Commun. math. Phys. 53, 299–302 (1977)

Communications in Mathematical Physics


# On the Decay of Correlations in $SO(n)$-symmetric Ferromagnets*

Oliver A. McBryan

Department of Mathematics, Cornell University, Ithaca, NY 14853, USA

Thomas Spencer**

Department of Mathematics, The Rockefeller University, New York, NY 10021, USA

**Abstract.** We prove that for low temperatures $T$ the spin-spin correlation function of the two-dimensional classical $SO(n)$-symmetric Ising ferromagnet decays faster than $|x|^{-\mathrm{const}\,T}$ provided $n \geqq 2$. We also discuss a nearest neighbor continuous spin model, with spins restricted to a finite interval, where we show that the spin-spin correlation function decays exponentially in any number of dimensions.

## I. Introduction and Results

The Mermin-Wagner theorem [1] states that at non-zero temperatures the two dimensional Heisenberg model has no spontaneous magnetization. Consequently the spin-spin correlation function decays to zero at large distances, although the Mermin-Wagner theorem gives no indication of the rate of decay. Similar results apply for the classical $SO(n)$-symmetric $(n \geqq 2)$ nearest neighbor Ising ferromagnets which we study here, see for example the paper of Mermin [2]. We establish a polynomial upper bound for the decay rate of the spin-spin correlation function for these models at very low temperatures. Fisher and Jasnow [3] have previously obtained a $\log^{-1}|x|$ decay.

To describe the $SO(n)$-symmetric ferromagnet, we consider the infinite lattice of unit spacing with sites labelled by indices $i \in \mathbb{Z}^2$. To each site $i$ we associate an $n$-component classical spin $s_i$ of unit length, $\|s_i\| = 1$. The spin-spin correlation function at inverse temperature $\beta = T^{-1}$ is

$$\begin{aligned} \langle s_0 \cdot s_x \rangle(\beta) &= Z^{-1} \prod_i \int d\Omega_i^{(n)} e^{\beta \sum_{\langle i,j \rangle} s_i \cdot s_j} s_0 \cdot s_x \,, \\ Z &= \prod_i \int d\Omega_i^{(n)} e^{\beta \sum_{\langle i,j \rangle} s_i \cdot s_j} \,, \end{aligned} \tag{1}$$

where $\sum_{\langle i,j \rangle}$ denotes a sum over nearest neighbor pairs, $\Omega_i^{(n)}$ is the invariant measure

* Work supported in part by NSF, Grant PHY76-17191

** A Sloan Fellow

on the unit sphere in $n$-dimensions and (1) is to be interpreted as the thermodynamic limit of corresponding finite volume quantities $\langle s_0 \cdot s_x \rangle(\beta, N)$, defined as in (1) but with sites restricted to a $(2N+1)\times(2N+1)$ periodic lattice. Let $C(x)$ denote the fundamental solution of Laplace's equation on the lattice: $-\Delta C(x) = \delta_{0,x}$, $C(0)=0$. Note that for large $|x|$

$$C(x) \approx -\frac{1}{2\pi}\log|x|\,.$$

**Theorem 1.** *For any $\varepsilon > 0$ and $\beta \geqq \beta_0(\varepsilon)$ sufficiently large*

$$\begin{aligned}\langle s_0 \cdot s_x \rangle(\beta) &\leqq \frac{n}{2} e^{(1-\varepsilon)\beta^{-1}C(x)} \\ &\leqq \frac{n}{2}|x|^{-(1-\varepsilon)/(2\pi\beta)}\,.\end{aligned} \tag{2}$$

*Remark.* For small $\beta$ the spin-spin correlation functions are known to decay exponentially.

We prove Theorem 1 for the case $n=2$ in Section II; other values of $n$ are handled by a straightforward generalization. For $n=2$ we use the representation $s = (\cos\phi, \sin\phi)$ so that (1) takes the form

$$\langle s_0 \cdot s_x \rangle(\beta) = Z^{-1} \prod_i \int_{-\pi}^{\pi} d\phi_i e^{\beta \sum_{\langle i,j \rangle} \cos(\phi_i - \phi_j)} \cos(\phi_0 - \phi_x)\,. \tag{3}$$

Our proof of Theorem 1 is motivated by the approximation [4]

$$\sum_{\langle i,j \rangle} (\cos(\phi_i - \phi_j) - 1) \simeq -\frac{1}{2} \sum_{\langle i,j \rangle} (\phi_i - \phi_j)^2 = \frac{1}{2}(\phi \Delta \phi)\,. \tag{4}$$

