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Schrödinger Operators with L^p_{loc}-Potentials

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Abstract. We discuss the question of when the closure of the Schrödinger operator, $-\Delta+V$, acting in $L^p(R^l, d^lx)$, generates a strongly continuous contraction semigroup. We prove a series of theorems proving the stability for $-\Delta: L^p \to L^p$ of the property of having a m-accretive closure under perturbations by functions in $L^q_{\text{loc}}(1<p\leqq q)$. The connection with form sums and the Trotter product formula are considered. These results generalize earlier results of Kato, Kalf-Walter, Semenov and Beliy-Semenov in that we allow more general local singularities, including arbitrary singularities at one point, and arbitrary growth at infinity. We exploit bilinear form methods, Kato's inequality and certain properties of infinitesimal generators of contractions.

1. Introduction and Results

Kato [1] showed that the L^2-operator sum, $-\Delta+V$, is essentially self-adjoint on $C_0^\infty(R^l)$ if $0\leqq V\in L^q_{\text{loc}}(R^l, d^lx)$, $q=2$. In particular, the Trotter product formula holds in this case. However, if $q<2$, it can happen that $\mathscr{D}(-\Delta)\cap\mathscr{D}(V)=\{0\}$, so that the operator sum $-\Delta+V$ is not densely defined. Nevertheless, in Semenov [3] and Beliy-Semenov [7, 12], an operator H is constructed so that the Trotter product formula

$$e^{-tH}=s\text{-lim}(e^{-tH_0/n}e^{-tV/n})^n$$

holds so long as $0\leqq V\in L^q(R^l, d^lx)$ $q\geqq 1$ or $0\leqq V\in L^q_{\text{loc}}(R^l, d^lx)$, $q\geqq 1$. (Here and below the symbol s-lim stands for an L^2 strong limit.) H was constructed as a form sum, and, in the second case, Kato's inequality was essentially employed. In addition we developed a criterion for a sum to have an m-accretive closure.

We recall that an operator A is called m accretive if and only if $-A$ generates a contraction semigroup e^{-tA}. We call D an m-accretive core for A if and only if the closure of $A\restriction D$ is m-accretive. If more than one Banach space is possible, e.g. $D=C_0^\infty(R^l)$ we will sometimes modify the phase m-accretive with a Banach space, e.g. L^p-m-accretive.

In the present paper, we wish to generalize the aforementioned results for Schrödinger operators $-\Delta+V: L^p(R^l)\to L^p(R^l)$, $1<p<\infty$.

Theorem 1.1. *Let l be an integer $\geqq 1$, and let $q>1$. Suppose that:*

(1) $V=V_+-V_-$; $V_\pm\geqq 0$.

(2) $V_\pm\in L^q_{\mathrm{loc}}(R^l)$.

(3) *For suitable fixed $b\geqq 0$ and $0\leqq a<\frac{1}{2}$ and all $u\in C_0^\infty(R^l)$:*

$\|V_-u\|_q\leqq a\|\Delta u\|_q+b\|u\|_q$.

Then the closure of $(-\Delta+V)\upharpoonright C_0^\infty(R^l): L^q\to L^q$ is a generator of bounded holomorphic semigroup.

Theorem 1.2. *Let $l\geqq 3$, $q>1$. Assume that*

(1) $V=V_+\geqq 0$.

(2) $V_+\in L^q_{\mathrm{loc}}(R^l\setminus\{0\})$.

Let $p\leqq r\equiv\min(q, l/2)$. Then $C_0^\infty(R^l\setminus\{0\})$ is an L^p-m-accretive core for $-\Delta+V$.

Theorem 1.3. *Let $l\geqq 3$ and $q>1$. Assume that*

(1) $V=V_+\geqq 0$.

(2) $V_+\in L^q_{\mathrm{loc}}(R^l\setminus\{0\})$.

