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Quasi-free "Second Quantization"

Lars-Erik Lundberg

Matematisk Institut HCØ, Copenhagen University, DK-2100 Copenhagen, Denmark

Abstract. Araki and Wyss considered in 1964 a map $A \to Q(A)$ of one-particle trace-class observables on a complex Hilbert-space $\mathscr{H}$ into the fermion C^*-algebra $\mathfrak{A}(\mathscr{H})$ over $\mathscr{H}$. In particular they considered this mapping in a quasi-free representation.

We extend the map $A \to Q(A)$ in a quasi-free representation labelled by T, $0 \leqq T \leqq I$, to all $A \in B(\mathscr{H})_{\mathrm{sa}}$ such that $\operatorname{tr}(TA(1-T)A) < \infty$ with $Q(A)$ now affiliated with the algebra. This generalizes some well-known results of Cook on the Fock-representation $T=0$.

1. Introduction

Let $\mathfrak{A}(\mathscr{H})$ denote the fermion C^*-algebra over a complex Hilbert space $\mathscr{H}$, i.e. there exists a conjugate linear mapping $f \mapsto a(f)$ of $\mathscr{H}$ into $\mathfrak{A}(\mathscr{H})$, whose range generates $\mathfrak{A}(\mathscr{H})$ as a C^*-algebra such that $a(f)a(g)^* + a(g)^*a(f) = \langle f, g \rangle I$, $a(f)a(g) + a(g)a(f) = 0$ for all $f, g \in \mathscr{H}$ and where $\langle \cdot, \cdot \rangle$ denotes the inner product on $\mathscr{H}$.

A gauge-invariant quasi-free state ω_T of $\mathfrak{A}(\mathscr{H})$ is uniquely defined by the n-point functions $\omega_T(a(f_n)^* \dots a(f_1)^* a(g_1) \dots a(g_m)) = \delta_{nm} \det(\langle g_i, Tf_j \rangle)$ where $T \in B(\mathscr{H})$ and $0 \leqq T \leqq I$. Denote by $\mathscr{H}_T$, π_T and Ω_T the Hilbert-space, the representation, and the cyclic unit-vector associated with ω_T via the GNS-construction, i.e. $\omega_T(x) = (\Omega_T, \pi_T(x)\Omega_T)$, $x \in \mathfrak{A}(\mathscr{H})$.

Let A be a self-adjoint (s.a.) finite-rank operator on $\mathscr{H}$, i.e. there exists an orthonormal set $\{u_n\}_{n=1}^N$ in $\mathscr{H}$ and $\alpha_n \in \mathbb{R}$ such that $Af = \sum_{n=1}^{N} \alpha_n u_n \langle u_n, f \rangle$ for $f \in \mathscr{H}$. Araki and Wyss [1] considered the following map Q of finite-rank s.a. operators on $\mathscr{H}$ into $\mathfrak{A}(\mathscr{H})_{\mathrm{sa}}$, $A \mapsto Q(A) = \sum_{n=1}^{N} \alpha_n a(u_n)^* a(u_n)$, which has the following properties:

$$Q(A) + Q(B) = Q(A+B)\,, \tag{1.1}$$

$$[Q(A), a(f)^*] = a(Af)^*\,, \tag{1.2}$$

$$i[Q(A), Q(B)] = Q(i[A, B])\,. \tag{1.3}$$

They showed in particular that the map $A \mapsto Q(A)$ extends to all s.a. trace-class operators on $\mathscr{H}$ and by complexification to all trace-class operators on $\mathscr{H}$ (see also Araki [2]).

Let us now consider $\pi_T(\mathfrak{A}(\mathscr{H}))$ and put

$$Q_T(A) = \pi_T(Q(A)) - \omega_T(Q(A)), \tag{1.4}$$

where A is a s.a. finite rank operator, i.e. $(\Omega_T, Q_T(A)\Omega_T) = 0$. One easily verifies that $\omega_T(Q(A)) = \operatorname{tr}(TA)$. For convenience we put $\pi_T(a(f)) = a_T(f)$. Equations (1.1)–(1.3) now imply

$$Q_T(A) + Q_T(B) = Q_T(A+B), \tag{1.5}$$

$$[Q_T(A), a_T(f)^*] = a_T(Af)^*, \tag{1.6}$$

$$i[Q_T(A), Q_T(B)] = Q_T(i[A,B]) - 2\operatorname{Im}\operatorname{tr}(TAB)1. \tag{1.7}$$

A simple calculation gives

$$(\Omega_T, Q_T(A)Q_T(B)\Omega_T) = \operatorname{tr}(TA(1-T)B), \tag{1.8}$$

which suggests an alternative form of (1.7)

$$i[Q_T(A), Q_T(B)] = Q_T(i[A,B]) - 2\operatorname{Im}\operatorname{tr}(TA(1-T)B)1, \tag{1.9}$$