In this approximation, and allowing the limit of integration in (3) to extend to infinity, we obtain the Gaussian correlation

$$\begin{aligned}\langle \cos(\phi_0 - \phi_x) \rangle_{-\Delta,\infty}(\beta) &\equiv \prod_i \int_{-\infty}^{\infty} d\phi_i e^{\frac{1}{2}\beta\phi\Delta\phi} \cos(\phi_0 - \phi_x) \Big/ \left( \prod_i \int_{-\infty}^{\infty} d\phi_i e^{\frac{1}{2}\beta\phi\Delta\phi} \right) \\ &= e^{\beta^{-1}C(x)}\,,\end{aligned} \tag{5}$$

which is essentially the bound of Theorem 1. It is difficult to justify the two approximations leading to (5). Because (3) is the integral of a periodic function of the $\phi_i$, the limits of integration may be extended to infinity without changing (3), however the quadratic approximation (4) is then unreasonable, since it makes sense only if $|\phi_i - \phi_j| \ll 2\pi$. In Theorem 2 (below) we show that there is a marked difference in behavior in correlations such as (5) depending on whether the integration range is finite or infinite. In fact, defining correlations on a $\nu$-dimensional lattice by

$$\langle \phi_0 \phi_x \rangle_{-\Delta,\mu}(\beta) \equiv \prod_i \int_{-\mu}^{\mu} d\phi_i e^{\frac{1}{2}\beta\phi\Delta\phi} \phi_0 \phi_x \Big/ \left( \prod_i \int_{-\mu}^{\mu} d\phi_i e^{\frac{1}{2}\beta\phi\Delta\phi} \right) \tag{6}$$

we will prove (see Section III) that:

**Theorem 2.** *For any finite* $\mu, \beta, v$ *there is an* $\tilde{m}>0$ *such that*

$$\langle \phi_0 \phi_x \rangle_{-\Delta,\mu}(\beta) \leqq \text{const}\, e^{-\tilde{m}|x|}, \quad |x| \to \infty .$$

The constant $\tilde{m}$ may be chosen at least as large as $\cosh^{-1}(1+m^2/4)$, $m = (v/2\pi\beta\mu^2)^{1/2} e^{-\beta\mu^2 v}$. In contrast to the exponential decay for finite $\mu$, (5) always gives a polynomial decay for $\mu=\infty$ in two dimensions.

## II. Polynomial Decay Bound for the Plane Rotator

*Proof of Theorem 1.* We use the representation (3) for $n=2$, replacing $\cos(\phi_0 - \phi_x)$ by $e^{i(\phi_0-\phi_x)}$ since $\langle \sin(\phi_0 - \phi_x)\rangle(\beta)=0$. Using the periodicity of the integrand, we make complex translations

$$\phi_j \to \phi_j + i a_j, \qquad a_j = \beta^{-1}(C(j) - C(j-x)),$$

in the numerator of (3). This means that we deform the path of integration and use the periodicity of the cosine to cancel the lateral contours. The above translation combined with the bound $|e^{iz}|=1$, $z$ real, yields

$$\langle s_0 \cdot s_x \rangle(\beta) \leqq e^{-(a_0 - a_x)} Z^{-1} \prod_i \int d\phi_i e^{\beta \sum_{\langle i,j\rangle} \cos(\phi_i - \phi_j)\cosh(a_i - a_j)}$$

$$\leqq e^{-(a_0-a_x)+\beta \sum_{\langle i,j\rangle} (\cosh(a_i - a_j) - 1)} \tag{7}$$

From properties of the fundamental solution $C(x)$ we prove below that

$$|a_i - a_j| \leqq 4\beta^{-1} \quad \text{if} \quad |i-j|=1, \quad \text{uniformly in} \quad x, i, j. \tag{8}$$

Thus for any $\varepsilon>0$ we can find $\beta_0(\varepsilon)$ such that for $\beta \geqq \beta_0$

$$\sum_{\langle i,j\rangle} (\cosh(a_i - a_j) - 1) \leqq \tfrac{1}{2}(1+\varepsilon) \sum_{\langle i,j\rangle} (a_i - a_j)^2 = \tfrac{1}{2}(1+\varepsilon)(a, -\Delta a)$$

$$= (1+\varepsilon)(2\beta)^{-1}(a_0 - a_x). \tag{9}$$

Nothing that $a_0 - a_x = -2\beta^{-1}C(x)$ we obtain the bound of Theorem 1 from (7) and (9).

Our proof of Theorem 1 has been formal in that (3) should be interpreted as a limit of finite volume quantities $\langle s_0 \cdot s_x \rangle(\beta, N)$, defined as in (3) but with sites in a $(2N+1)\times(2N+1)$ periodic lattice $L$. All of the steps above are valid for $\langle s_0 \cdot s_x\rangle(\beta, N)$ provided we replace $C(x)$ everywhere by the corresponding fundamental solution for the lattice $L$:

$$C_N(x) \equiv (2N+1)^{-2} \sum_{\substack{k \in L^* \\ k \neq 0}} (\cos k \cdot x - 1)/(4 - 2\cos k_1 - 2\cos k_2), \tag{10}$$

where $L^* = \{k = (k_1, k_2) | k_i = 2\pi(2N+1)^{-1} r_i, r_i \text{ integers}, |r_i| \leqq N\}$. To prove (8) we use (10), $|\sin\theta| \leqq |\theta|$, and $1-\cos\theta \geqq 2\pi^{-2}\theta^2$, $|\theta| \leqq \pi$, to obtain for nearest neighbors $i, j$:

$$|C_N(i) - C_N(j)| \leqq \pi^2 (2N+1)^{-2} \sum_{\substack{k \in L^* \\ k \neq 0}} |k_1|/(k_1^2 + k_2^2) < 2 .$$

The bound (8) follows immediately, uniformly in $x$, $i$, $j$, $N$.

