Commun. math. Phys. 56, 1—9 (1977)

Communications in
Mathematical
Physics

Are There Geon Analogues in Sourceless Gauge-Field Theories?*

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Abstract. It has recently been shown that there is no finite-energy non-singular solution to the sourceless gauge-field equations in four-dimensional Minkowski space that does not radiate energy. However, this does not preclude the possibility of solutions which hold themselves together for a long time before radiating away their energy. If they existed, such objects would be analogous to the geons of general relativity. We show such objects do not exist.

I. Introduction

This is the second of two papers devoted to the study of finite-energy non-singular solutions of classical gauge field theories in four-dimensional Minkowski space. In the first of these papers [1], it was shown that the only such solution that did not radiate energy to infinity was the vacuum solution, $F^a_{\mu\nu}=0$. Recently, this result has been strengthened by Weder [2], who showed that it is impossible for any non-zero amount of energy to be permanently confined in any compact volume.

These are strong results, but they do not exclude the possibility of energy being confined to some compact volume for a very long time before eventually escaping. A priori, this is a live possibility; after all, gauge field theories have much in common with general relativity, and general relativity is known to possess solutions of precisely this character, the geons of Wheeler [3], Brill and Hartle [4].

The purpose of this paper is to kill this possibility, and to attempt to understand physically why gauge field theories differ from general relativity in this respect.

In Section 2 we prove some rigorous theorems that can be (very loosely) described as saying that any configuration of gauge fields initially confined to the interior of some sphere falls apart in the time it takes light to cross the sphere. Not only are the forces of classical gauge field theory not enough to confine, they are not even enough to restrain.

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* Work supported in part by the National Science Foundation under Grants No. PHY75—20427 and No. PHY76—14852

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These theorems require assumptions about the behavior of the fields at spatial infinity that can probably be weakened considerably by better analysis than we; however, we think they are weak enough already to include most cases of physical interest.

In Section 3, we try to understand the physics of these theorems by attempting to mimic in a gauge field theory the construction of a geon in general relativity. Here we make no attempt at generality; we restrict ourselves to an SU(2) gauge theory and make the analogues of all the drastic simplifying assumptions that go into the construction of a static spherical geon. Of course, the attempt fails, but only at the very last step, and the reason for its failure is enlightening: The gravitational force is always attractive; the Yang-Mills force, however, can be either attractive or repulsive, and it turns out that the repulsive interaction always wins.

In Section 4, we summarize our conclusions and make some speculations.

II. Some Theorems

Our basic input will be certain properties of the symmetric energy-momentum tensor[1]

$$\theta^{\mu\nu} = F^{a\mu\lambda}F^{a\nu}_{\lambda} - \tfrac{1}{4}g^{\mu\nu}F^{a}_{\lambda\varrho}F^{a\varrho\lambda}. \tag{1}$$

These are

$$\theta^{00} \geqq 0, \tag{2}$$

$$\partial_\mu \theta^{\mu\nu} = 0, \tag{3}$$

$$\theta^{\lambda}_{\lambda} = 0. \tag{4}$$

It will be convenient for us to choose a Lorentz frame such that the total three-momentum of our solution vanishes, and to choose the origin of coordinates at the center of energy.

Our first theorem deals with a very restricted set of solutions, those such that there exists a positive number ε such that

$$\lim_{r\to\infty} |r^{5/2+\varepsilon}F^{a}_{\mu\nu}| = 0, \tag{5}$$

at all times. (We make no assumption about the uniformity of the limit in time.) In particular, this excludes solutions that have a Coulomb field at large distances from the center; we shall take care of this case in our second theorem.

For solutions obeying Equation (5), we can define the moment of inertia at time t by

$$I(t) = \int d^3x r^2 \theta^{00}. \tag{6}$$

1 Notation: Greek indices range from 0 to 3; Latin indices from the beginning of the alphabet range from 1 to the dimension of the Lie algebra; Latin indices from the middle of the alphabet range from 1 to 3. Summation over repeated indices is implied. The signature of the metric tensor is $(+---)$. A^a_μ are the gauge fields, $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + c^{abc}A^b_\mu A^c_\nu$, where the c's are the structure constants of a compact Lie group