Let $H=H_0\dot{+}V$ be the form sum of $H_0=(-\Delta\upharpoonright C_0^\infty)^\sim$ and V. Then on L^2:

$e^{-tH}=s\text{-}\lim(e^{-tH_0/n}e^{-tV/n})^n$

for each $t>0$.

Theorem 1.4. *Let $l\geqq 2$. Assume that*

(1) $V\in L^2_{\mathrm{loc}}(R^l\setminus\{0\})$.

(2) $V\geqq[1-(1-l/2)^2]|x|^{-2}$.

Then $-\Delta+V: L^2\to L^2$ is essentially self-adjoint on $C_0^\infty(R^l\setminus\{0\})$.
In particular, for each $-\infty<t<\infty$,

$s\text{-}\lim[e^{-itH_0/n}e^{-itV/n}]^n=e^{-itH}$.

Remark 1. Theorem 1.1 in case $q=2$ (but with $a<\frac{1}{2}$ replaced by the weaker $a<1$) is due to Kato [1]. Theorems 1.2 and 1.3 are generalizations of results of Beily and Semenov [7, 12]. Theorems 1.3 and 1.4 are generalizations of a theorem of Kalf-Walter [14] (see also [2]). Simon [11] has also proven Theorem 1.4.

Remark 2. If condition (2) of Theorem 1.4, we require that

$V\geqq a|x|^{-2}$; $a>1-(1-l/2)^2$

then, we obtain supplementary information about H, namely $\mathscr{D}(H)\subset\mathscr{D}(|x|^2)$.

Remark 3. Our method of proof of Theorem 1.4 may be extended to the case of N-particle Hamiltonians with two particle potentials $V_{ij}\in L^2(R^l\setminus\{0\})$ that

$V_{ij}\geqq[1-(1-l/2)^2+\varepsilon]|x_{ij}|^{-2}$, $\quad\varepsilon>0$

for the "physical" dimension $l=1,2,3$ [15].

Remark 4. Since $L^q_{\rm loc} \subset L^p_{\rm loc}$ for $p<q$, under the hypothesis of Theorem 1.1, we have that C_0^∞ is an L^p-m-accretive core for $-\Delta+V_+$ (note the $+$) for any p with $1<p\leqq q$.

In the proofs of the above theorems, we use Kato's inequality, and contraction and holomorphic properties of the semigroup generated by Δ.

2. The Semigroup Extension of $-\Delta+V$ on L^p Defined via a Form Sum

Let $\mathscr{H}=L^2(R^l,d^lx)$, $l\geqq 1$ and let $H_0=[-\Delta\upharpoonright C_0^\infty(R^l)]^\sim$ where $\sim$ denotes operator closure. It is well-known that H_0 is a self-adjoint operator and that the generated semigroup $\exp(-tH_0)$ is a contraction on all L^p, $1\leqq p\leqq\infty$; i.e. for all $t\geqq 0$, $u\in\mathscr{H}\cap L^p$:

$$\|\exp(-tH_0)u\|_p\leqq\|u\|_p.$$

Let $S_p(t)$ be the unique bounded extension of $\exp(-tH_0)\upharpoonright\mathscr{H}\cap L^p$ to all of L^p. Then, for $1\leqq p<\infty$, $S_p(t)$ is a C_0-contraction semigroup, so that there is an operator $H_{0,p}$ with $S_p(t)=\exp(-tH_{0,p})$. In particular, $H_{0,2}\equiv H_0$.

Let V be a non-negative function in $L^1_{\rm loc}(R^l\diagdown\{0\})$ and use V to also denote the associated self-adjoint operator on L^2. As above define a C_0-contraction semigroup $Q_p(t):L^p\to L^p(1\leqq p<\infty)$ and V_p so that $Q_p(t)=\exp(+tV_p)$; $V_2\equiv V$.

