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Dilations of Dynamical Semi-Groups

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Abstract. We prove the existence of isometric and unitary dilations of a class of semi-groups of completely positive maps on an algebra of operators on a Hilbert space. The result has relevance to the problem of embedding an open quantum mechanical system in a closed one.

§ 1. Introduction

Empirical semi-group laws for the irreversible evolution of the state of a quantum mechanical system have been remarkably successful in a variety of applications [1, 2, 8, 14]. This has encouraged some workers to propose axioms for dynamical semi-groups [10, 12, 7]. From the point of view of fundamental theory such semi-groups are by themselves unsatisfactory: the conventional position is that the laws of quantum theory prescribe the time-reversible evolution of a closed system, and irreversible behaviour enters only when the evolution is restricted to an open sub-system. The time-reversible evolution of a closed system is described by a strongly-continuous one-parameter group of unitary operators on a Hilbert space. The question then arises: is a given irreversible dynamical semi-group the restriction to an open subsystem of a time-reversible evolution of a closed system? The purpose of this paper is to formulate this question mathematically and to answer it in the affirmative for a class of dynamical semi-groups which have interesting applications.

From the mathematical point of view we prove results for semi-groups of completely positive normal maps of W^*-algebras which are analogues of Szökefalvi-Nagy's dilation theorem [17] for semi-groups of contractions on Hilbert spaces and Stroescu's dilation theorem [16] for semi-groups of contractions on Banach spaces. Some results in this direction were obtained by Davies [5]; his proof was based on his theory [4] of quantum jump processes. We adopt his construction of a semi-group of isometries but our proof uses only the perturbation theory of semi-groups on a Banach space.

§ 2. Dilations of Dynamical Semi-Groups

A *dynamical semi-group* on a W^*-algebra M is a semi-group $\{T_t : t \geqq 0\}$ of completely positive normal maps of M into itself such that:

(i) $T_0 = i_M$, (ii) $T_t(1) = 1$ for all $t \geqq 0$.

A dynamical semi-group is said to be *weakly continuous* if $\lim_{t \to 0+} \langle T_t m, \varphi \rangle = \langle m, \varphi \rangle$ for all m in M and all φ in the pre-dual M_* of M; if T_t is weakly continuous then the pre-adjoint semi-group ${}_*T_t$, defined on M_*, is strongly continuous and hence has a densely-defined generator (Yosida [18], p. 233). (Whenever $A : M \to M$ is $\sigma(M, M_*)$-continuous we denote by ${}_*A : M_* \to M_*$ its pre-adjoint, defined by $\langle Am, \varphi \rangle = \langle m, {}_*A\varphi \rangle$ for all m in M and φ in M_*.) A dyanmical semi-group T_t is said to be *norm-continuous* if $\lim_{t \to 0+} \|T_t - 1\| = 0$ in which case T_t itself has a $\sigma(M, M_*)$-continuous bounded generator L so that $T_t = e^{tL}$. Lindblad [12] has shown that the generator L of a norm-continuous dynamical semi-group T_t on the algebra $\mathscr{B}(\mathscr{K})$ of all bounded operators on a separable Hilbert space $\mathscr{K}$ can be put in the form

$$L(m) = i[H, m] + V(m) - \tfrac{1}{2}\{V(1), m\} \tag{2.1}$$

for all m in $\mathscr{B}(\mathscr{K})$. Here H is a bounded self-adjoint operator on $\mathscr{K}$ and $V : \mathscr{B}(\mathscr{K}) \to \mathscr{B}(\mathscr{K})$ is a completely positive normal map so that, by Kraus [11], there exist bounded operators A_i, $i = 1, 2, \ldots$ on $\mathscr{K}$ such that

$$V(m) = \sum_{i=1}^{\infty} V_i(m), \quad V_i(m) = A_i^* m A_i, \tag{2.2}$$

for all m in $\mathscr{B}(\mathscr{K})$.