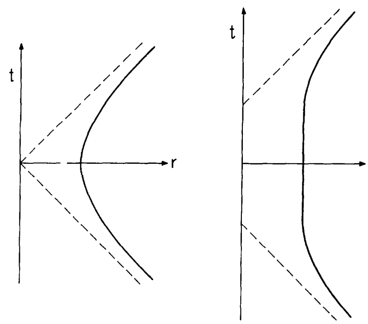

Fig. 1. a Radius of gyration as a function of time for an imploding and exploding spherical wave in free electromagnetism. **b** Radius of gyration as a function of time for a hypothetical object that holds itself together for some time before falling apart

and the radius of gyration by

$$\bar{r}(t)^2 = I(t)/E\,, \tag{7}$$

where, as usual

$$E = \int d^3x\theta^{00}\,. \tag{8}$$

Because the energy density is positive semidefinite [Eq. (2)], the radius of gyration is a measure of the spatial extent of our solution, and its time rate of change is thus a measure of whether the system is expanding or contracting. For example, it is easy to see that for an imploding-and-exploding spherical wave in free electromagnetism,

$$\bar{r}^2 = (t - t_0)^2 + r_0^2\,, \tag{9}$$

where t_0 and r_0 are constants. This situation is shown graphically in Figure 1a. This is to be contrasted with a hypothetical system that holds itself together for a time before radiating away its energy, shown in Figure 1b. Note that both of these figures are consistent with the theorems cited in Section 1 (both systems radiate), but the detailed time dependence of $\bar{r}$ is quite different in the two cases.

Theorem 1. *For any non-singular solution of the sourceless gauge-field equations obeying Equation* (5), *the radius of gyration obeys Equation* (9).

Proof. By Equation (3),

$$\partial_t^2 I = \int r^2 \partial_i \partial_j \theta^{ij} d^3x\,. \tag{10}$$

Integrating by parts and using Equation (4), we find

$$\partial_t^2 I = 2\int \theta^{00} d^3x + \int dS_i (r^2 \partial_j \theta^{ij} - x^i \theta^{ji})\,, \tag{11}$$

or

$$\partial_t^2 \bar{r}^2 = 2 . \tag{13}$$

This implies Equation (9).

We will now extend this result to include the possibility of a non-vanishing Coulomb field at large distances. Of course, we need only worry about distant electric or magnetic monopole fields; higher multipoles automatically obey Equation (5).

We define a general monopole field by

$$F^a_{oi} = E^a x_i / r^3 , \tag{14}$$

$$F^a_{ij} = H^a \varepsilon_{ijk} x^k / r^3 , \tag{15}$$

where E^a and H^a are bounded functions of spatial position at any time. (Note that this definition is invariant under arbitrary space-time dependent gauge transformations.)

We now consider solutions such that there exists a positive number ε and a monopole field $f^a_{\mu\nu}$ such that

$$\lim_{r\to\infty} r^{2+\varepsilon} |F^a_{\mu\nu} - f^a_{\mu\nu}| = 0 , \tag{16}$$

for all t.

Theorem 2. *Define*

$$I(R,t) = \int_{r \leqq R} d^3x r^2 \theta^{00} . \tag{17}$$

For any non-singular solution of the sourceless gauge-field equations obeying Equation (16),

$$\lim_{R\to\infty} \partial_t I(R,t) = 2E(t - t_0) . \tag{18}$$

where t_0 is a constant.

Proof. First we show that the limit exists:

$$\begin{aligned} \partial_t I(R,t) &= -\int_{r\leqq R} d^3x r^2 \partial_i \theta^{i0} \\ &= 2 \int_{r\leqq R} d^3x x_i \theta^{i0} - \int_{r=R} dS_i r^2 \theta^{i0} . \end{aligned} \tag{19}$$

The terms in θ^{0i} obtained by contracting the monopole field with itself vanish; the remaining terms fall off faster than r^{-4}; thus the limit exists. Now we evaluate the time derivative of the limit:

$$\begin{aligned} \partial_t \int d^3x x_i \theta^{0i} &= -\int d^3x x_i \partial_j \theta^{ij} \\ &= \int d^3x \theta^{00} - \int dS_j x_i \theta^{ij} , \end{aligned} \tag{20}$$

where the surface integral is at spatial infinity. Because θ_{ij} falls off like r^{-4} the surface term vanishes. Q.E.D.