(observe that $\operatorname{tr}(TATB)$ is real). Let us put $W_T(A) = e^{iQ_T(A)}$; then $W_T(sA)$, $s \in \mathbb{R}$, is a unitary one-parameter group on $\mathscr{H}_T$. Equations (1.6) and (1.9) now imply that

$$a_T(e^{isA}f) = W_T(sA)a_T(f)W_T(sA)^{-1}, \tag{1.10}$$

$$W_T(A)W_T(B)W_T(A)^{-1} = W_T(e^{iA}Be^{-iA})e^{ib_T(A,B)}, \tag{1.11}$$

$$b_T(tA, B) = -2\operatorname{Im}\int_0^t \operatorname{tr}(TA(I-T)e^{isA}Be^{-isA})ds. \tag{1.12}$$

Here we have used the fact that $\pi_T(\mathfrak{A}(\mathscr{H}))''$ is a factor (see Powers and Størmer [3]) to conclude that $b_T(A, B)$ is a real number. The one-parameter group property of $W_T(tA)$ implies that $b_T((t_1+t_2)A, B) = b_T(t_1A, B) + b_T(t_2, e^{it_1A}Be^{-it_1A})$, which is the cocycle equation. Equation (1.11) gives $W_T(tA)Q_T(B)W_T(tA)^{-1} = Q_T(e^{itA}Be^{-itA}) + b_T(tA, B)$ i.e. (1.9) implies $(d/dt)b_T(tA, B) = -2\operatorname{Im}\operatorname{tr}(TA(I-T)B) + b_T(tA, i[A, B])$ with the initial condition $b_T(0, B) = 0$. The solution is given by (1.12).

In this paper we show that the mapping $A \to W_T(sA)$ can be extended to $O_T(\mathscr{H}) = \{A \in B(\mathscr{H})_{\text{sa}};\ \operatorname{tr}(TA(1-T)A) < \infty\}$ and such that $W_T(sA)$ is a strongly continuous unitary one-parameter group on $\mathscr{H}_T$ fulfilling (1.10) and (1.11). Stones theorem then ensures the existence of a s.a. operator $Q_T(A)$ such that $W_T(sA) = e^{isQ_T(A)}$.

We furthermore construct a domain $\mathscr{D}_T$ in $\mathscr{H}_T$ such that $Q_T(A)\mathscr{D}_T \subset \mathscr{D}_T$ for all $A \in O_T(\mathscr{H})$ and the restriction of $Q_T(A)$ to $\mathscr{D}_T$ is essentially s.a. Formulas (1.5), (1.6), and (1.9) hold on $\mathscr{D}_T$ and (1.8) hold for all $A, B \in O_T(\mathscr{H})$.

We shall also briefly discuss the *-algebras generated by the complexified operators $Q_T(A)$.

In a second paper we apply these results to quantum field theory. In particular we show how the Luttinger, Thirring and Schwinger models fit into this framework.

2. Quasi-free Representations in Terms of the Fock-Representation

For later convenience we review some well-known properties of quasi-free states and representations (see for example [2]).

Let γ_t denote the gauge-automorphism group of $\mathfrak{A}(\mathscr{H})$ whose action on $a(f)$ is given by $\gamma_t(a(f))=a(e^{it}f)$. The quasi-free state ω_T $(T\in B(\mathscr{H})_{\mathrm{sa}}, 0\leqq T\leqq I)$ is gauge-invariant, i.e. invariant under the transposed action of γ_t.

Definition 2.1. Let $U_T(t)$ denote the unitary group on $\mathscr{H}_T$ implementing the gauge-automorphism γ_t of $\pi_T(\mathfrak{A}(\mathscr{H}))$ and leaving Ω_T invariant, i.e.

$$\pi_T(\gamma_t(x))=U_T(t)\pi_T(x)U_T(t)^{-1}\,,\qquad U_T(t)\Omega_T=\Omega_T\,. \tag{2.1}$$

In the case when T is a projection $T=P$, then ω_P is a pure state and π_P is irreducible. A state ω_T of $\mathfrak{A}(\mathscr{H})$ can always be expressed as a restriction of a pure state ω_{P_T} of $\mathfrak{A}(\mathscr{H}\oplus\mathscr{H})$ with P_T given by

$$P_T=\begin{pmatrix} T & T^{\frac{1}{2}}(I-T)^{\frac{1}{2}} \\ T^{\frac{1}{2}}(I-T)^{\frac{1}{2}} & I-T\end{pmatrix} \tag{2.2}$$

and

$$\omega_T(a(f_n)^*\dots a(g_m))=\omega_{P_T}(a(f_n\oplus 0)^*\dots a(g_m\oplus 0))\,. \tag{2.3}$$

