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The Cluster Expansion for Potentials with Exponential Fall-off*

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Abstract. Continuing the work of a previous paper, the Glimm-Jaffe-Spencer cluster expansion from constructive quantum field theory is adapted to treat quantum statistical mechanical systems of particles interacting by potentials that fall off exponentially at large distance. The Hamiltonian H_0+V need be stable in the extended sense that $H_0+4V+BN\geqq 0$ for some B. In this situation, with a mild technical condition on the potentials, the cluster expansion converges and the infinite volume limit of the correlation functions exists, at low enough density. These infinite volume correlation functions cluster exponentially. A natural system included in the present treatment is that of matter with the r^{-1} potential replaced by $e^{-\alpha r}/r$. The Hamiltonian is stable, but the system would collapse in the absence of the exclusion principle—the potential is unstable. Therefore this system cannot be handled by the classic work of Ginibre, which requires stable potentials.

1. Introduction

In a previous paper, [1], we adapted the Glimm-Jaffe-Spencer cluster expansion [8] to treat quantum statistical mechanical systems with finite range potentials. We now extend this program to include potentials that fall off exponentially. Under very general conditions we will obtain the infinite volume limit of correlation functions (in the Euclidean region) and their exponential clustering, at low density. We will later remark on some extensions of the present work to even more general potentials.

Matter (positive charged particles and negative charged identical fermions interacting with a r^{-1} potential) with the r^{-1} modified to $e^{-\alpha r}/r$, one of our matter-like systems, has been our main motivational example. For this system the Hamiltonian is stable; proofs of stability for the matter system [4, 5, 10] may be modified to show this. But the potential is not stable, [11] and in fact the system

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would collapse in the absence of the exclusion principle [3]. The classic work of Ginibre [7], does not apply, requiring stable potentials. For this system our method will yield, at any fixed temperature for low enough density, the infinite volume limit of expectations of products of spatially smeared Euclidean densities, and their exponential clustering.

In fact the cluster expansion we use in this paper is different from the one used in [1] and [8], and is a slightly simplified form of the expansion developed by Glimm, Jaffe, and Spencer in [9]. We could have used the expansion from [1] and a scheme like that in [6] to interpolate potentials, but the route taken in the present paper leads to more general results. We are indebted to [9] for some conceptual ideas, and a numerical estimate, but the reader is only assumed to be familiar with [1] and [8].

2. Notation

We follow the notation of [1] closely, but recall some of the equations for convenience. There are ℓ species of particles, each obeying either fermion or boson statistics, described by fields, $\phi_1, \phi_2, \ldots, \phi_\ell$. We set

$$H_{00} = \sum_{i=1}^{\ell} (2m_i)^{-1} \int dx (\nabla \phi^-_i)(\nabla \phi_i)\,, \tag{2.1}$$

$$H_0 = H_{00} - \sum_{i=1}^{\ell} \mu_i \int dx \phi^-_i \phi_i\,, \tag{2.2}$$

$$N_i = \int dx \phi^-_i \phi_i\,; \quad N = \sum_{i=1}^{\ell} N_i\,, \tag{2.3}$$

$$H = H_0 + V\,, \tag{2.4}$$

$$\varrho_i(x) = \phi^-_i \phi_i(x)\,, \tag{2.5}$$

$$V = 1/2 \sum_{i,j} \int dx dy :\varrho_i v_{ij} \varrho_j:. \tag{2.6}$$

We consider objects A of the form

$$A = a_1(t_1) \ldots a_s(t_s)\,, \tag{2.7}$$

$$a_i(t_i) = \sum_j \int dx f_{ij}(x) \varrho_j(x)\,. \tag{2.8}$$

For a given i, each f_{ij} is supported in a single cube. The f_{ij} are real, measurable, and $0 \leqq f_{ij} \leqq 1$; this allows our estimates to be taken to depend on A only through s, the number of factors.