Define $H=H_0+V$ as the self-adjoint operator obtained as the form sum [8, Chapt. VI]. It is easy to see (e.g. [3, 4, 7]) that the semigroup $\exp(-tH)$ is a contraction on each $L^p(1\leqq p\leqq\infty)$. *Moreover the extension* $R_p(t)$ of $\exp(-tH)\upharpoonright L^p\cap\mathscr{H}$ to L^p defines a C_0-contraction semigroup if $1<p<\infty$ (see [7] or Proposition 2.2 below). As above, $R_p(t)=\exp(-tH_p)$; $H_2\equiv H$.

We systematically use the following notation in the results below: $A=H_{0,p}$, $B=V_p$, $C=H_p$, $B_n=B$ on the space where $B\leqq n$, $B_n=0$ otherwise, $C_n=A+B_n$. We use R^l_+ to denote $R^l\diagdown\{0\}$ and $\mathscr{L}(L^p,L^q)$ denotes the space of bounded operators from L^p to L^q. $\mathscr{L}(L^p)\equiv\mathscr{L}(L^p,L^p)$.

Proposition 2.1. *For each $t>0$ and all $1<p<\infty$,*

$$\lim_{n\to\infty}\exp(-tC_n)=\exp(-tC)$$

in the strong $\mathscr{L}(L^p)$ topology.

Proof. The case $p=2$ follows from theorems of Kato [8, Chapt. VII, Theorem 3.13] and Trotter-Kato [9]. The general case follows from the following Proposition.

Proposition 2.2. *Let $S_k:L^1\cap L^\infty\to L^1\cap L^\infty$, $k=0,1,\ldots$, so that $\|S_kg\|_p\leqq\|g\|_p$, all k, $1\leqq p\leqq\infty$. If $S_kg\to S_0g$ for all $g\in\mathscr{H}$, then $S_kg\xrightarrow{L^p}S_0g$ for all $g\in L^p$, $1<p<\infty$.*

Proof. Let $f\in L^1\cap L^\infty$. By Hölder's inequality, for $1<p\leqq 2$

$$\begin{aligned}\|S_kf-S_0f\|_p^p&\leqq\|S_kf-S_0f\|_1^{2-p}\|S_kf-S_0f\|_2^{2p-2}\\&\leqq 2^{2-p}\|f\|_1^{2-p}\|S_kf-Sf\|_2^{2p-2}\end{aligned}$$

and for $2\leqq p<\infty$

$$\|S_kf-S_0f\|_p^p\leqq 2^{p-2}\|f\|_\infty^{p-2}\|S_kf-Sf\|_2^2.$$

Thus, for such f, $S_kf\to S_0f$ in L^p-norm. Since such f's are dense, $S_kg\to S_0g$ for any g in L^p.

Proposition 2.3. *The semigroups* $\exp(-tC), \exp(-tC_n) n=1,2,\ldots$ *are holomorphic on*

$$\Gamma_p = \{t \mid |\arg t| < (1-|1-2p^{-1}|)\pi/2\}$$

and uniformly bounded on Γ_p. *The convergence of* $\exp(-tC_n)$ *to* $\exp(-tC)$ *in the strong* L^p *topology is uniform on compact subsets* K *of* Γ_p *for* $1<p<\infty$.

Proof. The semigroups $\exp(-tB_n)$ are holomorphic on $\mathrm{Re}\,t>0$ and

$$\|\exp(-tB_n)g\|_p \leqq \|g\|_p \leqq \|g\|_p \quad 1 \leqq p \leqq \infty; \ \mathrm{Re}\,t>0. \tag{1}$$

The semigroup $S_p(t)=\exp(-tA)$ is a contraction on L^p for $1 \leqq p \leqq \infty$. Thus, by the Stein interpolation theorem (e.g. [13, Proposition II.5]):

$$\|S_p(z)g\|_p \leqq \|g\|_p \quad z \in \bar{\Gamma}_p \tag{2}$$

and $S_p(z)$ is analytic on Γ_p. By the Trotter formula, for each $t>0$ and any $g \in L^p$, $1<p<\infty$

$$\exp(-tC_n)g = L^p\text{-}\lim_{\varkappa\to\infty} [S_p(t/\varkappa)\exp(-B_n t/\varkappa)]^\varkappa g. \tag{3}$$