Note that if the limit of $I(R, t)$, as R goes to infinity existed, Theorem 2 would be identical to Theorem 1. The point is that even when the fields fall off too slowly for $\bar{r}^2$ to be well-defined, we can still define $\partial_t \bar{r}^2$ and use it as a measure of the gross expansion or concentration of the system. In particular, if we have a geon-like situation, a lump of energy (perhaps surrounded by a Coulomb field) holding itself together for a long time, we would expect $\partial_t \bar{r}^2$ to be small (compared to the size of the lump) for a long time. Theorem 2 tells us that this is impossible, unless by "a long time" we mean the time it takes light to cross the lump.

III. General Relativity vs. Yang-Mills Theory

It is clear from the above proof that it is the scale invariance (4) which prevents a Yang-Mills field from holding itself together. This is not a very satisfying physical reason, even if it is a general proof. To obtain a deeper understanding of what goes wrong, we will attempt to construct a Yang-Mills lump in exact analogy to the successful construction technique used by Wheeler [3], Brill and Hartle [4] in building electrogravitational and pure gravitational geons, respectively.

Let us briefly recall that a classical geon is a self-consistent approximate solution to either the source-free Einstein-Maxwell or the pure Einstein field equations. The topology of each spacelike hypersurface of the highly curved spacetime is E^3. To build a geon one assumes a metric background which is spherically symmetric and static outside some radius $r=r_0$ and flat inside. The effective source of this gravitational field is a zero rest mass field in a thin spherical shell concentrated at $r=r_0$. The background field has a "trapping potential" at $r=r_0$ and, to close the self-consistency, the effective energy density built out of the spherical shell of radiation is always positive—thus acting to bind. The details are outlined in Table 1.

We now attempt a similar construction for a SU(2) Yang-Mills field theory in Minkowski spacetime. First, we specify some background Yang-Mills field $\bar{A}^b_\mu$. The source for this background will be the current produced by a high-frequency perturbation a^b_μ of the background. Thus, the full potential is:

$$A^b_\mu = \bar{A}^b_\mu + a^b_\mu . \tag{21}$$

The total field strength becomes[2]

$$F^b_{\mu\nu} = \bar{F}^b_{\mu\nu} + \bar{D}_\mu a^b_\nu - \bar{D}_\nu a^b_\mu + (a_\mu \times a_\nu)^b \tag{22}$$

where the background covariant derivative is

$$\bar{D}_\nu f^a_\mu \equiv \mathscr{D}_\nu f^a_\mu + (\bar{A}_\nu \times f_\mu)^a . \tag{23}$$

We have included spatial Christoffel symbols in the $\mathscr{D}_\nu$ derivative.

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[2] To shorten our equations, we use the standard notation $(A \times B)^a \equiv \varepsilon^{abc} A^b B^c$

Table 1. Two successful attempts to construct static spherical geons and a failed attempt to construct a similar object in Yang-Mills theory

