Commun. math. Phys. 55, 187—194 (1977)

Communications in
Mathematical
Physics
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Pure Soliton Solutions of Some Nonlinear Partial Differential Equations

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Abstract. A general approach is given to obtain the system of ordinary differential equations which determines the pure soliton solutions for the class of generalized Korteweg-de Vries equations (cf. [6]). This approach also leads to a system of ordinary differential equations for the pure soliton solutions of the sine-Gordon equation.

§ 1

A numerical study [10] of the Korteweg-de Vries equation (KdV in short)

$$u_t + 6uu_x + u_{xxx} = 0 \tag{1}$$

showed that some solutions of this nonlinear partial differential equation decompose for large time t into solitary waves travelling with constant velocity (the so called solitons). A better understanding of this phenomenon was possible when Gardner et al. [5] developed their ingenious Schrödinger operator calculus which gave rise to the inverse scattering method for solving the KdV. Since then similar methods have been developed for other nonlinear partial differential equations ([1], [13], [19]).

In a fundamental paper [12] Lax gave a better explanation of this phenomenon by considering the gradients of the conservation laws of the KdV equation. As a result of this he gave a precise description of the 2-soliton case. An integral equation (Gelfand-Levitan eq.) for the N-soliton case was already provided by the inverse scattering method. Considering explicit solutions of this integral equation the authors of [6] obtained a system of ordinary differential equations for the N-soliton solution of the KdV. One advantage of this method is that it permits identification of solitons even during the interaction process.

In this paper a general approach to this system of ordinary differential equations is given. The only tool on which our method depends is the time invariance of the spectrum of a family of Schrödinger operators. No other spectral properties are used and we do not work with reflection and normalization coefficients. Therefore we have also found a description of the N-soliton case for the generalized KdV equations. Of course, the time history changes if we replace the KdV by its generalizations.

In the last chapter we use similar ideas to derive a system of $2N$ ordinary differential equations for the N-soliton solutions of the sine-Gordon equation.

The main difference between this paper and the work [12] of Lax is that we use conservation laws with nonlocal gradients instead of those with local gradients. Therefore our results do not depend on the ingenious construction of integrals described in [12] and [16].

§2

To make the paper self contained we start with a brief account of some results from Lax' fundamental paper [12].

We consider the complex Hilbert space $L^2 = L^2(dx)$, dx being the Lebesgue measure on $\mathbb{R}$. $(f,g) = \int \bar{f}(x)g(x)dx$ denotes the scalar product in L^2, ∂ is the differential operator and $L^2_{\mathbb{R}}$ stands for the real functions in L^2. For $u \in L^2$ the densely defined multiplication operator $f(x) \to f(x)u(x)$ is also denoted by u. Subscripts t, ε and x denote partial derivatives.

Let $B(u)$ be a family of selfadjoint operators depending on u and not explicitly depending on x, where u can be taken from a dense subspace of $L^2_{\mathbb{R}}$, and let $S(u) = \partial^2 + u$ be the Schrödinger operator. We consider a one-parameter family $u(t) \in L^2_{\mathbb{R}}$ (t called time) satisfying the operator evolution equation:

$$S(u(t))_t = u_t(t) = i[B(u(t)), S(u(t))] \tag{2}$$

where [,] denotes the commutator and $B(u)$ is required of such a form that

$$(u(t), u(t))_t = 0 \quad \text{(energy conservation)}.$$

In the following we always assume that there is a dense subspace D of $L^2_{\mathbb{R}}$ such that for any initial condition $u_0 = u(t_0) \in D$ there is a unique solution $u(t, u_0) \in D$ for (2) such that $u(t, u_0)$ is differentiable (with respect to u_0) in D. For $v_0 \in D$ we abbreviate

$$v(t) = v(t, v_0) = \frac{\partial}{\partial \varepsilon} u(t, u_0 + \varepsilon v_0)|_{\varepsilon = 0}. \tag{3}$$

Exact criterions for the uniqueness and existence of solutions of (2) can be found in [8]. In many examples D is a space of sufficiently smooth functions vanishing rapidly at ∞.