The objects of interest to us are expectations

$$\langle A \rangle_\Lambda = \mathrm{Tr}_\Lambda(\boldsymbol{T}\exp(-\int_0^\beta H^\Lambda(\tau)d\tau)A)/\mathrm{Tr}_\Lambda(\exp(-\beta H^\Lambda))\,. \tag{2.9}$$

The times correspond to imaginary real times—one is in the Euclidean region—of course if all the times in A are equal, the expectation $\langle A \rangle_\Lambda$ is the same as a real time expectation value. Λ is the large box one works in, Tr_Λ is the trace on the Fock space built on $L^2(\Lambda)$. *For* each $\xi \in R^3$, we denote the translation, in the obvious sense, of A by A_ξ. β is fixed throughout our discussion, dependences on β are suppressed. All our constants, $\{c_\alpha\}$, satisfy

$$0 < c_\alpha < \infty . \tag{2.10}$$

3. Results and Discussion

We assume our system has the following properties

a) There are a c_1, c_2, and c_0 such that

$$|v_{ij}(x)| \leqq c_1 \exp(-c_2|x|) \tag{3.1}$$

for $|x| \geqq c_0$.

b) There is a B such that

$$H_0 + 4V + BN \geqq 0 .$$

c) Each v_{ij} is in $L_{3/2}$. (The choice of L_2 instead of $L_{3/2}$ would lead to a more standard analysis.)

We then have the following basic theorems proven in the following sections.

Theorem 1. *There exists a μ_0 such that if $\mu_i \leqq \mu_0$, all i, then for any A as defined in Section 2,*

$$\lim_{\Lambda \to \infty} \langle A \rangle_\Lambda$$

exists. The limit is understood to be taken through any sequence of boxes centered at the origin whose minimum width goes to infinity.

We denote the limit in the theorem as $\langle A \rangle$.

Theorem 2. *There exists a μ_0 such that if $\mu_i \leqq \mu_0$, all i, then for any A and B as defined in Section 2,*

$$|\langle AB_\xi \rangle - \langle A \rangle \langle B \rangle| \leqq c_{A,B} \exp(-c(\mu_0)|\xi|) \tag{3.2}$$

for $|\xi|$ large enough. $c(\mu_0) \to c_2$ as $\mu_1, \mu_2, \ldots, \mu_\ell \to -\infty$.

We choose the μ_0 in Theorem 1 and Theorem 2 to be the same, at the expense of possibly not using the best value of μ_0 in Theorem 1. Theorem 3.4 and Proposition 3.1 from [1] also hold but we do not restate them.

We carry out the proofs using unit cubes and barriers of width 2/10 as in [1]. We assume instead of a) above the following condition:

a′) There are a c_1' and c_2' such that

$$|v_{ij}(x)| \leqq c_1' \exp(-c_2'|x|) \tag{3.3}$$

if $|x| \geqq 2/10$.

A length scaling argument then shows this is sufficient to yield our general results. This is equivalent to using larger cubes.

Remarks. 1) In condition b) above it is sufficient to have $(1+\varepsilon)V$ with $\varepsilon>0$ instead of $4V$.

2) The technical condition c) may be weakened, to include infinitely repulsive hard cores, for example.

3) It is not difficult to accommodate many-body potentials that satisfy suitable substitutes for a) and c).

4) It should be possible to treat potentials that fall off as a suitably high power rather than exponentially, yielding a weaker cluster property. One may have to modify the cluster expansion to obtain the best results here.

5) Suitably smeared reduced density matrices are also tractable.

6) The Mayer expansion may be shown to converge. The ratio of Z's for complex z may be studied as in [8]; the techniques of [1] are not sufficient here.

After we have developed the cluster expansion and proved convergence (Estimate 5.1), the proof of Theorem 1 proceeds as in [1]. In Theorem 2, (3.2) is deduced from the cluster expansion by a "doubling the measure" argument. See for example [8]. The statement $c(\mu_i)\to c_2$ in Theorem 2 is a consequence of tracing the effects of $\mu_i\to-\infty$ painfully through the convergence proof. Various remarks about this are inserted through the remainder of the paper.

In summary, the cluster expansion in statistical mechanics is a powerful tool in the study of low density systems. Some of the lines of possible development have been mentioned above. Constructive quantum field theory should continue to be a source of ideas for statistical mechanics.

4. The Cluster Expansion

Since, as mentioned above, the cluster expansion differs from that in [1], we will redevelop the expansion, with a minimal change in notation. R^3 is filled with closed unit cubes, $\{\Delta_i\}$, with disjoint interiors. The set of faces of these cubes, taken as closed, are called $\{S_\alpha\}$. The set of points within distance 1/10 of S_α is called η_α (the barrier α)

$$\eta_\alpha=\{x\in R^3:\mathrm{dist}(x,S_\alpha)\leqq 1/10\} \tag{4.1}$$

Λ, the large box we work in, is a union of cubes Δ_i. $\{\Delta_j: j\in J_1\}$ is a distinguished set of cubes.