Let $\mathscr{H}$ be a Hilbert space and let $\bar{M}$ be a von Neumann algebra contained in $\mathscr{B}(\mathscr{H})$. Let $e : M \to \bar{M}$ be an embedding of M in $\bar{M}$ such that $e(M)$ is a W^*-algebra on $\mathscr{H}$ (see Sakai [15], 2.7.5), and let $N : \bar{M} \to M$ be a conditional expectation such that $N \circ e = i_M$ (i.e. N is a completely positive normal map of $\bar{M}$ onto M such that (i) $\|N\| = 1$, (ii) $N(\bar{1}) = 1$, (iii) $N(m(e \circ N)(m')) = N((e \circ N)(m)m') = N(m)N(m')$ for all m, m' in $\bar{M}$). Let $\{G_t : t \geqq 0\}$ be a strongly continuous semi-group of isometries on $\mathscr{H}$ such that $G_t^* \bar{M} G_t \subseteqq \bar{M}$ for all $t \geqq 0$. Then $(G_t, e, \bar{M}, N)$ is said to be an *isometric dilation* of the dynamical semi-group (T_t, M) if for all $t \geqq 0$ and all a in M

$$(e \circ T_t)(a) = G_t^* e(a) G_t. \tag{2.3}$$

Remark. Equation (2.3) cannot hold for G_t unitary unless T_t is a homomorphism of M. Let $\{U_t : t \in \mathbb{R}\}$ be a strongly continuous group of unitary operators on $\mathscr{H}$ such that $U_t^* \bar{M} U_t \subseteqq \bar{M}$ for all $t \geqq 0$. Then $(U_t, e, \bar{M}, N)$ is said to be a *unitary dilation* of the dynamical semigroup (T_t, M) if

$$T_t(m) = N(U_t^* e(m) U_t) \tag{2.4}$$

for all $t \geqq 0$ and all m in M. Notice that if a dilation exists then so does a minimal one; in the isometric case take $\bar{M}$ to be $\{G_t^* e(M) G_t : t \geqq 0\}''$ and in the unitary case take $\bar{M}$ to be $\{U_t^* e(M) U_t : t \geqq 0\}''$.

First we prove the existence of isometric and unitary dilations of a norm-continuous dynamical semi-group T_t on the algebra $\mathscr{B}(\mathscr{K})$ of all bounded operators on a separable Hilbert space $\mathscr{K}$. Then we relax somewhat the conditions on both the semi-group and on the algebra.

Theorem 1. *Let $\mathscr{K}$ be a separable Hilbert space. Let $\{T_t : t \geqq 0\}$ be a norm-continuous dynamical semi-group on $\mathscr{B}(\mathscr{K})$. Then there exists an isometric dilation (G_t, e_1, M^1, N_1) of $(T_t, \mathscr{B}(\mathscr{K}))$.*

Proof. We have seen that the generator L of T_t has the form (2.1) where V is given by (2.2). Define $Z \in \mathscr{B}(\mathscr{K})$ by

$$Z = -iH - \tfrac{1}{2}V(1), \tag{2.5}$$

so that $\{B_t = e^{tZ} : t \geqq 0\}$ is a contraction semi-group on $\mathscr{K}$ and $(S_t : t \geqq 0\}$, defined by

$$S_t(m) = B_t^* m B_t \tag{2.6}$$

for all m in $\mathscr{B}(\mathscr{K})$, is a contraction semi-group on $\mathscr{B}(\mathscr{K})$ with generator L_0 given by

$$L_0(m) = Z^* m + mZ \tag{2.7}$$

for all m in $\mathscr{B}(\mathscr{K})$ so that

$$L = L_0 + V. \tag{2.8}$$

Hence T_t and S_t are connected by the perturbation formula (Kato [9], p. 495)