For every t the space $\{v(t) | v_0 \in D\}$ is dense in $L^2_{\mathbb{R}}$ because t_0 was arbitrary. Energy conservation implies $(u(t), v(t))_t = 0$. And because $B(u(t))$ is selfadjoint $S(u(t))$ has to be a family of unitarily equivalent operators

$$S(u(t)) = U(t) S(u_0) U(t)^{-1}$$

where $U(t)$ are unitary with $U_t(t) = iB(u(t))U(t)$.

This implies that the eigenvalues λ of $S(u(t))$ are time-independent and that the eigenvectors w_λ (i.e. $S(u)w_\lambda = \lambda w_\lambda$) of $S(u(t))$ are developing according to $(w_\lambda(t))_t = iB(u(t))w_\lambda(t)$. The content of the next lemma can be found in [12, p. 475]. For completeness we give a short proof.

Lemma 1. *Let $w_\lambda(t)$ be eigenvectors of $S(u(t))$ with norm 1. If*

$$(w_\lambda(t), v(t)w_\lambda(t)) = \int |w_\lambda(x,t)|^2 v(x,t)dx$$

exists, then it is independent of t.

Proof. Consider $S(u(t))w_\lambda(t) = \lambda w_\lambda(t)$, where $u(t) = u(t, u_0 + \varepsilon v_0)$. Differentiation with respect to ε at the point $\varepsilon = 0$ gives:

$$v(t)w_\lambda(t) + S(u(t))(w_\lambda(t))_\varepsilon = \lambda_\varepsilon w_\lambda(t) + \lambda(w_\lambda(t))_\varepsilon .$$

Forming the scalar product with $w_\lambda(t)$ we get $(w_\lambda(t), v(t)\, w_\lambda(t)) = \lambda_\varepsilon$. □

The next theorem is known in case of the KdV equation, i.e. for special $B(u)$ (see [6] Theorems 3.2 and 3.4).

Theorem 1. *Let for $t = t_0$ the solution $u(t)$ of* (2) *be of the form $u(t_0) = \sum_{n=1}^{N} \alpha_n |w_{\lambda_n}(t_0)|^2$. Then we have for all t*

$$u(t) = \sum_{n=1}^{N} \alpha_n |w_{\lambda_n}(t)|^2 . \tag{4}$$

Proof. According to Lemma 1 $\left(u(t) - \sum_{n=1}^{N} \alpha_n |w_{\lambda_n}(t)|^2, v(t)\right)$ is independent of t and therefore equal to 0 (evaluation at t_0). Now, the theorem follows from the fact that the $v(t)$ are dense in $L^2_{\mathbb{R}}$. □

Lemma 2. *If ω with $\omega^2 \in L^2$ is a solution of the ordinary nonlinear differential equation*

$$(\partial^2 + |\omega|^2)\omega = \lambda\omega \tag{5}$$

then there is a $c \in \mathbb{R}$ such that $u(x,t) = |\omega(x - ct)|^2$ is a solution of (2).

Proof. Let $u(t)$ be the solution of (2) fulfilling the initial condition $u(t_0) = |\omega_0|^2$, where $(\partial^2 + |\omega_0|^2)\omega_0 = \lambda\omega_0$. From Theorem 1 we obtain $u(t) = (\omega_0, \omega_0)|w_\lambda(t)|^2$ and $\omega = (\omega_0, \omega_0)^{1/2}\, w_\lambda(t)$ must be a solution of (5). The L^4-solutions of (5) are a one parameter family generated by translation. Therefore we have $u(x,t) = |\omega_0(x - \varphi(t))|^2$. With our evolution Equation (2) we get

$$\begin{aligned} u_t(x,t) &= -|\omega(x - \varphi(t))|_x^2 \varphi_t \\ &= i[B(|\omega(x - \varphi(t))|^2), L(|\omega(x - \varphi(t))|^2)] . \end{aligned} \tag{6}$$

Now, let a be such that $|\omega(a)|_x^2 \neq 0$ then evaluation of (6) at $x = \varphi(t) + a$ gives that φ_t is independent of t. □

The same proof goes through for the equation $(\partial^2 - |\omega|^2)\omega = \lambda\omega$. But some acquaintance with Schrödinger operators tells that this equation has no L^4-solutions. It is quite easy to determine the solutions of (5). Integration yields:

$$\omega(x) = (2\lambda)^{1/2} \{\cosh(\lambda^{1/2}(x - x_0))\}^{-1} . \tag{7}$$

Since Lemma 1 does not depend on the special form of $B(u)$ we have proved that (7) also determines the shape of the solitary waves of the generalized KdV equations.