The expectation $\langle A\rangle_\Lambda$ given in Equation (2.9) is rewritten in path space with the same notation as Equation (2.8) of [1].

$$\left(\int_\Lambda d\mu \exp\left(-\int_0^\beta V(\tau)d\tau\right)a_1(t_1)\ldots a_s(t_s)\right)\Big/\left(\int_\Lambda d\mu\exp\left(-\int_0^\beta V(\tau)d\tau\right)\right) \tag{4.2}$$

We refer to [1] for the definitions of the path space integrals. For simplicity we define a function U on path space

$$U=\int_0^\beta V(\tau)d\tau\,. \tag{4.3}$$

For a set $S \subset \Lambda$ we define $\partial S = (\bar{S} - \mathrm{Int}\, S) - \partial\Lambda$; and $\hat{S} = \{x \in S : \mathrm{dist}(x, \partial S) > 1/10\}$. E_α is the characteristic function of the subset of path space consisting of all n-paths such that no particle hits the barrier η_α. $H_\alpha = 1 - E_\alpha$. If N is a union of faces S_α, then

$$E_N = \prod_{S_\alpha \in N} E_\alpha$$

$$H_N = \prod_{S_\alpha \in N} H_\alpha . \tag{4.4}$$

Given a set of cubes $\{\Delta_j : j \in J\}$, a union of cubes X and a union of faces Γ, the pair (X, Γ) will be said to *isolate* J if

1) $(\Gamma \cap \mathrm{Int}\, X)^- = \Gamma$

2) each connected component of $X - \Gamma^c$ contains at least one $\Delta_j^{\mathrm{int}}, j \in J$.

[The bar in 1) indicates closure, Γ^c is the set of faces S_α in Λ, complementary to Γ, considered as a subset of R^3.] We have the crucial identity, for any J, as above

$$1 = \sum_{X, \Gamma} H_\Gamma E_{\Gamma^c \cap X} \tag{4.5}$$

where the sum is over pairs (X, Γ) that isolate J. $\Gamma^c \cap X$ is the union of faces in X not in Γ. *The identity of functions on path space given by Equation* (4.5) *substituted into the numerator of Equation* (4.2) *is exactly the cluster expansion of* [1], for a correct choice of J.

We now must discuss the interpolation of potentials. Given a union of cubes, X, in Λ, we interpolate the two body potential between its original form, and the potential with elimination of any interaction between a particle in X and a particle in $\Lambda - X$. This process introduces a parameter s. Specifically

$$\int : \varrho_i(x)\, \omega(x, y)\, \varrho_j(y) : \tag{4.6}$$

becomes

$$\int [\chi_X(x)\chi_X(y) + \chi_{\Lambda - X}(x)\chi_{\Lambda - X}(y) + s\chi_X(x)\chi_{\Lambda - X}(y) + s\chi_{\Lambda - X}(x)\chi_X(y)]$$
$$:\varrho_i(x)\omega(x, y)\varrho_j(y):. \tag{4.7}$$

For convenience we define $O(X, s)$ as an operation that carries (4.6) to (4.7). (This interpolation is the same as the interpolation of covariances in [9].) For an operator M built up as sums and integrals of objects like (4.6), such as U of (4.3), we have

$$e^{-M} = e^{-O(X, 0)M} + \int_0^1 ds (d/ds) e^{-O(X, s)M} . \tag{4.8}$$

The differentiation in (4.8) brings down from the exponent an operator that only involves interactions between particles in X with particles in $\Lambda - X$.