$$T_t(m) = S_t(m) + \int_0^t (S_{t-s} \circ V \circ T_s)(m) ds \tag{2.9}$$

for all m in $\mathscr{B}(\mathscr{K})$. The pre-adjoint semi-groups ${}_*T_t$ and ${}_*S_t$ on the pre-dual of $\mathscr{B}(\mathscr{K})$ (which we identify with the Banach space $\mathscr{I}(\mathscr{K})$ of trace-class operators on $\mathscr{K}$) satisfy

$${}_*T_t(\varrho) = {}_*S_t(\varrho) + \int_0^t ({}_*T_s \circ {}_*V \circ {}_*S_{t-s})(\varrho) ds \tag{2.10}$$

for all ϱ in $\mathscr{I}(\mathscr{K})$. Because of the particular form (2.2) of the perturbation V we can write the von Neumann series for (2.9) and (2.10) in an unfamiliar but useful way (cf. Davies [4, 5]).

Let X_∞ be the set of all sequences $\{(x_i, t_i) \in \mathbb{N} \times (0, \infty) : 0 < t_1 < t_2 \ldots\}$ regarded as a Borel subset of $\bigcup_{m=0}^{m=\infty} \left\{ \prod_{n=0}^{m} \mathbb{N} \times (0, \infty) \right\}$ in an obvious way, let Y_∞ be the Borel subset of X_∞ consisting of all sequences of finite length and, for each $t > 0$, let X_t be the Borel subset of X_∞ consisting of all finite sequences $\{(x_i, t_i) : 0 < t_1 < t_2 \ldots t_n \leqq t\}$. For each $t > 0$ there is a Borel isomorphism $\lambda_t : X_t \times Y_\infty \to Y_\infty$ defined by

$$\{(x_i, t_i)\}_{i=1}^n, \{(y_j, s_j)\}_{j=1}^m \mapsto (x_1, t_1), \ldots, (x_n, t_n), (y_1, s_1 + t), \ldots, (y_m, s_m + t). \tag{2.11}$$

The inverse map is given by

$$\{(y_i, s_i)\}_{i=1}^n \mapsto \{(y_i, s_i)\}_{i=1}^p, \{(y_i, s_i - t)\}_{i=p+1}^n, \tag{2.12}$$

where p is the unique integer such that $s_p \leqq t < s_{p+1}$. We denote by X_0 the subset consisting of the single sequence z of zero length. We define a measure μ_t on X_t given by the product measure constructed from counting measure on each component $\mathbb{N}$ and Lebesgue measure on each component $(0, \infty)$; we assign Dirac measure to the point $z \in X_t$. We define a measure μ_∞ on Y_∞ in an analogous fashion. For each $w \in X_t$ define $({}_*S_*V_*S)(w)$ by

$$({}_*S_*V_*S)(w) = {}_*S_{t_1} \circ {}_*V_{x_1} \circ {}_*S_{t_2-t} \circ {}_*V_{x_2} \ldots {}_*V_{x_n} \circ {}_*S_{t-t_n}; \tag{2.13}$$

where $w = \{(x_i, t_i) : 0 < t_1 < \ldots t_n \leqq t\}$, then the Neumann series

$$\begin{aligned} {}_*T_t(\varrho) = {}_*S_t(\varrho) &+ \int_0^t ({}_*S_{t_1} \circ {}_*V \circ {}_*S_{t-t_1})(\varrho) dt_1 \\ &+ \int_0^t \int_0^{t_2} ({}_*S_{t_1} \circ {}_*V \circ {}_*S_{t_2-t_1} \circ {}_*V \circ {}_*S_{t-t_2}(\varrho) dt_1 dt_2 \\ &+ \ldots \end{aligned} \tag{2.14}$$

can be written as

$${}_*T_t(\varrho) = \int_{X_t} ({}_*S_*V_*S)(w)(\varrho) d\mu_t(w), \tag{2.15}$$