Remark 2.2. One can identify $\mathscr{H}_T$, $\pi_T(\mathfrak{A}(\mathscr{H}))$ and Ω_T with a subspace of $\mathscr{H}_{P_T}$, $\pi_{P_T}(\mathfrak{A}(\mathscr{H}\oplus O))$ and Ω_{P_T} respectively. The commutant $\pi_T(\mathfrak{A}(\mathscr{H}))'$ is then identified with a part of $U_{P_T}(\pi)\pi_{P_T}(\mathfrak{A}(O\oplus\mathscr{H}))''$.

Definition 2.3. Let $\mathscr{F}(\mathscr{H})$ denote the anti-symmetric Fock-space over $\mathscr{H}$, i.e. $\mathscr{F}(\mathscr{H})=\bigoplus_{n=0}^{\infty}\mathscr{H}_n^a$ with $\mathscr{H}_0^a=\mathbb{C}$, $\mathscr{H}_1^a=\mathscr{H}$, and $\mathscr{H}_n^a$ is the antisymmetric part of $\otimes^n\mathscr{H}$. The Fock-vacuum $\Omega=\bigoplus_{n=0}^{\infty}\Omega_n$ is given by $\Omega_0=1$, $\Omega_n=0$ for $n\geqq 1$. Let furthermore $a_0(f)$ denote the Fock-representation in $\mathscr{F}(\mathscr{H})$ of $a(f)\in\mathfrak{A}(\mathscr{H})$ i.e. $a_0(f)\Omega=0, \forall f\in\mathscr{H}$.

The quasi-free state ω_0 is usually called the Fock-state and one can identify $\mathscr{H}_0$, $\pi_0(a(f))$ and Ω_0 with $\mathscr{F}(\mathscr{H})$, $a_0(f)$ and Ω respectively.

Definition 2.4. Let P be an orthogonal projection operator on $\mathscr{H}$ and J a conjugation commuting with P, i.e. $J^2=1$, $\langle Jf, Jg\rangle=\langle g,f\rangle$ for $f,g\in\mathscr{H}$ and $[J,P]=0$. Let us then define

$$a_P(f)=a_0((I-P)f)+a_0(JPf)^*\,,\qquad f\in\mathscr{H}\,. \tag{2.4}$$

It is easy verified that $a_P(f)$ gives a representation of $a(f)\in\mathfrak{A}(\mathscr{H})$ in Fock-space $\mathscr{F}(\mathscr{H})$ and one can identify $\mathscr{H}_P$, $\pi_P(a(f))$ and Ω_P with $\mathscr{F}(\mathscr{H})$, $a_P(f)$ and Ω respectively.

3. On Innerness of One-Particle *-Automorphisms in $\pi_T(\mathfrak{A}(\mathscr{H}))''$

In the introduction we considered the map $A\to W_T(sA)$ of s.a. finite-rank operators on $\mathscr{H}$ into strongly continuous one parameter groups in $\pi_T(\mathfrak{A}(\mathscr{H}))$ fulfilling (1.10) and (1.11).

Definition 3.1. Let $T \in B(\mathscr{H})_{sa}$ such that $0 \leqq T \leqq I$ and put $O_T(\mathscr{H}) = \{A \in B(\mathscr{H})_{sa};\ \mathrm{tr}(TA(1-T)A) < \infty\}$. $O_T(\mathscr{H})$ is a real linear vector space. We shall call $O_T(\mathscr{H})$ the vector space of one-particle observables.

Remark 3.2. If $A, B \in O_T(\mathscr{H})$ then $i[A, B] \in O_T(\mathscr{H})$ and $e^{iA}Be^{-iA} \in O_T(\mathscr{H})$.

Theorem 1. *There exists a map $A \mapsto W_T(sA)$ of $O_T(\mathscr{H})$ into $\pi_T(\mathfrak{A}(\mathscr{H}))''$ such that $W_T(sA)$ is a strongly continuous unitary one-parameter group, fulfilling (we put $a_T(f) = \pi_T(a(f))$)*

$$a_T(e^{isA}f)) = W_T(sA)a_T(f)W_T(sA)^{-1}, \quad \forall f \in \mathscr{H}. \tag{3.1}$$

$\Omega_T \in D((d/ds)W_T(sA)_{s=0})$, and $W_T(sA)$ are uniquely determined if we require that

$$(\Omega_T, (d/ds)W_T(sA)_{s=0}\Omega_T) = 0. \tag{3.2}$$

For $A, B \in O_T(\mathscr{H})$ the following identity holds

$$W_T(A)W_T(B)W_T(A)^{-1} = W_T(e^{iA}Be^{-iA})e^{ib_T(A,B)}, \tag{3.3}$$

$$b_T(tA, B) = -2\,\mathrm{Im}\int_0^t \mathrm{tr}(TA(I-T)e^{isA}Be^{-isA})ds. \tag{3.4}$$

The proof of this theorem will be divided into several lemmas. We will first prove it when T is a projection P and then reduce the general case to this by a method indicated in the previous section.