We now describe the cluster expansion. $Y_1, Y_2, \ldots, Y_n$ is a sequence of non-empty unions of cubes with disjoint interiors. All the Y_i except possibly Y_1 will be connected. We also define $X_1 = Y_1, X_{i+1} = X_i \cup Y_{i+1}$. $J_n = J_1 \cup \{j_2, \ldots, j_n\}$ with $\Delta_{j_i} \in Y_i$. Any such choice of $\{Y_i\}$ *and* $\{j_i\}$ we denote as a pair (ξ_n, J_n). There are parameters

$s_1, \ldots, s_n$ with $0 \leqq s_i \leqq 1$, the ordered set of s_i will be denoted by σ_n. The interpolated potentials $U(\xi_n, \sigma_n)$ are defined inductively

$$U(\xi_1, \sigma_1) = O(X_1, s_1)U$$
$$U(\xi_n, \sigma_n) = O(X_n, s_n)U(\xi_{n-1}, \sigma_{n-1}) . \tag{4.9}$$

We define

$$W(\xi_n, J_{n+1}, \sigma_n) = \prod_{i=1}^{n} \left(-\frac{d}{ds_i} U(\xi_i, \sigma_i) \right)_{j_i+1} \tag{4.10}$$

where the subscript indicates the localization of the interaction to $\Delta_{j_{i}+1}$, this localization of the term in parentheses then involves interaction between a particle in Δ_{j_i+1} with a particle in X_i.

We consider subsets of faces Γ_i, $\Gamma_1 \subset \Gamma_2 \subset \ldots \subset \Gamma_n$, and define

$$F(\xi_n, J_n) = \sum_{\Gamma_n} H_{\Gamma_n} E_{\Gamma_n^c \cap X_n} \tag{4.11}$$

where the sum is over Γ_n such that the pair $(Y_i, \Gamma_i - \Gamma_{i-1})$ isolate $j_i(J_1$, for $i=1)$. The expansion, finally, for (4.2) follows

$$\int_\Lambda d\mu \exp(-U)A / \int_\Lambda d\mu \exp(-U) = \sum_X K(X) \cdot \int_{(\Lambda - X)^\frown} d\mu \exp(-U) / \int_\Lambda d\mu \exp(-U) \tag{4.12}$$

where X appearing in the sum is required to contain the union of cubes in J_1. $K(X)$ is given by

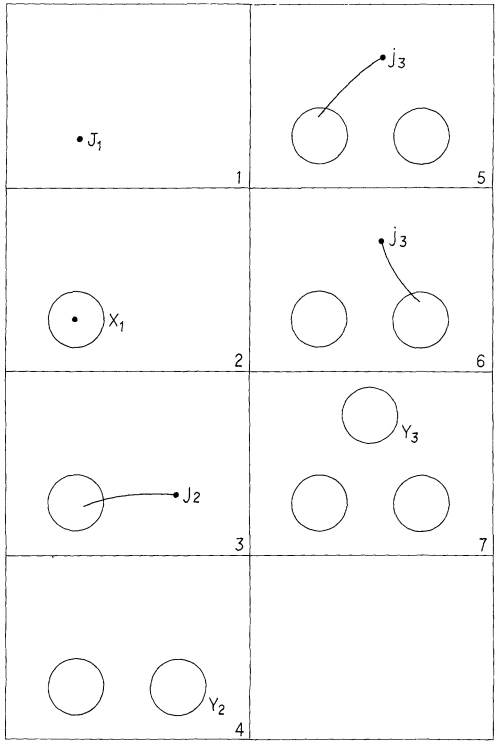
$$K(X) = \sum_{n=1} \sum_{(\xi_n, J_n)} \left(\prod_{i=1}^{n-1} \int_0^1 ds_i \right) \int_X d\mu F(\xi_n, J_n)$$
$$\cdot W(\xi_{n-1}, J_n, \sigma_{n-1}) \cdot \exp(-U(\xi_{n-1}, \sigma_{n-1})) \cdot A . \tag{4.13}$$

The $n=1$ term in the sum is understood as

$$\int_X d\mu F(\xi_1, J_1) \exp(-U) \cdot A . \tag{4.14}$$

In the sum over ξ_n and J_n in (4.13), the restrictions mentioned before Equation (4.9) hold, and $X_n = X$.

The expansion [Eqs. (4.12) and (4.13)] has been developed by iterative applications of (4.5) and (4.8). We enter a casual discussion to help the reader get a feeling for how this has been done, and refer to Figures 1 through 7. One desires the expectation of an operator A located at J_1, schematically represented in Figure 1. The use of Equation (4.5) yields a sum of terms isolating J_1. Figure 2 shows a region X_1 containing J_1 from this sum. There is a barrier of width 2/10 around the boundary of this region, along both sides of which particles obey Dirichlet data. If as in [1] the range of the potentials was less than 2/10 one could stop at this stage, as there would be no interaction between the interior and exterior regions. We rather interpolate the interaction between the interior and exterior regions, writing the result as a term with $s_1 = 0$ in which there is no mutual interaction, that contributes



Figs. 1 – 7

to the $n=1$ term in (4.13), and a differentiated term giving an interaction between particles at j_2 and the interior of X_1. This is illustrated in Figure 3. (4.5) is used again to isolate j_2 in a new component Y_2; Figure 4 illustrates the two regions $X_1 = Y_1$ and Y_2, together forming X_2.