and the adjoint series can be written as

$$T_t(m) = \int_{X_t} [({}_*S_*V_*S)(w)]^*(m) d\mu_t(w). \tag{2.16}$$

Define the operator G_t on $L^2(Y_\infty; \mathcal{K})$ for $t \geqq 0$ by

$$(G_t \psi)(w) = (\mathrm{BAB})(w_{\bar{t}}) \psi(w_t), \tag{2.17}$$

where

$$(w_{\bar{t}}, w_t) = \lambda_t^{-1}(w) \tag{2.18}$$

for $w \in Y_\infty$, and $(\mathrm{BAB})(w') \in \mathcal{B}(\mathcal{K})$ is defined by

$$(\mathrm{BAB})(w') = B_{t_1} A_{x_1} B_{t_2-t_1} A_{x_2} \ldots A_{x_n} B_{t-t_n} \tag{2.19}$$

for any $w' = \{(x_i, t_i) : 0 < t_1 < t_2 \ldots < t_n \leqq t\} \in X_t$.

We prove next that $\{G_t : t \geqq 0\}$ is a strongly continuous group of isometries on $L^2(Y_\infty; \mathcal{K})$. We have

$$\begin{aligned} (G_{t_1}(G_{t_2}\psi))(w) &= (\mathrm{BAB})(w_{\bar{t}_1})(G_{t_2}\psi)(w_{t_1}) \\ &= (\mathrm{BAB})(w_{\bar{t}_1})(\mathrm{BAB})(w_{t_1,\bar{t}_2})\psi(w_{t_1,t_2}) \\ &= (\mathrm{BAB})(w_{\overline{t_1+t_2}})\psi(w_{t_1+t_2}) \\ &= (G_{t_1+t_2}\psi)(w) \end{aligned} \tag{2.20}$$

where we have used the following immediate consequences of the definitions:

$$(\mathrm{BAB})(w_{\bar{t}_1})(\mathrm{BAB})(w_{t_1,\bar{t}_2}) = (\mathrm{BAB})(w_{\overline{t_1+t_2}}), \tag{2.21}$$

$$w_{t_1,t_2} = w_{t_1+t_2}. \tag{2.22}$$

We check that G_t is an isometry using (2.15) and the observation that the measure μ_∞ is the product of the measures μ_t and μ_∞ under the Borel isomorphism λ_t of $X_t \times Y_\infty$ with Y_∞:

$$\begin{aligned}\langle G_t\psi, G_t\psi\rangle &= \int_{Y_\infty} \langle (\mathrm{BAB})(w_{\bar{t}})\psi(w_t), (\mathrm{BAB})(w_{\bar{t}})\psi(w_t)\rangle d\mu_\infty(w) \\ &= \int_{Y_\infty}\int_{X_t} \mathrm{trace}([(\mathrm{BAB})(w_t)]\psi(w_t)\otimes\overline{\psi(w_t)}\,[(\mathrm{BAB})(w_{\bar{t}})])^* \\ &\quad \cdot d\mu_t(w_{\bar{t}})d\mu_\infty(w_t) \\ &= \int_{Y_\infty} \mathrm{trace}(\int_{X_t} ({}_*S_*V_*S)(w_{\bar{t}})(\psi(w_t)\otimes\overline{\psi(w_t)})d\mu_t(w_{\bar{t}}))d\mu_\infty(w_t) \\ &= \int_{Y_\infty} \mathrm{trace}({}_*T_t(\psi(w_t)\otimes\psi(w_t))d\mu_\infty(w_t) \end{aligned} \tag{2.23}$$

where we have used the positivity of the integrand to interchange the trace and integration operations.