Now, by (1) and (2), the functions $\mathscr{I}_{n,\varkappa} = [S_p(z/\varkappa)\exp(-B_n z/\varkappa)]^\varkappa g$ are analytic and uniformly bounded on Γ_p. By (3), they converge pointwise on the positive real axis.
Thus, by the Vitali convergence theorem, $\exp(-zC_n)g$ is analytic in Γ_p and:

$$\exp(-zC_n)g = L^p\text{-}\lim_{\varkappa\to\infty} \mathscr{I}_{n,\varkappa} g; \ z \in \Gamma_p, \ g \in L^p, \tag{4}$$

$$\|\exp(-zC_n)g\|_p \leqq \|g\|_p \quad z \in \Gamma_p, \ g \in L^p. \tag{5}$$

Using (5) and Proposition 2.1, we can repeat the above argument for the convergence of $\exp(-tC_n)$ to $\exp(-tC)$.

Proposition 2.4. *Let* V *be a positive function in* $L^q_{\mathrm{loc}}(R^l_+)$ *for some fixed* $q>1$. *Then, for any* $\varphi \in C_0^\infty(R^l_+)$, $t>0$ *and* $1<p\leqq q$

$$\exp(-tC)(A+B)\varphi = C\exp(-tC)\varphi.$$

Proof. By Proposition 2.3, $C_n \exp(-tC_n)$ converge strongly to $C\exp(-tC)$ in L^p for each $t>0$ and $1<p<\infty$. Also $\exp(-tC_n)A\varphi$ and $\exp(-tC_n)B\varphi$ converge respectively to $\exp(-tC)A\varphi$ and $\exp(-tC)B\varphi$. Thus we need only show that

$$\|\exp(-tC_n)(B_n - B)\varphi\|_p \to 0 \quad \varphi \in C_0^\infty(R^l_+).$$

But since $\|\exp(-tC_n)\| \leqq 1$, and V is in $L^q_{\mathrm{loc}}(R^l_+)$, this is evident.

Proposition 2.5. *Let* V *be a positive function in* $L^q_{\mathrm{loc}}(R^l_+)$ *for some fixed* $q>1$. *Then, for each* $1<p\leqq q$ $C_0^\infty(R^l_+) \subset \mathscr{D}(C)$ *and*

$$C \restriction C_0^\infty(R^l_+) = A + B \restriction C_0^\infty(R^l_+).$$

Proof. Let $\varphi_\varkappa = \exp(-C/\varkappa)\varphi$, $\varphi \in C_0^\infty(R^l_+)$. Then, as $\varkappa\to\infty$, $\varphi_\varkappa \xrightarrow{L^p} \varphi$ and, by Proposition 2.4, $C\varphi_\varkappa \xrightarrow{L^p} (A+B)\varphi$. Since C is closed, the result is proven.

Remark. Using the same argument, it can be shown that $C \restriction C_0^\infty(R^l \setminus S) = -\Delta + V \restriction C_0^\infty(R^l \setminus S)$ on L^p, $1<p\leqq q$ if V is positive and in $L^q_{\mathrm{loc}}(R^l \setminus S)$ where S is an arbitrary closed set of measure zero.

3. Kato's Inequality and the Generation of an L^p-Semigroup by the Closure of $-\Delta+V$

Proposition 3.1. *Let $V_+\in L^q_{loc}(R^l)$ for some $q\neq\infty$. Then $(A+B+1)C_0^\infty(R^l)$ is dense in L^q.*

Proof. Suppose that $(A+B+1)C_0^\infty$ is not dense in L^q.