§3

The Korteweg-de Vries Equation (1) came up in the study of shallow water waves and has many applications in physics ([9] or [11] for references).

It is well known ([5], [12]) that the Schrödinger operators $S(u(t)) = \partial^2 + u(t)$ are unitarily equivalent if $u(t)$ is a solution of the KdV. The infinitesimal generator of (2) is then given by

$$iB(u) = -4\partial^3 - 3u\partial - 3\partial u. \tag{8}$$

Solutions of (1) being of the form $u(x,t) = s_\lambda(x-ct)$ are called travelling waves. Obviously they are solutions of an ordinary differential equation. Lemma 2 tells us that a travelling wave has to be a translation of (7). Inserting (7) in the ordinary differential equation describing the travelling waves gives the well known result $c = 4\lambda$. Other nonlinear evolution equations leading to (2) are called *generalized* KdV equations ([12], [15]). For these equations it was observed that the solutions often decompose for $t \to \infty$ into travelling waves of the form (7) the so called solitons.

Here we are interested in *pure N-soliton solutions*, which means that for $t \to \infty$ the total energy (u, u) of u is carried by N solitons and that the decomposition into solitons is stable. To be more precise we require:

$$u(x,t) = \sum_{n=1}^{N} s_{\lambda_n}(x - 4\lambda_n t + \theta_n) + \Delta(x,t) \tag{9}$$

where the error term $\Delta(x,t) = \Delta$ is such that

$$\lim_{t \to \infty} \int |\Delta(x,t)|^2 dx = 0 \tag{9A}$$

for $t \to \infty$ the eigenvectors of $(\partial^2 + u(t))$ are converging in $L^4(dx)$ to the eigenvectors of $(\partial^2 + u(t) - \Delta(t))$. (9B)

$(v(t), v(t))$ is a bounded function in t, where $v(t)$ is defined by (3). (9C)

Theorem 2. *Let $u(x,t)$ be a pure N-soliton solution of a generalized Korteweg-de Vries equation. Then $u(x) = u(x, t_0)$ must be for any time t_0 a solution of the following system of N ordinary differential equations*

$$\frac{d^2\omega_n(x)}{dx^2} + u(x)\omega_n(x) = \lambda_n \omega_n(x) \tag{10}$$

where $u(x) = \sum_{n=1}^{N} (\omega_n(x))^2$.

Proof. Let $u(x,t)$ be as in (9) and consider

$$h(x,t) = u(x,t) - \sum_{n=1}^{N} \alpha_n \{w_{\lambda_n}(x,t)\}^2$$

where w_{λ_n} are the normed eigenvectors of $(\partial^2 + u)$ with eigenvalues λ_n, and where α_n is the integral over the square of the function given by (7). Because of Lemma 2 and the asymptotic behaviour of u one obtains $\lim_{t\to\infty}(h(\ ,t), h(\ ,t))=0$. Now, energy conservation and Lemma 1 imply that $(h(\ ,t), v(t))$ is independent of time. Taking the limit for $t\to\infty$ one gets with the help of our stability relations $(h(\ ,t), v(t))=0$; which has $h(x,t)=0$ as a consequence since the $v(t)$ are dense. By the definition $\omega_n=\alpha_n^{1/2}w_{\lambda_n}$ the desired result is obtained. □

A partial differential equation for the time development of a soliton is easily obtained. For fixed n we express u by (10) in terms of ω_n and insert this in (8). Thus we obtain as in [6] in case of the KdV the evolution equation

$$\omega\omega_t=3\omega_{xx}\omega_x-\omega\omega_{xxx}-6\lambda\omega\omega_x. \tag{11}$$

For the KdV equation Theorem 2 is already contained in [6, Theorems 3.2 and 3.4]. There it was proved by considering explicit pure soliton solutions. As in [6] we call from now on the square of the real eigenvectors of $S(u(t))$ *solitons*.

§4

Now, we shall use ideas similar to those of the preceding chapters to obtain a system of ordinary differential equations describing the shape of a pure soliton solution of the sine-Gordon equation ([1–3], [7] and [19]):

$$u_{xt}+4\sin(u)=0 \tag{12}$$

which results out of a variable transformation for the nonlinear Klein-Gordon equation $\Phi_{tt}-\Phi_{xx}+\sin(\Phi)=0$. Again our analysis does not depend on any knowledge of the explicit pure soliton solutions given in [3] and [7]. The principal tool will be the commutator formalism for (12) which was developed in [1] (see also [2], [11], [13] and [19]).