But $T_t(1)=1$ implies trace $({}_*T_t(\varrho))=\mathrm{trace}(\varrho)$ so

$$\langle G_t\psi, G_t\psi\rangle = \int_{Y_\infty} \langle\psi(w_t), \psi(w_t)\rangle d\mu_\infty(w_t) = \langle\psi,\psi\rangle\,. \tag{2.24}$$

Since we have shown that $\{G_t : t \geqq 0\}$ is a semi-group of isometries it is enough to check that it is weakly continuous at the origin on elements of the form $f(\cdot)k$ where $f(\cdot)\in L^2(Y_\infty)$ and $k\in\mathscr{K}$. This follows using the observation that $\mu_t\{X_t\setminus\{z\}\}=te^t$.

Now take M^1 to be $L^\infty(Y_\infty;\mathscr{B}(\mathscr{K}))$ which is a W^*-algebra with pre-dual $M^1_*=L^1(Y_\infty;\mathscr{I}(\mathscr{K}))$ (Sakai [15], 1.22.13); the mapping $f\otimes a\to f(\cdot)a$ can be extended uniquely to a W^*-isomorphism of $L^\infty(Y_\infty)\overline{\otimes}\mathscr{B}(\mathscr{K})$ onto $L^\infty(Y_\infty;\mathscr{B}(\mathscr{K}))$. The predual of $L^\infty(Y_\infty)\overline{\otimes}\mathscr{B}(\mathscr{K})$ is $L^1(Y_\infty)\otimes_\gamma\mathscr{I}(\mathscr{K})$, the projective tensor product, which we identify with $L^1(Y_\infty;\mathscr{I}(\mathscr{K}))$. We make use of the embedding with $e_1:\mathscr{B}(\mathscr{K})\to M^1$ defined by

$$e_1(a)=1\otimes a\,, \tag{2.25}$$

where 1 is the constant function in $L^\infty(Y_\infty)$; we use the conditional expectation $N_1: M^1\to\mathscr{B}(\mathscr{K})$ defined by

$$N_1(m)=m(z)\,. \tag{2.26}$$

We note that

$$\begin{aligned}({}_*e_1)(\phi) &= \int_{Y_\infty} \phi(w)d\mu_\infty(w)\,, \\ ({}_*N_1)(\varrho) &= \delta_z\otimes\varrho\,. \end{aligned} \tag{2.27}$$

Next we check that $G_t^*M^1G_t\subseteqq M^1$ for all $t\geqq 0$. For this we require the explicit form of the action of G_t^* on a vector ψ; we get this by inspecting $\langle G_t\psi,\phi\rangle$ for arbitrary ϕ:

$$\begin{aligned}\langle G_t\psi,\phi\rangle &= \int_{Y_\infty}\int_{X_t} (\mathrm{BAB})(w_{\bar{t}})\psi(w_t), \phi(\lambda_t(w_{\bar{t}}), w_t))d\mu_t(w_{\bar{t}})d\mu_\infty(w_t) \\ &= \int_{Y_\infty}\int_{X_t} \psi(w_t), [(\mathrm{BAB})(w_{\bar{t}})]^*\phi(\lambda_t(w_{\bar{t}}, w_t))d\mu_t(w_{\bar{t}})d\mu_\infty(w_t)\,.\end{aligned}$$

Hence G_t^* is given by

$$(G_t^*\phi)(w) = \int_{X_t} [(\mathrm{BAB})(w')]^* \phi(\lambda_t(w', w)) d\mu_t(w') . \tag{2.28}$$

In what follows we use the notation w^t to denote $\lambda_t(w', w)$ where $w' \in X_t$ is a running variable of integration and remark that $w^t_{\bar{t}} = w'$, and $w^t_{\,t} = w$. Now we take $a(\cdot) \in L^\infty(Y_\infty; \mathscr{B}(\mathscr{K}))$ and compute $G_t^* a(\cdot) G_t$ as an element of $\mathscr{B}(L^2(Y_\infty; \mathscr{K}))$ and show that it lies in $L^\infty(Y_\infty; \mathscr{B}(\mathscr{K}))$:

$$\begin{aligned}(G_t^* a G_t \psi)(w) &= \int_{X_t} [(\mathrm{BAB})(w')]^* (a G_t \psi)(w^t) d\mu_t(w') \\ &= \int_{X_t} [(\mathrm{BAB})(w')]^* a(w^t)(\mathrm{BAB})(w^t_{\bar{t}}) \psi(w^t_t) d\mu_t(w') \\ &= \int_{X_t} [(\mathrm{BAB})(w')]^* a(w^t)(\mathrm{BAB})(w') \psi(w) d\mu_t(w') \\ &= \int_{X_t} [({}_*S_* V_* S)(w')]^* a(w^t) d\mu_t(w') \psi(w) .\end{aligned} \tag{2.29}$$

But

$$(G_t^* a G_t)(w) = \int_{X_t} [({}_*S_* V_* S)(w')]^* a(\lambda_t(w', w)) d\mu_t(w') \tag{2.30}$$

lies in $L^\infty(Y_\infty; \mathscr{B}(\mathscr{K}))$ and so $G_t^* M^1 G_t \subseteq M^1$.

Now put $a(\cdot) = 1(\cdot) \otimes m$ where $m \in \mathscr{B}(\mathscr{K})$; we have

$$\begin{aligned}(G_t^* e_1(m) G_t)(w) &= 1(w) \otimes \left(\int_{X_t} [({}_*S_* V_* S)(w')]^* d\mu_t(w') \right) m \\ &= 1(w) \otimes T_t(m)\end{aligned} \tag{2.31}$$

by (2.16). Thus we have proved

$$e_1(T_t(m)) = G_t^* e_1(m) G_t . \tag{2.32}$$

Theorem 2. *Let $\mathscr{K}$ be a separable Hilbert space. Let $\{T_t : t \geqq 0\}$ be a norm-continuous dynamical semi-group on $\mathscr{B}(\mathscr{K})$. Then there exists a unitary dilation $(U_t, e, \bar{M}, N)$ of $(T_t, \mathscr{B}(\mathscr{K}))$.*

Proof. Let (G_t, e_1, M^1) be the isometric dilation of $(T_t, \mathscr{B}(\mathscr{K}))$ of Theorem 1. Then by Cooper [3] (see also Masani [13]) there exists a Hilbert space $\mathscr{H}$, an isometric embedding $W : L^2(Y_\infty, \mathscr{K}) \to \mathscr{H}$ and a strongly continuous group $\{U_t : t \in \mathbb{R}\}$ of unitary operators on $\mathscr{H}$ such that for $t \geqq 0$ we have for all ψ in $L^2(Y_\infty; \mathscr{K})$

$$W G_t \psi = U_t W \psi . \tag{2.33}$$

It follows that for $t \geqq 0$ we have

$$G_t = W^* U_t W , \tag{2.34}$$

and

$$G_t^* = W^* U_t^* W . \tag{2.35}$$

Put $\bar{M} = \{U_t^* e_2(M^1) U_t : t \geqq 0\}''$ where $e_2 : M^1 \to \mathscr{B}(\mathscr{H})$ is defined by

$$e_2(a) = W a W^* \tag{2.36}$$

and $N_2 : \bar{M} \to \mathscr{B}(L^2(Y_\infty; \mathscr{K}))$ be the conditional expectation given by

$$N_2(m) = W^* m W . \tag{2.37}$$

Then we have to show that $N_2(\bar{1}) = 1$ and that $N_2(\bar{M}) \subseteq M^1$. By (2.34) and (2.35) we have for $t \geqq 0$ and x in M^1

$$\begin{aligned} N_2(U_t^* e_2(x) U_t) &= W^* U_t^* W x W^* U_t W \\ &= G_t^* x G_t , \end{aligned} \tag{2.38}$$

which we saw is in M^1. For $n > 1$ and $t_i \geqq 0$, $i = 1, 2, \ldots, n$, we define a_n by

$$a_n = N_2(U_{t_1}^* e_2(x_1) U_{t_1} U_{t_2}^* e_2(x_2) U_{t_2} \ldots U_{t_n}^* e_2(x_n) U_{t_n}) . \tag{2.39}$$