Let M be its closure. Since $M\neq L^q$, there exists $f\in L^{q'}(q'=(1-q^{-1})^{-1})$ so that $f\neq 0$ and

$$\langle f,(-\Delta+V_++1)\varphi\rangle=0\,;\qquad \varphi\in C_0^\infty(R^l).$$

Rewriting $-\Delta$ as a map from L^1_{loc} into $(C_0^\infty(R^l))'$ we have that $\Delta f=V_+f+f$. Since $f\in L^{q'}$ and $V_+\in L^q_{loc}$, $V_+f+f\in L^1_{loc}$. Thus Kato's inequality holds:

$$\Delta|f|\geqq \mathrm{Re}((\mathrm{sgn}f)\Delta f)=V_+|f|+|f|\geqq|f|$$

so that $(-\Delta+1)|f|\leqq 0$ and $f=0$.

This contradiction shows that $M=L^q$.

Remarks. 1. The proof of Proposition 3.1 is, in fact, a slight modification of Kato's proof [1, 10] that $(-\Delta+V_++1)C_0^\infty$ is dense in L^2 when $V_+\geqq 0$, $V_+\in L^2_{loc}(R^l)$.

2. Since $L^p_{loc}\subset L^q_{loc}$ if $p\leqq q$, $(A+B+1)C_0^\infty(R^l)$ is dense in any L^p, $1\leqq p\leqq q$.

Proposition 3.2. *Let $V_+\in L^q_{loc}(R^l)$ for some $q>1$. Then, $A+B$ with domain $\mathscr{D}(A)\cap\mathscr{D}(B)$ is closable. It closure, C, generates a contraction semigroup on all $L^p(1\leqq p<\infty)$ and $C_0^\infty(R^l)$ is a core for C.*

Proof. Apply Propositions 2.5 and 3.1.

Proposition 3.3. *Let $V=V_+-V_-$; $0\leqq V_\pm\in L^q_{loc}(R^l)$ for some $q>1$. Suppose that for some $b\geqq 0$, $a\in[0,1/2)$ and all $u\in C_0^\infty(R^l)$:*

$$\|V_-u\|_q\leqq a\|\Delta u\|_q+b\|u\|_q.$$

Then

$$\|V_-u\|_q\leqq a\|(\Delta-V_+)u\|_q+b\|u\|_q$$

for all $u\in C_0^\infty(R^l)$.

Proof. This proposition is a direct consequence of a lemma of Davies and Faris [6, Lemma 2].

Proof of Theorem 1.1. Apply Propositions 3.2 and 3.3.

Proposition 3.4. *Let $l\geqq 3$. Let $V_+\in L^q_{loc}(R^l_+)$ for some $q>1$. Let $q_0=\min(q,l/2)$. Then the range of $(A+B+1)\restriction C_0^\infty(R^l_+)$ is dense $L^p(R^l)$ for any $1<p\leqq q_0$.*

Proof. The argument used in the proof of Proposition 3.1 may be generalized in the following way: Let M be the closure of $(A+B+1)C_0^\infty(R^l_+)$. By the Hahn-Banach theorem, for any $v\neq 0$ in $L^p\setminus M$, there exists an $f\in L^{p'}$ $(p'=(1-p^{-1})^{-1})$ with $\langle f,v\rangle=1$, $\langle f,u\rangle=0$ for all $u\in M$. Since $V_+\geqq 0$, by using Kato's inequality as in the proof of Proposition 3.1, we have

$$\langle|f|,(-\Delta+1)\varphi\rangle\leqq 0\,;\varphi\in C_0^\infty(R^l_+),\ \varphi\geqq 0\,.$$

Suppose that for any $\varphi \in \mathscr{S}(R^l)$ with $\varphi \geqq 0$, we can construct a sequence $\varphi_n \geqq 0$ with $\varphi_n \in C_0^\infty(R^l_+)$ so that

$$\lim_{A\to\infty} \langle |f|, (-\Delta+1)\varphi_n\rangle = \langle |f|, (-\Delta+1)\varphi\rangle .$$

Then noting that $(-\Delta+1)^{-1}$ takes $\{\varphi \in \mathscr{S} | \varphi \geqq 0\}$ into itself we have that $\langle |f|, g\rangle \leqq 0$ for any $g \in \mathscr{S}$ with $g \geqq 0$, for take $\varphi = (-\Delta+1)^{-1} g \geqq 0$ in the above. Thus $f=0$ and the proof is complete.