To prove Theorem 1 for $n>2$, we parameterize the $n$-sphere by angles $\theta^{(1)}, \ldots, \theta^{(m-2)}, \phi, |\theta^{(r)}| \leqq \pi/2, |\phi| \leqq \pi$ in such a way that only the components $s^{(1)}, s^{(2)}$ of a unit spin vector $s$ involve $\phi$. We then treat $\langle s_0^{(1)} s_x^{(1)} + s_0^{(2)} s_x^{(2)} \rangle = (2/n)\langle s_0 \cdot s_x \rangle$ as for the case $n=2$, translating only the variables $\phi_i$. Alternatively one may apply correlation inequalities which compare $N=2$ with $N \geqq 3$. See [5, 6].

## III. Exponential Decay Rate for the Square Well Model

*Proof of Theorem 2.* The change of variable $\phi \to \phi/\mu$ reduces the problem to the case $\mu=1$. We abbreviate $\langle \phi_0 \phi_x \rangle_{-\Delta,1}(\beta)$ by $\langle \phi_0 \phi_x \rangle$ and note that it is the limit as $p \to \infty$ through even integers of expectations $\langle \phi_0 \phi_x \rangle_p$ defined by replacing $\int_{-1}^{1} d\phi_i$ at each site in (6) by $\int_{-\infty}^{\infty} d\phi_i e^{-\phi_i^p}$. Using integration by parts we have for any $m>0$

$$\beta(-\Delta+m)\langle \phi_0 \phi_x \rangle_p = \delta_{0,x} - \{(p-1)\langle \phi_0 \phi_x^{p-1} \rangle_p - \beta m \langle \phi_0 \phi_x \rangle_p\}.$$

Since

$$0 \leqq (-\Delta+m)^{-1}(x,y) \leqq O(1) e^{-\tilde{m}|x-y|}, \qquad \tilde{m} \equiv \cosh^{-1}(1+m^2/4),$$

Theorem 2 will follow if we can find $m>0$ such that

$$(p-1)\langle \phi_0 \phi_x^{p-1} \rangle_p - \beta m \langle \phi_0 \phi_x \rangle_p \geqq 0, \quad \text{all} \quad p. \tag{11}$$

By Griffith's inequality [7] the left side of (11) is positive if

$$(p-1)\langle \phi_0^{p-2} \rangle_p \geqq \beta m. \tag{12}$$

We again use Griffiths inequality to eliminate the ferromagnetic couplings:

$$\begin{aligned}(p-1)\langle \phi_0^{p-2} \rangle_p &\geqq (p-1) \int d\phi e^{-\nu\beta\phi^2} e^{-\phi^p} \phi^{p-2} / \int d\phi e^{-\nu\beta\phi^2} e^{-\phi^p} \\ &\geqq (p-1) e^{-\nu\beta-1} \int_{-1}^{1} d\phi \phi^{p-2} \Big/ \int_{-\infty}^{\infty} d\phi e^{-\nu\beta\phi^2} \\ &= (2/e)(\nu\beta/\pi)^{1/2} e^{-\nu\beta}.\end{aligned}$$

Thus the choice $m=(\nu/2\pi\beta)^{1/2} e^{-\nu\beta}$ satisfies (12) for all $p$.

*Remark.* Brascamp and Lieb have shown using inequalities for log concave functions that $\langle \phi_0 \phi_x \rangle_{-\Delta,\mu}$ has no long range order, see [8].

## References

1. Mermin, N.D., Wagner, H.: Phys. Rev. Letters **17**, 1133 (1960)
2. Mermin, N.D.: J. Math. Phys. **8**, 1061 (1967)
3. Fisher, M., Jasnow, D.: Phys. Rev. B**3**, 895 (1970)
4. Berezinskii, V.L.: Soviet Phys. JETP **32**, 493 (1971)
5. Dunlop, F.: Commun. math. Phys. **49**, 247 (1967)
6. Kunz, H., Pfister, C., Vuillermot, P.
7. Ginibre, J.: Commun. math. Phys. **16**, 310 (1975)
8. Brascamp, H., Lieb, E.: J. Funct. Anal. **22**, 366 (1976)

Communicated by A. Jaffe

Received October 27, 1976