We have

$$a_n = G_{t_1}^* x_1 G_{t_2}^* G_{t_1} x_2 G_{t_3}^* G_{t_2} \ldots G_{t_{n-1}} x_n G_{t_n} . \tag{2.40}$$

where we have used the observation that for all $s, t > 0$

$$W^* U_t U_s^* W = G_s^* G_t . \tag{2.41}$$

(For $t > s$ we have $W^* U_t U_s^* W = G_{t-s}$ but $G_s G_{t-s} = G_t$ so that $G_{t-s} = G_s^* G_t$ since G_s is an isometry; an analogous calculation works for $s > t$.) We have to show that a_n lies in M^1. In order to be able to use induction we define b_n for $n \geqq 1$ by

$$b_n = G_{t_1}^* x_1 G_{t_2}^* G_{t_1} x_2 G_{t_3}^* G_{t_2} \ldots x_n G_{t_{n+1}}^* G_{t_n} \tag{2.42}$$

and notice that $b_n|_{t_{n+1}=0} = a_n$.

We have by direct calculation of the kind used in the proof of Theorem 1

$$(b_1 \phi)(w) = \int_{X_{t_1}} \int_{X_{t_2}} \bar{b}_1(w', w''; w) \phi(w^{t_1 t_2}{}_{t_1}) d\mu_{t_1}(w') d\mu_{t_2}(w'') \tag{2.43}$$

where

$$\bar{b}_1(w', w''; w) = [(\mathrm{BAB})(w')]^* x_1(w^{t_1}) [(\mathrm{BAB})(w'')]^* (\mathrm{BAB})(w^{t_1 t_2}{}_{\bar{t}_1}) . \tag{2.44}$$

Suppose that for $n \geqq 1$ we have

$$\begin{aligned} (b_n \phi)(w) = \int_{X_{t_1}} \ldots \int_{X_{t_{n+1}}} & \bar{b}_n(w', w'', \ldots, w^{(n+1)}; w) \phi(w^{t_1 t_2}{}_{t_1} \ldots {}^{t_{n+1}}{}_{t_n}) d\mu_{t_1}(w') \\ & \ldots d\mu_{t_{n+1}}(w^{(n+1)}) ; \end{aligned} \tag{2.45}$$

then

$$\begin{aligned} (b_{n+1} \phi)(w) &= \int_{X_{t_1}} \ldots \int_{X_{t_{n+1}}} \bar{b}_n(w', \ldots, w^{(n+1)}; w) (x_{n+1} G_{t_{n+2}}^* G_{t_{n+1}} \phi)(w^{t_1 t_2}{}_{t_1} \ldots {}^{t_n}{}_{t_{n-1}} {}^{t_{n+1}}{}_{t_n}) \\ & \quad d\mu_{t_1}(w') \ldots d\mu_{t_{n+1}}(w^{(n+1)}) \\ &= \int_{X_{t_1}} \int_{X_{t_{n+2}}} \bar{b}_{n+1}(w', w'', \ldots, w^{(n+2)}; w) \phi(w^{t_1 t_2}{}_{t_1} \ldots {}^{t_{n+1}}{}_{t_n} {}^{t_{n+2}}{}_{t_{n+1}}) \\ & \quad d\mu_{t_1}(w') \ldots d\mu_{t_{n+2}}(w^{(n+2)}) \end{aligned} \tag{2.46}$$

where

$$\begin{aligned} & \overline{b_{n+1}}(w', \ldots, w^{(n+2)}; w) = \bar{b}_n(w', \ldots, w^{(n+1)}; w) x_{n+1}(w^{t_1 t_2}{}_{t_1} \ldots {}^{t_{n+1}}{}_{t_n}) \\ & \times [(\mathrm{BAB})(w^{(n+2)})]^* [(\mathrm{BAB})(w^{t_1 t_2}{}_{t_1} \ldots {}^{t_{n+2}}{}_{\bar{t}_{n+1}})] . \end{aligned} \tag{2.47}$$