Let P denote an orthogonal projection on $\mathscr{H}$ and decompose $A \in O_P(\mathscr{H})$ as follows

$$A = A_0 + A_1, \quad A_0 = (1-P)A(1-P) + PAP, \quad A_1 = PA(1-P) + (1-P)AP,$$

i.e. $[P, A_0] = 0$ and A_1 is Hilbert-Schmidt (H.S.).

Lemma 3.3. *Let $U(s) = e^{-isA_0}e^{isA}$. The following representation of $U(s)$ holds:*

$$U(s) = \sum_{n=0}^{\infty} R_n(s), \tag{3.5}$$

with $R_0(s) = I$, $R_n(s) = i\int_0^s A_1(s')R_{n-1}(s')ds'$, $n = 1, 2, \ldots$, $A_1(s) = e^{-isA_0}A_1e^{isA_0}$ and the convergence is with respect to H.S. norm and is uniform in s for compact subsets of $\mathbb{R}$.

Proof. Differentiation and integration of $U(s)$ gives

$$U(s) = 1 + i\int_0^s A_1(s')U(s')ds'. \tag{3.6}$$

Iteration of (3.6) in N steps gives $U_N(s) = \sum_{n=0}^{N} R_n(s)$. The H.S. convergence in the limit follows from the following estimates

$$\|R_1(s)\|_2 \leqq \|A_1\|_2|s|,\ \|R_2(s)\|_2 \leqq \|A_1\| \int_0^s \|R_1(s')\|_2 ds' \leqq \|A_1\|_2^2/2! \cdot |s|^2,\ \|R_n(s)\|_2 \leqq (\|A_1\|_2^n/n!)|s|^n, \tag{3.7}$$

where $\| \ \|_2$ denotes the H.S. norm □

Remark 3.4. It follows that $Pe^{isA}(1-P)$ is H.S. $\forall s \in \mathbb{R}$, i.e. $Pe^{isA}(1-P) = e^{isA} \circ PU(s)(1-P)$ which obviously is H.S.

As mentioned in the previous section one can identify $\mathscr{H}_P$, $\pi_P(a(f))$ and Ω_P with $\mathscr{F}(\mathscr{H})$, $a_P(f)$ and Ω respectively. We shall do this in the following. Equation (3.1) then takes the form

$$a_P(e^{isA}f) = W_P(sA)a_P(f)W_P(sA)^{-1}. \tag{3.8}$$

The defining equation for $a_P(f)$, (2.4) implies that (3.8) is equivalent to

$$a_0(L(s)f) + a_0(M(s)f)^* = W_P(sA)a_0(f)W_P(sA)^{-1}, \tag{3.9}$$

where $L(s)$ is complex linear, $M(s)$ is complex anti-linear and they are explicitly given by $L(s) = L_1(s) + L_2(s)$, $M(s) = M_1(s) + M_2(s)$ with

$$L_1(s) = (1-P)e^{isA}(1-P), \qquad L_2(s) = PJe^{isA}JP, \tag{3.10}$$

$$M_1(s) = JPe^{isA}(1-P), \qquad M_2(s) = (1-P)e^{isA}JP \tag{3.11}$$

We note if there exists a unitary operator $W_P(sA)$ in $\mathscr{F}(\mathscr{H})$ such that (3.9) holds then

$$[a_0(L(s)f) + a_0(M(s)f)^*]W_P(sA)\Omega = 0 \tag{3.12}$$

for all $f \in \mathscr{H}$.

Theorem 1 will be proved by actually first constructing $\chi_s = W_P(sA)\Omega$ explicitly. This method of proof goes back to Friedrichs [4] (see also Shale and Stinespring [5] and Araki [2]).