Property	Wheeler Electrogravitational geon [3]	Brill-Hartle Pure gravitational geon [4, 5]	Possible gauge-field analogue
Full field equations	$G_{\mu\nu}=T^{EM}_{\mu\nu}$ $\mathscr{D}^{\mu}F_{\mu\nu}=0$	$G_{\mu\nu}=0$	$D^{\mu}\,F_{\mu\nu}=0$
Field split	$g_{\mu\nu},\,A_{\mu}$	$g_{\mu\nu}=\gamma_{\mu\nu}+h_{\mu\nu}$	$A^{b}_{\mu}=\bar{A}^{b}_{\mu}+a^{b}_{\mu}$
Background field	$ds^2=-e^{\nu(r)}\,dt^2+e^{\lambda(r)}\,dr^2+r^2d\Omega^2$		$\bar{A}^{a}_{\mu}=\delta^{o}_{\mu}\delta^{a}_{1}f(r)$
Background equations[a]	$G_{\mu\nu}=8\pi\langle T^{EM}_{\mu\nu}\rangle$	$\bar{G}_{\mu\nu}=\frac{1}{4}\langle\bar{\mathscr{D}}_{\mu}\underline{h}_{\alpha\beta}\bar{\mathscr{D}}_{\mu}\underline{h}^{\alpha\beta}\rangle$	$\bar{D}^{\mu}\bar{F}^{a}_{\mu\nu}=\langle a^{\mu}\times\bar{D}_{\nu}a_{\mu}\rangle^{a}$
Perturbation equation	$\mathscr{D}^{\alpha}\mathscr{D}_{a}A_{\mu}-R_{\mu\nu}A^{\nu}=0$	$\bar{\mathscr{D}}^{\alpha}\bar{\mathscr{D}}_{\alpha}\underline{h}_{\mu\nu}+2\bar{R}_{\alpha\mu\beta\nu}\underline{h}^{\alpha\beta}=0$	$\bar{D}^{\alpha}\bar{D}_{\alpha}a^{b}_{\mu}+2(\bar{F}_{\mu\nu}\times a^{\nu})^{b}=0$
Gauge condition	$\mathscr{D}^{\alpha}A_{\alpha}=0$	$\mathscr{D}^{\alpha}\underline{h}_{\alpha\mu}=0 \qquad \underline{h}^{\alpha}_{\alpha}=0$	$\bar{D}^{\alpha}a_{\alpha}=0$
Nonzero perturbation	A_{ϕ}	$h_{0\phi}\equiv h_0 \qquad h_{r\phi}\equiv h_1$	$a_{\phi}=a^{2}_{\phi}+i\,a^{3}_{\phi}$
Variable separation	$A_{\phi}=e^{i\omega t}\,R(r)\,\Phi_l(\theta)+\text{c.c.}$	$h_1\,e^{1/2(\nu-\lambda)}\,r^{-1}=e^{i\omega t}\,R(r)\Phi_l(\theta)+\text{c.c.}$	$a_{\phi}=e^{i\omega t}\,R(r)\,\Phi_l(\theta)$
Radial equation[b]	$d^2R/dr^{*2}+[\omega^2-V(r)]R=0$	$d^2R/dr^{*2}+[\omega^2-V(r)]R=0$	$d^2R/dr^2+[\omega^2-V(r)]R=0$
Effective potential	$V(r)=e^{\nu}l(l+1)r^{-2}$	$V(r)=e^{\nu}l(l+1)r^{-2}-(3/2r)de^{1/2(\nu-\lambda)}/dr^{*}$	$V(r)=l(l+1)r^{-2}-f^2-2\omega f$
Trapping orbits	yes	yes	yes
Can trapped orbits be source of background?	yes	yes	no

[a] Here we set $G=c=1$ and use [5]; the metric perturbation $\underline{h}_{\alpha\beta}=h_{\alpha\beta}-\frac{1}{2}h\bar{\gamma}_{\alpha\beta}$

[b] In the gravitational case, it is convenient to use r^* as a radial coordinate where $dr^*=e^{1/2(\lambda-\nu)}dr$

Expanding the nonlinear field equations up to quadratic powers of a^b_μ we find the following two sets of equations:

$$\bar{D}^\mu \bar{D}_\mu a^b_\nu - 2(\bar{F}_{\mu\nu} \times a^\mu)^b = 0 \tag{24}$$

$$\bar{D}^\mu \bar{F}^a_{\mu\nu} = [a^\mu \times \bar{D}_\nu a_\mu - 2\bar{D}^\mu (a_\mu \times a_\nu)]^a \tag{25}$$

where we have imposed the background field gauge on the perturbation:

$$\bar{D}^\mu a^b_\mu = 0 . \tag{26}$$

That is, the perturbations a^b_ν satisfy a linear wave equation in the background $\bar{A}^b_\mu$ which is created as a result of a current quadratic in the perturbation. This current can be averaged over several wavelengths to form an effective source for the field $\bar{A}^b_\mu$. The second term in (25) is of higher order and can be dropped[3] (just as in the geon construction) if we impose the further condition:

$$\bar{A}^{b\mu} a^c_\mu = 0 . \tag{27}$$

In this case, the effective current becomes:

$$\langle j^a_\nu \rangle = \langle a^\mu \times \bar{D}_\nu a_\mu \rangle^a . \tag{28}$$

In order for the analogy to be complete, we assume there exists a Coulomb potential outside of a thin shell at $r = r_0$ and a constant potential inside, with the potential everywhere pointing in a constant direction in isospin space:

$$\begin{aligned} \bar{A}^a_\mu &= \delta^0_\mu \delta^a_1 f(r) \\ f(r) &= \begin{cases} e r^{-1}, & r \geqq r_0 \\ e r_0^{-1}, & r \leqq r_0 \end{cases} . \end{aligned} \tag{29}$$