Such a sequence is not difficult to construct. In fact, let λ and μ be fixed C^∞ functions with $0 \leqq \mu, \lambda \leqq 1$ so that $\lambda(x)=1$ if $|x|>1$, $\lambda(x)=0$ if $|x|=\frac{1}{2}$; $\mu(x)=1$ if $|x|<1$, $\mu(x)=0$ if $|x|>2$. Let

$$\omega_n(x) = \lambda(nx)\mu(n^{-1}x).$$

Then $\omega_n \in C_0^\infty(R^l_+)$, $0 \leqq \omega_n \leqq 1$, $\omega_n(x)=1$ if $1/n<|x|<n \omega_n(x)=0$, if $|x|>2n$ or $|x|<1/2n$ and moreover:

$$|\nabla\omega_n(x)| \leqq D|x|^{-1}; |\Delta\omega_n(x)| \leqq D|x|^{-2}.$$

Now let $\varphi \in \mathscr{S}$, $\varphi \geqq 0$ and define $\varphi_n = \omega_n \varphi$. Then

$$\Delta(\omega_n\varphi) = \omega_n \Delta\varphi + 2\nabla\omega_n \cdot \nabla\varphi + \varphi\Delta\omega_n$$

so

$$\langle |f|, (\Delta-1)\varphi_n\rangle = I^{(n)} + I_1^{(n)} + I_2^{(n)},$$

where $I^{(n)} = \langle |f|, \omega_n(\Delta-1)\varphi\rangle$ clearly converge to $\langle |f|, (\Delta-1)\varphi\rangle$, $I_1^{(n)} = 2\langle |f|, \nabla\omega_n \cdot \nabla\varphi\rangle$ and $I_2^{(n)} = \langle |f|, (\Delta\omega_n)\varphi\rangle$. Now, by Hölder's inequality

$$|I_1^{(n)}| \leqq 2D\|f\|_{q'} \left[\int_{\substack{|x|<1/n \\ \text{or } |x|>n}} |x|^{-q} |\nabla\varphi|^q d^l x \right]^{1/q}$$

which goes to zero as $n\to\infty$ since $\varphi \in \mathscr{S}$ and $q \leqq l/2 < l$. Here $q' = (1-q^{-1})^{-1}$. Again by Hölder's inequality:

$$|I_2^{(n)}| \leqq D(A_n + B_n)$$

with $A_n = \|f\|_{q'} \left[\int_{|x|>n} |x|^{-2q} |\varphi|^q d^l x \right]^{1/q}$ which goes to zero as $n\to\infty$ since $\varphi \in \mathscr{S}$ and:

$$B_n = \left[\int_{|x|<1/n} |f(x)|^{q'} d^l x \right]^{1/q'} \left[\int_{(2n)^{-1} \leqq |x| \leqq 1/n} |\varphi(x) x^{-2}|^q d^l x \right]^{1/q}$$

which goes to zero as $n\to\infty$ since the first term goes to zero and the second is bounded when $q \leqq l/2$.

Proof of Theorem 1.2. Apply Propositions 2.5 and 3.4.

Remark. It follows from Proposition 3.4 that if $l \geqq 4$, $-\Delta + V_+ : L^2 \to L^2$ is essentially self-adjoint on $C_0^\infty(R^l \setminus S)$ whenever $V_+ \in L^2_{\text{loc}}(R^l \setminus S)$ where $S = \{a_0, \ldots, a_\nu\}$.

Theorem 3.1. *Suppose that the hypotheses of Theorems* 1.1 *or 1.2 hold. Then for* $1 < p < \infty$

$$\exp(-tC) = \lim_{n\to\infty} [\exp(-tA/n)\exp(-tB/n)]^n \tag{6}$$

in the strong L^p-topology.