We begin with a brief account of the results needed from those papers. We are interested in real valued solutions u of (12) with $u_x\in L^2(dx)$. For this, one considers the operator $\Lambda(u): E\to E$ given by

$$\Lambda=\Lambda(u)=\frac{1}{2}\begin{pmatrix}-2\partial, & -u_x\\ -u_x, & 2\partial\end{pmatrix} \tag{13}$$

where E are the vectors $w=\begin{pmatrix}w_1\\ w_2\end{pmatrix}$, $w_1, w_2\in L^2(dx)$ equiped with the scalar product $(w,\tilde{w})=(w_1,\tilde{w}_1)+(w_2,\tilde{w}_2)$. If u evolves according to (12) then we have

$$\Lambda_t\Lambda=R\Lambda-\Lambda R \tag{14}$$

with

$$R=R(u)=\begin{pmatrix}\cos(u), & \sin(u)\\ \sin(u), & -\cos(u)\end{pmatrix} \tag{15}$$

Λ^+ is given by $\Lambda^+=T\Lambda T$ where

$$T=T^{-1}=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}. \tag{16}$$

As a consequence of (14) to (16) the eigenvalues of Λ are real and independent of time. If we normalize the eigenvectors w_λ of $\Lambda w_\lambda = \lambda w_\lambda$ by

$$(Tw_\lambda, w_\lambda) = 1 \tag{17}$$

then the time evolution of w_λ is given by:

$$(w_\lambda)_t = \lambda^{-1} \cdot \{Rw_\lambda - (Tw_\lambda, Rw_\lambda) w_\lambda\}. \tag{18}$$

As in §2 we define

$$v(t) = v(t, v_0) = \frac{\partial}{\partial \varepsilon} u(t, u_0 + \varepsilon v_0)|_{\varepsilon=0}$$

and we assume that $\{v(t)_x | v_0$ possible perturbation$\}$ is dense in $L^2(dx)$, where $u(t, u_0 + \varepsilon v_0)$ is the solution of the sine-Gordon equation (12) for the perturbed initial value $u_0 + \varepsilon v_0$. We consider the derivative of $\lambda = (Tw_\lambda, \Lambda w_\lambda)$ with respect to ε at $\varepsilon = 0$ and we obtain with (17) and (13):

$$\left.\frac{\partial \lambda}{\partial \varepsilon}\right|_{\varepsilon=0} = -\frac{1}{2}\left(Tw_\lambda, \begin{pmatrix} 0 & v_x \\ v_x & 0 \end{pmatrix} w_\lambda\right) = -\frac{1}{2}\left(w_\lambda, \begin{pmatrix} v_x & 0 \\ 0 & v_x \end{pmatrix} w_\lambda\right).$$

Since λ does not depend on t this implies that

$$\int v_x(x,t)\{|w_\lambda(x,t)_1|^2 + |w_\lambda(x,t)_2|^2\} dx \tag{19}$$

is time independent. Here $(w_\lambda)_1$, $(w_\lambda)_2$ are the first and second components of w_λ. Fortunately from (12) we can show that (u_x, u_x) is time independent. Therefore we can argue exactly as in Theorem 1 and Lemma 2 to obtain:

Theorem 3. *Let for* $t = t_0$ *the solution* $u(t)$ *of* (12) *be such that*

$$u_x(t_0) = \sum_{n=1}^{N} \alpha_n (|(w_{\lambda_n}(t_0))_1|^2 + |(w_{\lambda_n}(t_0))_2|^2).$$

Then we have for all t

$$u_x(t) = \sum_{n=1}^{N} \alpha_n (|(w_{\lambda_n}(t))_1|^2 + |(w_{\lambda_n}(t))_2|^2). \tag{20}$$

Lemma 3. *Let* $\varepsilon = \pm 1$. *If* $\omega = \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}$ *is a solution of the following system*

$$-(\omega_1)_x - \frac{1}{2} u_x \omega_2 = \lambda \omega_1 \tag{21A}$$

$$+(\omega_2)_x - \frac{1}{2} u_x \omega_1 = \lambda \omega_2 \tag{21B}$$

where $u_x = \varepsilon(|\omega_1|^2 + |\omega_2|^2)$, *then there is a* $c \in \mathbb{R}$ *such that*

$$u(x,t)_x = \varepsilon(|\omega_1(x - ct)|^2 + |\omega_2(x - ct)|^2)$$

is the derivative of a solution of (12).