Now particles inside and outside X_2 are separated by a barrier, but the potential still may reach across the barrier. Interpolating again yields a term with no mutual interaction between the interior and exterior of X_2, included in the $n=2$ term in (4.13), and a differentiated term involving interaction between particles at j_3 and particles inside X_2. Figures 5 and 6 illustrate two different possibilities that will be important to distinguish in the estimates of the next sections. Figure 7 isolates j_3 in a new region Y_3, giving the three regions comprising X_3.

If one is familiar with the cluster expansion of [1], the present expansion is thus quite straightforward, although the notation is complex.

5. Convergence

Define $\|X\| = \sup_{x, x' \in X} |x - x'|$. We will prove

$$\sum_{\|X\| \geqq D} |K(X)| \leqq c_A \exp(-c(\mu_0) D) \tag{5.1}$$

where $c(\mu_0) \to c_2'$ as $\mu_0 \to -\infty$. The convergence of (4.12) is implied by (5.1) and

Lemma 5.1. *if* $\Lambda_1 \subset \Lambda$

$$|\mathrm{Tr}_{\Lambda_1}(\exp(-\beta H^{\Lambda_1})/\mathrm{Tr}_{\Lambda}(\exp(-\beta H^{\Lambda})| \leqq 1 .$$

This is a simple consequence of the Minimax Principle. $K(X)$ is estimated by

Lemma 5.2. *Let B be a product of r functions on path space each depending on the n-paths at a single time, then*

$$\left|\int_X d\mu F(\xi_n, J_n) B \exp(-U(\xi_{n-1}, \sigma_{n-1}))\right| \leqq \|F(\xi_n, J_n)B\|_{2,X}$$

$$\cdot\left(\int_X d\mu E_{\partial Y_1 \cup \ldots \cup \partial Y_n} \exp(-2U(\xi_{n-1}, \sigma_{n-1})\right)^{1/2} \cdot 2^{6|X|} \tag{5.2}$$

where

$$\|(\cdot)\|_{p,X} = \left(\int_X d|\mu|\, |(\cdot)|^p\right)^{1/p} .$$

This estimate is a slight generalization of the result in Section 4 of [1]. Furthermore

$$\int_X d\mu E_{\partial Y_1 \cup \ldots \cup \partial Y_n} \exp(-2U(\xi_{n-1}, \sigma_{n-1})) \leqq \exp(c_5|X|) \tag{5.3}$$

by the hypothesis b) for the Hamiltonian, combined with the observation that the left hand side of (5.3) is the trace of the exponential of a convex combination of operators of the form

$$(H_0 + 2V)^{\bigcup_{i \in S_1} \hat{Y}_i} \oplus \ldots \oplus (H_0 + 2V)^{\bigcup_{i \in S_r} \hat{Y}_i}$$

where $(S_1, \ldots, S_r)$ is a partition of $(1, 2, \ldots, n)$.

This is a very helpful feature of the expansion we are using.

We next describe the choice of B in Lemma 5.2. Let η be a map from $\{1, 2, \ldots, n-1\}$ into itself such that $\eta(i) \leqq i$ for $i = 1, \ldots, n-1$. Let $(V(t))_{j', j}$ be the interaction potential energy of paths in Δ_j, at time t with paths in $\Delta_{j'}$ at time t. The definition (4.10) is equivalent to

$$W(\xi_{n-1}, J_n, \sigma_{n-1}) = \left(\prod_{i=1}^{n-1} \int_0^{\beta} dt_i\right) \sum_{\eta} f(\eta, \sigma_{n-2}) \cdot$$

$$\sum_{j'_1 \in Y_{\eta(1)}} \cdots \sum_{j'_{n-1} \in Y_{\eta(n-1)}} \prod_{i=1}^{n-1} (V(t_i))_{j'_i, j_{i+1}} \tag{5.4}$$

where $j \in Y$ means $\Delta_j \subset Y$ and $f(\eta, \sigma_{n-2}) = \prod_{i=1}^{n-1} s_{i-1} s_{i-2} \cdots s_{\eta(i)}$. By convention $s_{i-1} \ldots s_{\eta(i)} = 1$ if $\eta(i) = i$. Thus we choose B in Lemma 5.2 to be $A \prod_{i=1}^{n-1} (V(t_i))_{j'_i, j_i+1}$.

