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Foliations of Space-Times by Spacelike Hypersurfaces of Constant Mean Curvature

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Abstract. The foliations under discussion are of two different types, although in each case the leaves are C^2 spacelike hypersurfaces of constant mean curvature. For manifolds, such as that of the Friedmann universe with closed spatial sections, which are topologically $I \times S^3$, I an open interval, the leaves will be spacelike hypersurfaces without boundary and the foliation will fill the manifold. In the case of the domain of dependence of a spacelike hypersurface, S, with boundary B, the leaves will be spacelike hypersurfaces with boundary, B, and the foliation will fill $D(S)$.

It is shown that a local energy condition ensures that the constant mean curvature increases monotonically with time through such foliations and that, in the case of a foliation whose leaves are spacelike hypersurfaces without boundary in a manifold where this energy condition is satisfied globally, the foliation is unique.

In the Robertson-Walker cosmologies there is a geometrically preferred time coordinate, namely the mean curvature of the isotropic spacelike sections which foliate the manifolds. It is the purpose of the following to show that this property of the mean curvature of the leaves of such a foliation holds true in more general circumstances. The more difficult question of the existence of such foliations will not be dealt with in this paper. It seems likely that, although it presumably fails in general, it will hold in many cases of interest. The following (restricted) definitions will be adopted.

Definition 1. A space-time is a four dimensional, C^∞, pseudo-Riemannian, time-orientable manifold of signature $(+ - - -)$.

Definition 2. A C^k foliation, F, of a manifold, M, by spacelike hypersurfaces without boundary is a map $F: M \to I$, where I is an interval, with the properties

(i) F is a C^k map.

(ii) F is onto.

(iii) If $t \in I$, then $F^{-1}(t)$ is a C^k spacelike hypersurface without boundary. $F^{-1}(t)$ is called a leaf of the foliation F.

The main theorems follow.

Theorem 1. *Suppose F is a C^2 foliation of a manifold M, topologically $S^3 \times I$ with I an open interval, by acausal spacelike hypersurfaces of constant mean curvature which are diffeomorphically S^3 and that M satisfies Einstein's equations and the strong energy condition*[1]*. Then if c is any fixed future directed timelike curve, with arc length parameter t, intersecting all leaves of F, we have $d\Omega/dt \geqq 0$, where $\Omega(t)$ is the mean curvature of the spacelike hypersurface intersected by c at $c(t)$. If $d\Omega/dt = 0$ for any t, then also the spacelike hypersurface intersected by c at $c(t)$ is a moment of time symmetry.*

Proof. Label the leaves of the foliation by t and the orthogonal trajectories of the foliation by x^i $i=1, 2, 3$ so that the metric of M can be written

$$ds^2 = l^2 dt^2 + g_{ij} dx^i dx^j. \tag{1}$$

By the Gauss-Codazzi equations [1], we have for the mean curvature, Ω, of a hypersurface $t = \text{constant}$

$$\Omega = (-g^{ij} g_{ij,0})/2l \tag{2}$$

where $x^0 \equiv t$ and

$$\nabla^2 l = \Omega,_0 + l({}^4R^0_0 - \Omega^{jl}\Omega_{jl}) \tag{3}$$

where a_{ij} is the metric induced on the hypersurface by the space-time metric, ∇^2 is its covariant Laplacian operator, ${}^4R^0_0$ is the component of the Ricci tensor of M normal to the hypersurface and Ω_{ij} is the second fundamental form of the hypersurface.

Let A be one of the leaves of F; applying Green's theorem we have

$$0 = \int_A (l\nabla^2 l + \nabla l \cdot \nabla l) d^3V \tag{4}$$

where $d^3V = \sqrt{|-a_{ij}|}\, d^3x$.

If we now assume that $\Omega,_0 < 0$ on A then from (3) and the strong energy condition it follows that $l\nabla^2 l < 0$ which is inconsistent with (4). Therefore we must have $\Omega,_0 \geqq 0$ on A.

If $\Omega,_0(t) = 0$ then (3) states that $\nabla^2 l$ is non-negative and so we must have, in this case, $\nabla l \equiv 0$ and ${}^4R^0_0 - \Omega^{jl}\Omega_{jl} \equiv 0$. That is, $l(t)$ is constant, as a function of the x^i and ${}^4R^0_0(t) \equiv \Omega_{ij}(t) \equiv 0$.

Corollary. *If, in addition, the metric on M is analytic and if each leaf of the foliation is a Cauchy surface for the manifold, then $\Omega(t) = \Omega(s)$ iff $t = s$ unless the manifold is static, such that its metric may be written in the form*

$$ds^2 = dv^2 + a_{ij} dx^i dx^j$$

with $\partial a_{ij}/\partial v = 0$.