Lemma 3.5. *There is an $\chi_s \in \mathscr{F}(\mathscr{H})$ such that for small s*

$$[a_0(L(s)f) + a_0(M(s)f)^*]\chi_s = 0, \qquad \forall f \in \mathscr{H}. \tag{3.13}$$

Proof. We first construct a vector χ_s such that (3.13) holds for all $f \in (1-P)\mathscr{H}$, and then we show that this vector solves (3.13) for all $f \in \mathscr{H}$. For $f \in (1-P)\mathscr{H}$ (3.13) takes the form

$$[a_0(L_1(s)f) + a_0(M_1(s)f)^*]\chi_s = 0, \qquad \forall f \in (1-P)\mathscr{H}. \tag{3.14}$$

The operator $L_1(s): (1-P)\mathscr{H} \to (1-P)\mathscr{H}$ is easily seen to have a bounded inverse (at least for small s). This means that (3.14) is equivalent to

$$[a_0(g) + a_0(M_1(s)L_1(s)^{-1}g)^*]\chi_s = 0, \qquad \forall g \in (1-P)\mathscr{H}. \tag{3.15}$$

$M_1(s)$ is H.S. by Remark 3.4; hence $K(s) \equiv M_1(s)L_1(s)^{-1}$ also is H.S. and therefore has a spectral representation

$$K(s)g = \sum_{n=1}^{\infty} \lambda_n(s)v_n(s)\langle g, u_n(s)\rangle, \qquad \lambda_n(s) \geqq 0, \tag{3.16}$$

where $\sum_{n=1}^{\infty} \lambda_n(s)^2 < \infty$ and $\{v_n(s)\}_{n=1}^{\infty}$, $\{u_n(s)\}_{n=1}^{\infty}$ are orthonormal sets in $P\mathscr{H}$, $(1-P)\mathscr{H}$ respectively. Let us define $\chi_s^N \in \mathscr{F}(\mathscr{H})$ by

$$\chi_s^N = \prod_{n=1}^{N} e^{-\lambda_n(s)a_0(u_n(s))^* a_0(v_n(s))^*}\Omega. \tag{3.17}$$

One easily verifies that (3.15) holds with $\chi_s = \chi_s^N$ for all $g \in \operatorname{span}\{u_n(s)\}_{n=1}^N$. A simple calculation gives

$$\|\chi_s^{N_1} - \chi_s^{N_2}\|^2 = \left| \prod_{n=1}^{N_1} (1+\lambda_n(s)^2) - \prod_{n=1}^{N_2} (1+\lambda_n(s)^2) \right|, \tag{3.18}$$

which shows that $\mathrm{s}-\lim_{N\to\infty} \chi_s^N$ exists in $\mathscr{F}(\mathscr{H})$. Let us then define

$$\chi_s = c(s) \prod_{n=1}^{\infty} e^{-\lambda_n(s) a_0(u_n(s))^* a_0(v_n(s))^*} \Omega, \tag{3.19}$$

$$c(s) = e^{i\varphi_s} \prod_{n=1}^{\infty} (1+\lambda_n(s)^2)^{-\frac{1}{2}}, \qquad \varphi_s \in \mathbb{R}, \tag{3.20}$$

then χ_s is a normalized vector fulfilling (3.15) for all $g \in \operatorname{span}\{u_n(s)\}_{n=1}^{\infty}$. If $(1-P)\mathscr{H} \ni g \perp \operatorname{span}\{u_n(s)\}_{n=1}^{\infty}$ it easily follows that (3.15) holds and (3.14) also holds. If one makes an analogous construction for $f \in P\mathscr{H}$ one just has to make the replacement $L_1(s) \to L_2(s)$, $M_1(s) \to M_2(s)$. A formula similar to (3.19) is then obtained. To compare the two formulas we anti-commute the creation operators in one of the formulas. The two vectors are now seen to coincide (up to a phase) because

$$M_1(s) L_1(s)^{-1} = -(M_2(s) L_2(s)^{-1})^*,$$

which follows from (3.10) and (3.11).

Lemma 3.6. *There exists a strongly continuous unitary one-parameter group* $V(s)$, *unique up to a phase* $e^{i\varphi s}$, $\varphi \in \mathbb{R}$ *such that* $a_P(e^{isA} f) = V(s) a_P(f) V(s)^{-1}$, $\forall f \in \mathscr{H}$.