$\|F(\xi_n, J_n)B\|_{2,X}$ is estimated by the Holder inequality. We write (with a simplified notation) $F \cdot B = F \cdot A \prod_i V_i$, and further decompose the product

$$F \cdot B = F \cdot A \prod_i E_{\partial X_i} V_i$$

using the fact that $FE_{\partial X_i} = F$. Thus

$$\|F \cdot B\|_2 \leqq \|F\|_6 \cdot \|A\|_6 \cdot \|\prod_i E_{\partial X_i} V_i\|_6$$

$\|A\|_6$ is treated as in [1] [see Eq. (4.4)]. The following two lemmas handle the other two terms.

Lemma 5.3.

$$\|F(\xi_n, J_n)\|_{6,X} \leqq c_6 \exp(-c_7(\mu_0)(|X| - n + 1 - |J_1|)$$

with $c_7(\mu_0) \to \infty$ *as* $\mu_0 \to -\infty$.

This is a simple consequence of the result in Appendix D in [1].

Lemma 5.4. *Assume* Δ_{j_i}, $i = 1, \ldots, n-1$, *are pairwise distinct, then*

$$\int d^{n-1}t \left\| \prod_{i=1}^{n-1} E_{\partial X_i}(V(t_i))_{j_i', j_{i+1}} \right\|_{6,X} \leqq e^{c_4|X|} \prod_{i=1}^{n-1} c_3(\mu_0) \exp(-c_2'' \operatorname{dist}(j_i', j_{i+1}))$$

$c_2'' < c_2'$, $c_3(\mu_0) \to 0$ *as* $\mu_0 \to -\infty$. $\operatorname{dist}(j', j) = \inf_{\substack{x \in \Delta_j \\ x' \in \Delta_{j'}}} |x' - x|$.

c_2'' may be chosen close to c_2' at the expense of $c_3(\mu_0)$ which can be tolerated more when $|\mu_0|$ is very large. This leads to $c(\mu_0) \to c_2'$ in (5.1) and $c(\mu_i) \to c_2$ in Theorem 2 as $\mu_0 \to -\infty$. Lemma 5.4 is proved in the appendix. The hypothesis (3.1) is essential to the proof of this lemma.

On collecting these estimates, we obtain

$$\sum_{\|X\| \geqq D} |K(X)| \leqq \exp(-c(\mu_0) D) \sum_{n=1} \sum_{\eta} \int d\sigma_{n-1} f(\eta, \sigma_{n-2}) \sum_{J_{n-1}', J_n} \cdot \sum_{Y_1, \ldots, Y_n} c_A' \exp\left(-c_8(\mu_0)\left(\sum_{i=1}^n |Y_i| - n + 1 - |J_1|\right)\right) \prod_{i=1}^{n-1} c_9(\mu_0) \cdot \exp(-c_2''' \operatorname{dist}(j_i', j_{i+1})) \tag{5.5}$$

where Y_i is restricted by $Y_i \ni j_i$, $Y_{\eta(i)} \ni j_i'$, $J_{n-1}' = (j_1', \ldots, j_{n-1}')$. $c_8(\mu_0) \to \infty$ as $\mu_0 \to -\infty$, $c_9(\mu_0) \to 0$ as $\mu_0 \to -\infty$. The factor $\exp(-c(\mu_0)D)$ has been obtained at the expense of the constants c_A', $c_8(\mu_0)$, $c_9(\mu_0)$, and $c_2''' < c_2''$.

