lambda\varphi$ is homotopic to zero. If $G = SO(3)$, $H = SO(2)$ the degree of characteristic map $\varphi: S^1 \to SO(2)$ is equal to 2 and $\xi(\lambda)$ is trivial if and only if the map $\lambda: SO(2) \to K$ is homotopic to the map $\lambda^{-1}: SO(2) \to K$. In the case $G = SO(4)$, $H = SO(3)$ the characteristic map $\varphi: S^2 \to SO(3)$ is homotopic to zero and $\xi(\lambda)$ is always trivial.

We have completely analysed the case when $\alpha(G)$ acts on M transitively. One can perform such analysis also if the isotropy subgroup $H(b)$ depends continuously on the point $b \in M$ using some results of transformation group theory. In the last case all isotropy subgroups are conjugate and the action of $\alpha(G)$ on M generates a fibration of the space M onto orbits of $\alpha(G)$; this fibration is associated with the principal fibration with the group $N(H(b))/H(b)$ where $N(H(b))$ is the normalizer of $H(b)$ in G. (Main results of transformation group theory used in present paper can be found in [4].) For brevity the complete analysis of this case will be omitted; we shall impose an additional condition that the coset space $B = M/\alpha(G)$ is contractible (then the fibration of M on orbits is trivial). One can prove that Theorems 1–3 remain correct in the case under consideration. The proofs require only minor modifications. In particular by the proof of Theorem 1 the space $E(\lambda)$ must be defined as the coset space of $B \times G \times K$ with respect of action of $H(b)$ transforming $(m, g, k) \in B \times G \times K$ into $(m, hg, \lambda(h)k)$.

Let us return to the general case. It is well known that orbits having minimal isotropy subgroup (non-singular orbits) fill in an open dense subset $M_1 \subset M$ and $H(b)$ depends continuously on $b \in M_1$. One can find an open dense subset $M_2 \subset M_1$ in such a way that $M_2/\alpha(G)$ is contractible. It was noted in Section 1 that one can consider all fields on M_2 only, but to study fields which are continuous at all points of M one must regard the automorphisms of principal fibrations with the base M. These automorphisms can be described in the most interesting case when the coset space $M/\alpha(G)$ is one-dimensional. Then the space $M/\alpha(G)$ is homeomorphic to one of (a) a circle, (b) on open interval, (c) a half-open interval, or (d) a closed interval (see [8]). In cases (a) and (b) the subgroup $H(b)$ depends continuously on $b \in M$ (all orbits are non-singular). In the case (c) there exists one singular orbit. We shall consider the case (d) for definiteness. In this case we identify $M/\alpha(G)$ with the closed interval $[0;1]$; the identification map of M onto $M/\alpha(G)=[0;1]$ will be denoted by q. The orbits $q^{-1}(t)$ are non-singular if $0<t<1$ and singular if $t=0$ or $t=1$. One can find such a map f of $[0;1]$ into M that $qf(t)=t$, the isotropy subgroups $H(f(t))=H$ don't depend on t if $0<t<1$, $H(f(0))=H_0\supset H$, $H(f(1))=H_1\supset H$ (see [8]). Every homomorphism $\tau\in\mathscr{A}(\xi,\alpha)$ where $\xi(E^\xi,M,K,p)$ is a principal fibration determines an action of $G\times K$ onto E^ξ; the coset space of this action is also $M/\alpha(G)=[0;1]$. Using once more the results of [8] we obtain such a map $\tilde{f}$ of $[0;1]$ into E^ξ that $p\tilde{f}=f$ and the isotropy subgroups of $G\times K$ at points $\tilde{f}(t)\in E^\xi$ don't depend on t if $0<t<1$. The homomorphisms $\lambda_0=\lambda_{\tau,\tilde{f}(0)}:H_0\to K$ and $\lambda_1=\lambda_{\tau,\tilde{f}(1)}:H_1\to K$ coincide on H, namely $\lambda_0=\lambda_1=\lambda$ where $\lambda=\lambda_{\tau,\tilde{f}(t)}:H\to K$, $0<t<1$. These homomorphisms depend on the choice of $\tilde{f}$; the family of pairs (λ_0,λ_1) obtained by various choice of $\tilde{f}$ will be denoted by $\Lambda(\tau)$. [It is easy to verify that by means of change of $\tilde{f}$ one can replace the pair (λ_0,λ_1) by the pair (λ'_0,λ'_1) if there exists such continuous function $k(t)\in K$ that $\lambda'_0=k(0)\lambda_0k^{-1}(0)$, $\lambda'_1=k(1)\lambda_1k^{-1}(1)$ and $k(t)\lambda k^{-1}(t)$ does not depend on t for $0<t<1$.]

Theorem 1'. *For every homomorphisms $\lambda_0:H_0\to K$, $\lambda_1:H_1\to K$ coinciding on H one can construct such principal fibration $\xi(E^\xi,M,K,p)$ and such $\tau\in\mathscr{A}(\xi,\alpha)$ that $(\lambda_0,\lambda_1)\in\Lambda(\tau)$.*

Theorem 2'. *The homomorphisms τ, $\tau'\in\mathscr{A}(\xi,\alpha)$ are gauge equivalent if and only if $\Lambda(\tau)=\Lambda(\tau')$.*

The proofs of these theorems are analogous to the proofs of Theorems 1, 2.

Theorem 1', 2' can be used in particular in the case $M=S^n$, $G=SO(k)\times SO(n+1-k)$.

If the space $M/\alpha(G)$ is homeomorphic to a half-open interval [the case (c)] then the situation is simpler.

Let $H=H(b)$ be an isotropy subgroup at the point $b\in M$ belonging to the singular orbit. Using the results of [8] one can construct for every homomorphism $\lambda:H\to K$ a principal fibration $\xi(\lambda)$ $(E(\lambda),M,K,p(\lambda))$, a homomorphism $\tau\in\mathscr{A}(\xi(\lambda),\alpha)$ and a point $e\in E(\lambda)$ in such a way that $\lambda=\lambda_{\tau,e}$. If $\tau\in\mathscr{A}(\xi,\alpha)$, $\tau'\in\mathscr{A}(\xi,\alpha)$ where $\xi(E,M,K,p)$ is a principal fibration and the point $e\in E$ satisfies $p(e)=b$ then the homomorphisms τ and τ' are gauge equivalent if and only if the homomorphisms $\lambda_{\tau,e}:H\to K$ and $\lambda_{\tau',e}:H\to K$ are conjugate.

We see that the geometrical language is very convenient to describe the subgroups of the symmetry group of the gauge theory. The invariant Yang-Mills fields (the invariant connections in principal fibrations) also can be studied by means of geometrical considerations. Probably it is most useful to combine the analytical approach of Section 1 and the geometrical approach of Section 2.

In particular the results of Section 2 can be used to describe the continuous invariant Yang-Mills fields on M in the case when $M/\alpha(G)$ is one-dimensional. For example if the spherically symmetric Yang-Mills field on $E^3\setminus\{0\}$ can be continuously extended on E^3 the homomorphism $\lambda:H\to K$ determining the type of spherical symmetry can be extended on G [here $H=SO(2)$ or $H=U(1)$, $G=SO(3)$ or $G=SU(2)$].

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