Proof. Let α_s denote the one-parameter *-automorphism group of $\mathfrak{A}(\mathscr{H})$ whose action on $a(f)$ is given by $\alpha_s(s(f)) = a(e^{isA} f)$ and let us define an operator $V(s)$ on $\pi_P(\mathfrak{A}(\mathscr{H}))\Omega$ by

$$V(s)\pi_P(x)\Omega = \pi_P(\alpha_s(x))\chi_s, \qquad x \in \mathfrak{A}(\mathscr{H}). \tag{3.21}$$

It is easily verified that $V(s)$ defines an isometry by using that $a_P(e^{isA} f)\chi_s = 0$, $\forall\, f \in (1-P)\mathscr{H}$ and $a_P(e^{isA} g)^* \chi_s = 0$, $\forall g \in P\mathscr{H}$, which follows from Lemma 3.5. The irreducibility of the representation implies that the range of $V(s)$ is dense in $\mathscr{F}(\mathscr{H})$, i.e. $V(s)$ extends to a unitary operator on $\mathscr{F}(\mathscr{H})$. Equation (3.21) implies that $a_P(e^{isA} f) = V(s(a_P(f) V(s)^{-1}$. The irreducibility implies that $V(s)$ is unique up to a phase, which is just the phase in formula (3.20). It follows from a theorem by Kadison [6] that this phase can be chosen such that $V(s)$ becomes a strongly continuous one-parameter group of unitaries and we are then left with a phase $e^{i\varphi_s}$, $\varphi \in \mathbb{R}$.

We shall from now on assume that $e^{i\varphi_s}$ in (3.20) is chosen such that Lemma 3.6 holds.

Lemma 3.7. *The phase* φ_s *is analytic in a neighbourhood of zero and* $\Omega \in \mathscr{D}((d/ds)V(s)_{s=0})$ *and provided* $\varphi_0' = 0$ *(3.2) holds with* $W_P(sA) = V(s)$.

Proof. We first note that $c(s)$ in (3.20) can be extended to an analytic function in a neighbourhood of zero, i.e. consider

$$(\psi, V(s)\Omega) = c(s) \left(\psi, \prod_{n=1}^{\infty} e^{-\lambda_n(s) a_0(u_n(s))^* a_0(v_n(s))^*} \Omega \right), \tag{3.22}$$

where ψ is an analytic vector of the generator of $V(s)$.

But

$$\sum_{n=1}^{\infty} e^{-\lambda_n(s)a_0(u_n(s))^*a_0(v_n(s))^*}\Omega$$
$$= e^{-\Sigma\lambda_n(s)a_0(u_n(s))^*a_0(v_n(s))^*}\Omega \equiv e^{-\mathscr{K}(s)}\Omega = \Sigma(-\mathscr{K}(s))^k\Omega/k!$$

and $\mathscr{K}(s)^k\Omega \in \mathscr{H}_a^{2k} \subset \mathscr{F}(\mathscr{H})$ is just the appropriately antisymmetrized tensor product of k copies of $K(s)$.

The fact that $K(s)$ has an analytical extension in a neighbourhood of zero then shows that $c(s)$ is analytic, which implies the first statement in the lemma. Let us now compute $(d/ds)V(s)\Omega|_{s=0}$ explicitly and verify that it is a vector in $\mathscr{F}(\mathscr{H})$. We have

$$(d/ds)K(s)|_{s=0} \equiv K = JPiA(1-P)\,, \tag{3.23}$$

which is H.S. and therefore has a spectral representation

$$Kg = \sum_{n=1}^{\infty} \lambda_n v_n \langle g, u_n\rangle\,, \qquad \lambda_n \geqq 0 \tag{3.24}$$

with $\Sigma\lambda_n^2 < \infty$ and $\{v_n\}_{n=1}^{\infty}$, $\{u_n\}_{n=1}^{\infty}$ are orthogonal sets in $P\mathscr{H}$, $(1-P)\mathscr{H}$ respectively. Equation (3.19) then gives

$$(d/ds)V(s)\Omega|_{s=0} = i\varphi_0'\Omega - \sum_{n=1}^{\infty} \lambda_n a_0(u_n)^* a_0(v_n)^*\Omega\,, \tag{3.25}$$

which clearly is a vector in $\mathscr{F}(\mathscr{H})$ and by choosing $\varphi_0'=0$ we find that $(\Omega, (d/ds)V(s)\Omega)_{s=0}=0$.

Lemma 3.8. *Let us define a mapping $A \mapsto Q_P(A)$ of $O_P(\mathscr{H})$ into s.a. operators on $\mathscr{F}(\mathscr{H})$ by $W_P(sA)=e^{isQ_P(A)}$. Then for A_1, $A_2 \in O_P(\mathscr{H})$ we have $Q_P(A_2)\Omega \in \mathscr{D}(Q_P(A_1))$ and*

$$(\Omega, Q_P(A_1)Q_P(A_2)\Omega) = \mathrm{tr}(TA_1(1-T)A_2)\,. \tag{3.26}$$

Proof. We note that (3.25) implies that

$$iQ_P(A_2)\Omega = -\Sigma\lambda_{2n} a_P(u_{2n})^* a_P(Jv_{2n})\Omega\,. \tag{3.27}$$