We perform the sum over J_{n-1}', J_n, $Y_1, \ldots, Y_n$ in the following order

$$\sum_{Y_1} \left(\sum_{j_1' \in Y_{\eta(1)}} \sum_{j_2} \sum_{Y_2 \ni j_2} \right) \cdots \left(\sum_{j_{n-1}' \in Y_{\eta(n-1)}} \sum_{j_n} \sum_{Y_n \ni j_n} \right) \tag{5.6}$$

and by the estimates

$$\sum_{Y_i \ni j_i} \exp(-c_{10}|Y_i|) \leqq \begin{cases} c_{12}|J_1| & \text{if } i=1 \\ c_{12} & \textit{otherwise} \end{cases} \tag{5.7}$$

$$\sum_{j_{i+1}} \exp(-c_2''' \operatorname{dist}(j_i', j_{i+1})) \leqq c_{14} \tag{5.8}$$

deduce

$$\sum_{\|X\| \geqq D} |K(X)| \leqq c_A'' \exp(-c(\mu_0)\, D) \sum_{n=1} c_{15}^{n-1}(\mu_0) \sup_{|Y_1|, \ldots, |Y_{n-1}|} \cdot \exp\left(-\sum_{i=1}^{n-1} |Y_i|\right) \sum_{\eta} \int d\sigma_{n-1} f(\eta, \sigma_{n-2}) |Y_{\eta(1)}| \ldots |Y_{\eta(n-1)}| \tag{5.9}$$

where $c_{15}(\mu_0) \to 0$ as $\mu_0 \to -\infty$. The proof of (5.1) is completed by

Lemma 5.5. *Given* $u_1, \ldots, u_{n-1} \geqq 0$, *n arbitrary,*

$$\sum_{\eta} \int d\sigma_{n-1} f(\eta, \sigma_{n-2}) u_{\eta(1)} \ldots u_{\eta(n-1)} \leqq \exp\left(\sum_{i=1}^{n-1} u_i\right)$$

Proof of Lemma 5.5. $\sum_{\eta} \int d\sigma_{n-1} f(\eta, \sigma_{n-2}) u_{\eta(1)} \ldots u_{\eta(n-1)}$

$$\leqq \int_0^1 ds_1 \ldots \int_0^1 ds_{n-1} \sum_{\eta} f(\eta, \sigma_{n-2}) u_{\eta(1)} \ldots u_{\eta(n-1)} \cdot \exp\left(\sum_{i=1}^{n-1} \sum_{k=1}^{i} s_i \ldots s_k u_k\right).$$

Perform the s integrals in the indicated order using

$$\int_0^1 ds v \exp(sv) \leqq \exp(v)$$

for $v \geqq 0$. Lemma 5.5 is the result. We are indebted to [9] for this procedure.

Appendix

Proof of Lemma 5.4, namely:

$$\int d^{n-1}t \left\| \prod_{i=1}^{n-1} E_{\partial X_i}(V(t_i))_{j_i', j_{i+1}} \right\|_{6,X} \leqq \exp(c_4|X|) \prod_{i=1}^{n-1} c_3(\mu_0) \exp(-c_2'' \operatorname{dist}(j_i', j_{i+1})) \tag{A.1}$$

under the condition that $\Delta_{j_{i+1}}$ are pairwise distinct for $i = 1, 2, \ldots, n-1$. $c_2'' < c_2'$ and $c_3(\mu_0) \to 0$ as $\mu_0 \to -\infty$.

As a function on path space

$$|E_{\partial X_i}(V(t_i))_{j_i',j_{i+1}}| \leq c_1' \exp(-c_2' \operatorname{dist}(j_i', j_{i+1})) \cdot \varrho(\Delta_{j_i'}, t_i)\varrho(\Delta_{j_{i+1}}, t_i) \tag{A.2}$$

where $\varrho(\Delta, t) = \sum_{k=1}^{\ell} \int d^3x \chi_\Delta(x) \varrho_k(x)$ with χ_Δ denoting the characteristic function of Δ. Inequality (A.2) follows from the hypothesis (3.3). By combining (A.2) with

$$\int d^{n-1}t \left\| \prod_{i=1}^{n-1} \varrho(\Delta_{j_i'}, t_i)\varrho(\Delta_{j_{i+1}}, t_i) \right\|_{6,X} \leq \bar{c}_3^{n-1}(\mu_0) \exp(c_4|X|) \prod_{\Delta_j} [2n(\Delta_j)]! \tag{A.3}$$

where $\bar{c}_3(\mu_0) \to 0$ as $\mu_0 \to -\infty$ and $n(\Delta_j) = |\{\Delta_{j_i'} : j_i' = j, i = 1, \ldots, n-1\}| + 1$, the proof of (A.1) is completed. c_2'' is constrained to be strictly less than c_2' because some configurations of $\Delta_{j_i'}$ lead to large values of $n(\Delta_j)$ for some j's, and these are to be controlled by a factor

$$\prod_{i=1}^{n-1} \exp(-(c_2' - c_2'') \operatorname{dist}(j_i', j_{i+1}))$$

using the hypothesis that the cubes $\Delta_{j_{i+1}}$ are pairwise distinct.