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[1] Recall that the strong energy condition, [2], requires that

$$g_{\alpha\beta} t^\alpha t^\beta > 0 \Rightarrow T_{\alpha\beta} t^\alpha t^\beta \geqq \tfrac{1}{2} T g_{\alpha\beta} t^\alpha t^\beta$$

where $T_{\alpha\beta}$ is the energy momentum tensor of M, T is its trace and $g_{\alpha\beta}$ is the metric tensor of M. A manifold M is said to satisfy the strong energy condition if this condition holds at all points of M.

Proof. Since $\Omega,_0 \geqq 0$, $\Omega(t)=\Omega(s)$ implies that $\Omega,_0(u)=0 \forall u\in[t,s]$. If we assume that this interval is not trivial, then we also have

$$\nabla l(u)=\Omega_{ij}(u)=0\forall u\in[t,s].$$

This implies that $g_{ij,0}(u)=0\forall u\in[t,s]$ and so, as $x^0=\frac{1}{2}(t+s)$ is a Cauchy surface for the manifold, that $g_{ij,0}(u)=0\forall u$.

The metric is then put into the desired form by setting $v=\int l\,dt$.

A geometrical lemma is required in order to prove the uniqueness theorem.

Lemma 1. *Suppose S and T are two spacelike hypersurfaces, given by $y^0=f(y^i)$ and $y^0=g(y^i)$ respectively. Suppose that they touch at a point P and that there is a neighbourhood U of P such that $g(y^i)\leqq f(y^i)$ if $(y^0,y^i)\in U$. Then, if the mean curvatures of S and T at P are Ω and Π respectively; it follows that $\Pi\geqq\Omega$.*

Proof. We know that $f|_p=g|_p$ and $f,_i|_p=g,_i|_p$ for $i=1,2,3$ so, the induced metrics in S and T are the same at P as are their normals. A straightforward calculation from the definition of the mean curvature of a hypersurface yields

$$\Omega-\Pi=\xi_0 a^{ij}(f,_{ij}-g,_{ij})|_p$$

where ξ_α is the future directed unit normal to both S and T at P. Now ξ_0 is positive, a^{ij} is negative definite and $(f,_{ij}-g,_{ij})|_p$ is positive semidefinite so $\Pi\geqq\Omega$.

Theorem 2. *Suppose that F is a C^2 foliation of a manifold M, topologically $S^3\times I$ with I an open interval, by acausal spacelike hypersurfaces of constant mean curvature which are diffeomorphically S^3 and that M satisfies Einstein's equations and the strong energy condition. Suppose further that if S and T are two leaves of the foliation then $\Omega(S)=\Omega(T)\Leftrightarrow S=T$. Then, if L is an acausal C^2 spacelike hypersurface of constant mean curvature, diffeomorphically S^3, in M, it must be a leaf of the foliation F. Consequently, F is the unique C^2 foliation of M of this type.*

Proof. By Theorem 1 the mean curvature Ω of the leaves of F may be used as a time coordinate for M, for if S and T are two leaves of F and if $S\subseteqq I^+(T)$ then $\Omega(S)>\Omega(T)$. Label the orthogonal trajectories of the foliation smoothly by y^i. Then L has the representation $\Omega=\Omega(y^i)$. By the compactness of L there exist points Pmax and Pmin in L such that

$$\Omega(P\max)\geqq\Omega(q)\geqq\Omega(P\min)\forall q\in L.$$

Applying Lemma 1

$$\Omega(q)\leqq\Omega(P\min)\leqq\Omega(P\max)\leqq\Omega(q).$$

It follows that L is the leaf of the foliation, F,

$$\Omega=\Omega(P\min).$$

A similar result to that of Theorem 1 may be proved for foliations of the domain of dependence of a compact spacelike hypersurface with boundary.

Definition 3. Given a compact, acausal spacelike hypersurface S with boundary B, a C^k foliation, F, of the domain of dependence of S, $D(S)$, by spacelike hypersurfaces with boundary B is a map F: Int $D(S) \to I$, where I is an open interval, with the properties

(i) F is a C^k map.

(ii) F is onto.

(iii) If $t \in I$, then $F^{-1}(t) U B$ is a C^k spacelike hypersurface with boundary B, it is called a leaf of the foliation F.

Theorem 3. *Suppose F is a C^2 foliation of $D(S)$, the domain of dependence of a compact, acausal spacelike hypersurface S with boundary B, where S is homeomorphic to the closed, three dimensional unit disc. Suppose further that the leaves of F are acausal, spacelike hypersurfaces of constant mean curvature, homeomorphic to S, and that the strong energy condition holds in $D(S)$, with the underlying manifold, M, satisfying Einstein's equations. Then, in the sense of theorem 1, $\Omega,_0 > 0$. That is, the mean curvature is a strictly monotonic increasing function of time through the foliation. The proof is an immediate extension of that of Theorem 1.*

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