This potential produces only one nonzero component of field strength:

$$F^1_{0r} = -\partial_r f . \tag{30}$$

The demand that the background be aligned in the 1 direction then implies that only $\langle j^1_\mu \rangle$ is nonzero. This can be seen to be equivalent to demanding:

$$a^1_\mu = 0 . \tag{31}$$

Let us combine a^2_μ and a^3_μ into a single complex field a_μ:

$$a_\mu \equiv a^2_\mu + i a^3_\mu . \tag{32}$$

Then the Equation (24) becomes:

$$(\Box - \bar{A}^{1\mu} \bar{A}^1_\mu) a_\nu + i[2\mathscr{D}^\mu (\bar{A}^1_\mu a_\nu) + a_\nu \mathscr{D}^\mu \bar{A}^1_\mu] = 0 \tag{33}$$

or using (29):

$$(\Box - f^2) a_\nu + 2 i f \partial_t a_\nu = 0 \tag{34}$$

where $\Box = \mathscr{D}^\mu \mathscr{D}_\mu$ is the coordinate d'Alembertian.

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[3] For the rules on manipulating spacetime covariant derivatives inside averaging brackets see Misner et al. [5], p. 969, and Isaacson [6]. Here we extend these rules to include gauge covariant derivatives

Following Wheeler's ansatz [3] we will assume a standing wave in a thin torus of radius $r=r_0$ with only a_ϕ nonzero [thus satisfying (27)], the spherical shell later to be constructed by use of the rotation group. In this case only the charge density ϱ is nonzero:

$$\langle j_0^1\rangle \equiv 4\pi\varrho = \langle a^\mu(i\partial_0+f)a_\mu^*\rangle . \tag{35}$$

If we separate variables by taking the decomposition:

$$a_\phi = e^{i\omega t}R(r)\theta_l(\theta) \tag{36}$$

then our final equations become:

$$(\Box - 2f\omega - f^2)a_\phi = 0 \tag{37}$$

$$\langle j_0^1\rangle \equiv -\left.\frac{\langle(\omega+f)a_\phi a_\phi^*\rangle}{r^2\sin^2\theta}\right|_{r=r_0} \tag{38}$$

where, as in Wheeler's construction of the electrogravitational geon,

$$\Box = \partial_t^2 - \partial_r^2 - \frac{1}{r^2}[\partial_\theta^2 - \cot\theta\partial_\theta] \tag{39}$$

Using the standard separation constant, $l(l+1)$, we are left with a radial equation:

$$\partial_r^2 R + [\omega^2 - V(r)]R = 0 \tag{40}$$

$$V(r) = \frac{l(l+1)}{r^2} - f^2 - 2\omega f . \tag{41}$$

Unlike the geon radial equations (Table 1) our effective potential *depends on* ω. This difference is essential. It is easy to see that the potential has two turning points for $\omega^2>0$ only if $\omega<0$ and:

$$|\omega| < \frac{e}{r_0} - \frac{l(l+1)}{er_0}\text{(trapping)} . \tag{42}$$

Furthermore, this can occur only if $e^2>l(l+1)$, i.e. only for the case where the charged attraction can overcome the centrifugal barrier.

On the other hand, the charged current of the shell (38) can act as a source of the assumed background (29) only if $\omega<0$ and

$$|\omega| > \frac{e}{r_0}\text{(self consistency)} . \tag{43}$$

Thus we are led to a contradiction between (42) and (43). In words, a background field of one sign of charge can only trap charge of the other sign, but then that trapped charge cannot serve as the source for the background: a spherical shell SU(2) Yang-Mills lump cannot be constructed.

IV. Conclusions and Speculations

If one is willing to accept the time dependence of the radius of gyration as a measure of the gross behaviour of a solution of the field equations, we have shown in Section 2 that the gross behavior of solutions of an arbitrary sourceless gauge field theory is the same as that of solutions of free electromagnetism. Of course, this does not mean that on a more detailed level such theories are essentially the same as electromagnetism, but it is sufficient to show that there is no hope of constructing a Yang-Mills analogue of a geon.

We think we have isolated the essential physical reason for this in the example of Section 3. Nearby portions of the gravitational field attract one another; however, by continuity, nearby small portions of the Yang-Mills field must always point in the same direction in internal space, and therefore must repel each other. (Like charges repel.) We believe it is this selfrepulsion at small distances which overwhelms all other effects and prevents the construction of a geon analogue.

Acknowledgements. We thank José Cervero, Charles Misner, Richard Treat, and Gordon Woo for stimulating discussions.

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Communicated by A. Jaffe

Received June 17, 1977