If we apply $W_P(sA_1)$ to (3.27) we get

$$i\,W_P(sA_1)Q_P(A_2)\Omega = -\Sigma\lambda_{2n} a_P(e^{isA_1}u_{2n})^* a_P(e^{isA_1}Jv_{2n})W_P(sA_1)\Omega\,. \tag{3.28}$$

The s-derivative at $s=0$ is easily seen to exist in $\mathscr{F}(\mathscr{H})$. In fact we get

$$\begin{aligned}&-Q_P(A_1)Q_P(A_2)\Omega = -\Sigma\lambda_{2n}a_P(iA_1u_{2n})^*a_0(v_{2n})^*\Omega - \Sigma\lambda_{2n}a_0(u_{2n})^*\\ &a_P(iA_1Jv_{2n})\Omega + \Sigma\Sigma\lambda_{1m}\lambda_{2n}a_0(u_{1m})^*a_0(v_{1m})^*a_0(u_{2n})^*a_0(v_{2n})^*\Omega\,.\end{aligned} \tag{3.29}$$

Equation (3.30) finally gives

$$\begin{aligned}(\Omega, Q_P(A_1)Q_P(A_2)\Omega) &= \Sigma\lambda_{2n}\langle JiA_1u_{2n}, v_{2n}\rangle = i\,\overline{\mathrm{tr}(A_1JK_2)}\\ &= \mathrm{tr}(PA_1(1-P)A_2)\,.\end{aligned} \tag{3.30}$$

Lemma 3.9. *If A, $B \in O_P(\mathscr{H})$ then*

$$W_P(tA)W_P(sB)W_P(tA)^{-1} = W_P(e^{itA}sBe^{-itA})e^{ib(tA,sB)} \tag{3.31}$$

where $b_P(tA, sB) = -2\,\mathrm{Im}\int_0^t \mathrm{tr}(TA(I-T)e^{iAr}sBe^{-iAr})dr$.

Proof. Equation (3.31) follows directly from (3.1) and the irreducibility of π_P. By taking the vacuum expectation value of (3.31) and differentiating with respect to t and s at $t=s=0$ we get

$$(d/dt)b_P(tA, B)_{t=0} = i(\Omega, [Q_P(A), Q_P(B)]\Omega) = -2\,\mathrm{Im}\,\mathrm{tr}(TA(I-T)B)\,. \tag{3.32}$$

The cocycle equation can now be solved as in the introduction.

We have thus completed the proof of Theorem 1 when T is an orthogonal projection. In order to prove the general case of a $T: 0 \leqq T \leqq I$ we recall Remark 2.2 which says that one can identify $\mathscr{H}_T$ with a subspace of $\mathscr{F}(\mathscr{H} \oplus \mathscr{H})$, $\pi_T(\mathfrak{A}(\mathscr{H}))$ with $\pi_{P_T}(\mathfrak{A}(\mathscr{H} \oplus O))$ and $\pi_T(\mathfrak{A}(\mathscr{H}))'$ with a part of $U_{P_T}(\pi)\pi_{P_T}(\mathfrak{A}(O \oplus \mathscr{H}))''$.

Let us now identify $A \in O_T(\mathscr{H})$ with $A \oplus O \in O_{P_T}(\mathscr{H} \oplus \mathscr{H})$. One easily verifies that

$$\mathrm{tr}(P_T(A \oplus O)(1-P_T)(B \oplus O)) = \mathrm{tr}(TA(1-T)B)\,. \tag{3.33}$$

Let α_s denote the one-parameter $*$-automorphism group of $\mathfrak{A}(\mathscr{H} \oplus \mathscr{H})$ whose action on $a(f \oplus g)$ is given by $\alpha_s(a(f \oplus g)) = a(e^{isA}f \oplus g)$, i.e. $\mathfrak{A}(O \oplus \mathscr{H})$ is left invariant. We have thus reduced the general case to the case of a projection and now we identify $W_T(A)$ with $W_{P_T}(A \oplus O)$. Finally $W_T(A)$ belongs to $\pi_T(\mathfrak{A}(\mathscr{H}))''$ because the automorphism α_s leaves $\pi_T(\mathfrak{A}(\mathscr{H}))'$ invariant in our identifications. The uniqueness follows from the factor nature which was proved by Powers and Størmer [3] and which also is obvious in the explicit representation in Fock-space given above. This completes the proof of Theorem 1.

4. Some Properties of the Quantization Map $A \to Q_T(A)$

Let $A \in O_T(\mathscr{H})$ and consider the map $A \to W_T(sA)$ described in Theorem 1. The generator of the strongly continuous unitary one-parameter group $\{W_T(sA)\}_{s\in\mathbb{R}}$ is denoted by $Q_T(A)$; i.e., $W_T(sA) = e^{isQ_T(A)}$.