Proof of (A.3). Write the left hand side of (A.3) in terms of a trace of products of annihilation and creation operators and evaluate it as a sum of quantities labelled by graphs by using

$$\begin{aligned} &\operatorname{Tr}_X \left(\boldsymbol{T} \exp\left(-\int_0^\beta H_0^X(\tau) d\tau \right) \prod_{k=1}^{P} \phi_{i_k}^{\#}(x_k, t_k) \right) \\ &= \operatorname{Tr}_X(\exp(-\beta H_0^X)) \sum_P \pm \prod_{\gamma \in P} \operatorname{Tr}_X \left[\boldsymbol{T} \exp\left(-\int_0^\beta H_0^X(\tau) d\tau \right) \right. \\ &\quad \left. \cdot\, \phi_{i_{\gamma_1}}^{\#}(x_{\gamma_1}, t_{\gamma_1}) \phi_{i_{\gamma_2}}^{\#}(x_{\gamma_2}, t_{\gamma_2}) \right] / \operatorname{Tr}_X(\exp(-\beta H_0^X)) \end{aligned} \tag{A.4}$$

where $\phi^{\#}$ is either ϕ or ϕ^-, P runs over all possible partitions of $\{1, 2, \ldots, p\}$ into unordered pairs $\gamma = (\gamma_1, \gamma_2)$. The times t_k, as usual, are dummy and serve only to define the ordering of the operators. If $i_{\gamma_1} \neq i_{\gamma_2}$ the corresponding trace in the right hand side of (A.4) vanishes. It also vanishes if $\phi_{i_{\gamma_1}}^{\#}, \phi_{i_{\gamma_2}}^{\#}$ are both ϕ's or both ϕ^-'s. The remaining cases satisfy

$$\left| \operatorname{Tr}_X \left[\boldsymbol{T} \exp\left(-\int_0^\beta H_0^X(\tau) d\tau \right) \phi_k(x, t) \phi_k^-(x', t') \right] / \operatorname{Tr}_X(\exp(-\beta H_0^X)) \right| \leq \begin{array}{ll} q(x - x', \quad t - t', \mu_0) & \text{if} \quad t' < t \\ q(x - x', -t + t' + \beta_0, \mu_0) & \text{if} \quad t' > t \end{array} \tag{A.5}$$

where

$$q(x, t, \mu_0) = (2\pi)^{-3/2} \int d^3k \cdot \exp(-t(k^2 - \mu_0))/(1 - \exp(-\beta(k^2 - \mu_0)) e^{ik \cdot x}. \tag{A.6}$$

This may be seen by noting that the left hand side of (A.5) is the measure of paths going from x to y in times $t-t'+n\beta$ if $t-t'>0$, $-t+t'+(n+1)\beta$ if $t-t'<0$, $n=0, 1, \ldots$. The paths are constrained to remain in X for these times. This is majorized by the path integral obtained by giving away the restriction that the paths remain in X. The latter, by explicit computation, is equal to the right hand side of (A.6).

The analyticity properties of the integrand in (A.6) show that $q(x, t, \mu_0)$ decays exponentially in $|x|$ away from zero, uniformly in t. Using this, the graphs arising in the evaluation of the left hand side of (A.3) may be counted using the method of Dimock and Glimm, [2], Lemma 2.6. Individual graphs may be estimated in terms of local L_2 norms [e.g., see (A.7) below] of (A.5) by the Cauchy Schwarz inequality. The reader is referred to [2] for more details. The constant $\bar{c}_3(\mu_0)$ in (A.3) is obtained by keeping track of the μ_0 dependence of the local L_2 norms.

$$\left[\int_{\Delta_j} d^3x \int_{\Delta_j} d^3x' |q(x-x', t, \mu_0)|^2\right]^{1/2} \leqq [\int d^3x |q(x, t, \mu_0)|^2]^{1/2}$$
$$= [\int d^3k |\check{q}(k, t, \mu_0)|^2]^{1/2} . \qquad (A.7)$$

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