Proof. If $1 < p \leqq q$, then (6) follows from Theorems 1.1 or 1.2 and the Trotter theorem. To obtain (6) for $p > q$, it suffices to use Proposition 2.2 modified by replacing the assumption $S_\varkappa g \to Sg$ *in* $\mathscr{H}$ by an assumption of convergence in L^q.

Proof of Theorem 1.4. Let $\beta_0 = 1-(1-l/2)^2$ and let $V_\varepsilon = V + \varepsilon|x|^{-2}$ for some $\varepsilon \in (0,1)$. We begin by showing that $-\Delta + V_\varepsilon$ is essentially self-adjoint on $C_0^\infty(R^l_+)$.

As in the proof of Proposition 3.4, we obtain

$$\langle |f|, (-\Delta + (\beta_0+\varepsilon)|x|^{-2} + E)\varphi\rangle \leqq 0$$

for all $f \in [-\Delta + V_\varepsilon + E[C_0^\infty(R^l_+)]]^\perp$, $E > 0$ and $0 \leqq \varphi \in C_0^\infty(R^l_+)$.

We use the fact that $-\Delta + \beta|x|^{-2}$ is essentially self-adjoint on $C_0^\infty(R^l_+)$ for $\beta \geqq \beta_0$ and the inequality [2]:

$$\|(-\Delta + (\beta_0+\varepsilon)|x|^{-2} + E)u\|_2 \geqq \varepsilon \| \, |x|^{-2}u\|_2 \,; u \in C_0^\infty(R^l_+)\,. \tag{7}$$

Let $Z = -\Delta + (\beta_0+\varepsilon)|x|^{-2}$ with $\mathscr{D}(Z) = C_0^\infty(R^l_+)$. By (7), the closure of Z, obeys $\mathscr{D}(Z^\sim) \subset \mathscr{D}(|x|^{-2})$.

Let $\{\varrho_n\}$ be a sequence of functions obeying

$$0 \leqq \varrho_n \in C_0^\infty(R^l)\,; \int_{R^l} \varrho_n(x) d^l x = 1$$

$$\operatorname{supp} \varrho_n = \{x \mid |x| \in [\alpha_n, \beta_n]\}\,; \alpha_n, \beta_n \to 0\,.$$

Given $\varphi \geqq 0$ in $\mathscr{D}(Z^\sim)$, let $\varphi_n = \omega_n \varphi$, $\varphi_{n,\varkappa} = \varphi_n * \varrho_\varkappa$ where ω_n is the sequence constructed in the proof of Proposition 3.4. Clearly $0 \leqq \varphi_{n,\varkappa} \in C_0^\infty(R^l_+)$ and

$$\lim_{\varkappa\to\infty} \langle |f|, (Z^\sim + E)\varphi_{n,\varkappa}\rangle = \langle |f|, (Z^\sim + E)\varphi_n\rangle\,.$$

Since $\mathscr{D}(Z^\sim) \subset \mathscr{D}(|x|^{-2})$

$$\lim_{n\to\infty} \langle |f|, (Z^\sim + E)\varphi_n\rangle = \langle |f|, (Z^\sim + E)\varphi\rangle$$

by using the argument from Proposition 3.4.

Noting that $(Z^\sim + E)^{-1}$ is positivity preserving [4, Theorem 5.1], we conclude by the standard argument [1] already used that $-\Delta + V_\varepsilon$ is essentially self adjoint on $C_0^\infty(R^l_+)$.

By a lemma of Davies and Faris [6, Lemma 2],

$$\|(-\Delta + V_\varepsilon)u\|_2 \geqq \varepsilon \| \, |x|^{-2}u\|_2$$

so $-\Delta + V_\varepsilon - \varepsilon|x|^{-2} = -\Delta + V$ is essentially self-adjoint on $C_0^\infty(R^l_+)$.

Theorem 1.4 is thus proven.

Added Note. The methods of this note have been extended by the author to deal with many particle Schrödinger operators. These results will appear in Ann. Ins. H. Poincaré.

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