But (2.45) holds for $n=1$ and hence by (2.46) for all $n\geqq 1$; evaluating $(b_n\phi)(w)$ at $t_{n+1}=0$ we have

$$(a_n\phi)(w)=\int_{X_{t_1}}\dots\int_{X_{t_n}} b_n(w',\dots,w^{(n)},z;w)\phi(w^{t_1t_2}{}_{t_1}\dots{}^{t_n}{}_{t_{n-1}t_n})d\mu_{t_1}(w')\dots d\mu_{t_n}(w^{(n)})\,. \quad (2.48)$$

But it follows directly from the definitions that

$$w^{t_1t_2}{}_{t_1}\dots{}^{t_n}{}_{t_{n-1}t_n}=w \quad (2.49)$$

so that

$$(a_n\phi)(w)=\bar{a}_n(w)\phi(w) \quad (2.50)$$

where

$$\bar{a}_n(w)=\int_{X_{t_1}}\dots\int_{X_{t_n}} b_n(w',\dots,w^{(n)},z;w)d\mu_{t_1}(w')\dots d\mu_{t_n}(w^{(n)}) \quad (2.51)$$

which lies in M^1, and by continuity we have $N(\bar{M})\subseteq M^1$. We complete the proof by putting $e=e_2\circ e_1$, $N=N_1\circ N_2$; then $N(\bar{1})=1$ and

$$N(U_t^*e(m)U_t)=T_t(m)\,, \quad (2.52)$$

and it is easily checked that N is a conditional expectation.

Remark. The map $t\to U_t^*\cdot U_t$ is weakly continuous. It cannot be norm-continuous even though $t\to T_t$ is unless T_t is a homomorphism of M. Indeed, suppose $t\to T_t$ is strongly continuous with generator L, suppose $t\to U_t^*\cdot U_t$ is strongly continuous with generator δ, and $Z=\mathcal{D}(\delta)\cap M$ is a core for L (that is, $L=(L|_Z)^-$); then for $x\in\mathcal{D}(\delta)\cap M$ we have

$$L(x)=(N\circ\delta\circ e)(x) \quad (2.53)$$

so that L is a derivation and hence T_t is a homomorphism (Evans [6]).

Inspecting the proofs of Theorems 2 and 3 we see that they still work if we relax somewhat the hypotheses on the continuity of $t\to T_t$ and on the algebra M. We have in fact proved the following

Theorem 3. *Let T_t be a weakly continuous dynamical semi-group on $\mathcal{B}(\mathcal{H})$ where $\mathcal{H}$ is a separable Hilbert space. Suppose that*

(i) *there exists a strongly continuous contraction semi-group $B_t=e^{Zt}$ on $\mathcal{H}$ whose generator Z is a bounded perturbation of a self-adjoint operator, and a completely positive normal map $V:\mathcal{B}(\mathcal{H})\to\mathcal{B}(\mathcal{H})$ such that*

$$T_t(m)=S_t(m)+\int_0^t (T_{t-s}\circ V\circ S_s)(m)ds$$

for all m in $\mathcal{B}(\mathcal{H})$,

(ii) *V has a decomposition $V(m)=\int_X A_x^*mA_x d\nu(x)$ where (X,ν) is a σ-finite measure space and $x\to A_x$ is weakly measurable.*

Then if M is a von Neumann algebra on $\mathcal{H}$ such that A_x lies in M for ν a.e. x in X and if $B_t^*MB_t\subseteq M$ for all $t\geqq 0$ the conclusions of Theorems 1 and 2 hold.

Remark. The unitary dilation theorem for a family of completely positive maps indexed by the elements of a group which was recently proved by Evans [6] does not overlap with the above results.

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