In Lemma 3 we have to admit negative coefficients which were excluded in Lemma 2 because of the special structure of the Schrödinger operator.

For obtaining the relation between c (speed of the travelling wave) and λ we determine the real solutions of (21) explicitly. Multiplication of (21 A) with $\varepsilon\omega_1$ and (21B) with $\varepsilon\omega_2$ and subtraction gives

$$u_{xx}=2\varepsilon\lambda(\omega_2^2-\omega_1^2). \tag{22}$$

Inserting u_x this results in:

$$2\lambda u_x+u_{xx}=4\varepsilon\lambda\omega_2^2 \tag{23A}$$

$$2\lambda u_x-u_{xx}=4\varepsilon\lambda\omega_1^2. \tag{23B}$$

(21 A)$\cdot\omega_2$ minus (21B)$\cdot\omega_1$ gives with the help of (22)

$$(\omega_1\omega_2)_x+\frac{1}{4\varepsilon\lambda}u_x u_{xx}=0.$$

This compared with (23A)$\cdot$(23B), i.e.

$$16\lambda^2(\omega_1\omega_2)^2=4\lambda^2u_x^2-(u_{xx})^2$$

results in

$$(u_x)^4=16\lambda^2(u_x)^2-4(u_{xx})^2. \tag{24}$$

The solutions of (24) with $u_x\in L^2(dx)$ are

$$u_x=\varepsilon\, 8\lambda\frac{e^{\pm 2\lambda(x-x_0)}}{1+e^{\pm 4\lambda(x-x_0)}}=\varepsilon\, 4\lambda\{\cosh(\pm 2\lambda(x-x_0))\}^{-1} \tag{25}$$

or

$$u=\mp\varepsilon\, 4\tan^{-1}\{e^{\pm 2\lambda(x-x_0)}\}. \tag{26}$$

Insertion of $u(x-ct)$ in (12) provides us with the desired relation:

$$c=\lambda^{-2}. \tag{27}$$

The travelling wave solutions determined by Lemma 3 are the so called *kinks* and *antikinks* (cf. [9]).

Therefore with similar arguments (as in Theorem 2) about the asymptotic behaviour of u one can interprete the system (21) as the differential equations which determine the shape of the N-kink-antikink solutions of the sine-Gordon equation.

Theorem 4. *A pure field of N kinks and antikinks with asymptotic speeds* $\frac{1}{\lambda_i^2}$ *solves the following system of ordinary differential equations:*

$$-(\omega_{\lambda_i})_{1_x}-\frac{1}{2}u_x(\omega_{\lambda_i})_2=\lambda_i(\omega_{\lambda_i})_1 \qquad i=1,\dots,N \tag{28A}$$

$$+(\omega_{\lambda_i})_{2_x}-\frac{1}{2}u_x(\omega_{\lambda_i})_1=\lambda_i(\omega_{\lambda_i})_2 \qquad i=1,\dots,N \tag{28B}$$

where $u_x=\sum_{n=1}^{N}\varepsilon_i\{(\omega_{\lambda_i})_1^2+(\omega_{\lambda_i})_2^2\}$ *and* $\varepsilon_i=\pm 1$.

The evolution equation which gives the relation between the time development of the kinks and antikinks and the constants of integration of (28) is:

$$(\omega_{\lambda_i})_t = \frac{1}{\lambda_i}(R\omega_{\lambda_i} - \beta_i \omega_{\lambda_i}) \tag{29}$$

where $\beta_i = (T\omega_{\lambda_i}, \omega_{\lambda_i})^{-1}(T\omega_{\lambda_i}, R\omega_{\lambda_i})$.

(29) differs from (18) because we have renormalized the eigenvectors such that the coefficients in (20) become ± 1.

Explicit soliton solutions of (12) were given in [3] and [7].

Acknowledgment. I am indebted to R. M. Miura for many helpful suggestions.

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Communicated by J. Moser

Received September 1, 1976