By $\mathfrak{A}_0(\mathscr{H})$ we denote the $*$-algebra generated by $a(f)$, $f \in \mathscr{H}$, and put $\mathscr{D}_T^0 = \pi_T(\mathfrak{A}_0(\mathscr{H}))\Omega_T$. It is clear that $\overline{\mathscr{D}_T^0} = \mathscr{H}_T$.

Lemma 4.1. *$\mathscr{D}_T^0 \subset \mathscr{D}(Q_T(A))$ for all $A \in O_T(\mathscr{H})$. Furthermore $Q_T(A_{n-1}) \dots Q_T(A_1)\mathscr{D}_T^0 \subset \mathscr{D}(Q_T(A_n))$ for all $A_1, \dots, A_n \in O_T(\mathscr{H})$, $n = 2, 3, \dots$.*

Proof. Lemma 3.7 and Lemma 3.8 proves that $\Omega_T \in \mathscr{D}(Q_T(A))$ and that $Q_T(A_1)\Omega_T \in \mathscr{D}(Q_T(A_2))$ if we make the identification of $W_T(sA)$ with $W_{P_T}(sA \oplus O)$. A direct generalization of this shows that $Q_T(A_{n-1}) \dots Q_T(A_1)\Omega_T \subset \mathscr{D}(Q_T(A_n))$. The extension of this to the whole of $\mathscr{D}_T^0$ follows easily by remembering equation (3.1) and the boundedness of $A_1, \dots, A_n$.

Definition 4.2. Let $\mathscr{D}_T$ denote the domain in $\mathscr{H}_T$ obtained by acting with monomials $\prod_{i=1}^{n} Q_T(A_i), A_1, \dots, A_n \in O_T(\mathscr{H})$ on $\mathscr{D}_T^0$ $n = 1, 2, \dots$, i.e. $Q_T(A)\mathscr{D}_T \subset \mathscr{D}_T$ for all $A \in O_T(\mathscr{H})$.

Theorem 2. *The map $A \to Q_T(A)$ of $O_T(\mathscr{H})$ into self-adjoint operators affiliated with $\pi_T(\mathfrak{A}(\mathscr{H}))''$ has the following properties: The restriction of $Q_T(A)$ to $\mathscr{D}_T$ is essentially self-adjoint, and*

$$\overline{Q_T(A)+Q_T(B)}=Q_T(A+B)\,, \tag{4.1}$$

$$\overline{[Q_T(A), a_T(f)^*]}=a_T(Af)^*\,, \tag{4.2}$$

$$\overline{i[Q_T(A), Q_T(B)]}=Q_T(i[A,B])-2\,\mathrm{Im}\,\mathrm{tr}(TA(1-T)B)1\,. \tag{4.3}$$

The last term in (4.3) will be referred to as the Schwinger term. These relations are generalizations of (1.5), (1.6) and (1.9).

Proof. Let $\varphi \in \mathscr{H}_T$, $\psi \in \mathscr{D}_T$ and consider $(\varphi, W_T(sA)\psi)$. By doing computations similar to the ones in the proof of Lemma 3.8 one can verify that $(\varphi, W_T(sA)\psi)$ has an analytical extension around $s=0$. Thus ψ is an analytical vector for $Q_T(A)$, i.e. $\mathscr{D}_T$ is a dense set of analytic vectors and the restriction of $Q_T(A)$ to $\mathscr{D}_T$ is ess. s.a. by Nelson's theorem. The verification of (4.1) is done by first verifying it on Ω_T which is a consequence of (3.27), (3.24) and (3.23). In order to verify it on an arbitrary vector in $\mathscr{D}_T$ we first note that (4.2) follows from (3.1) by differentiation at $t=0$. That (4.1) holds on $\mathscr{D}_T$ then follows from (4.2). Finally (4.3) follows by letting $A \to tA$, $B \to sA$ in (3.3) and differentiate with respect to t and s on $\mathscr{D}_T$ at $t=s=0$.

Definition 4.3. Let us for $A, B \in O_T(\mathscr{H})+iO_T(\mathscr{H})$, define

$$\langle A, B\rangle_T=\mathrm{tr}(TA^*(1-T)B)\,, \tag{4.4}$$

$$\gamma_T(A,B)=\langle A,B\rangle_T-\langle A,B\rangle_{1-T}\,. \tag{4.5}$$

For $A, B \in O_T(\mathscr{H})$ we easily verify that

$$-2\,\mathrm{Im}\,\mathrm{tr}(TA(1-T)B)=i\gamma_T(A,B) \tag{4.6}$$

i.e. if we complexify the map $A \to Q_T(A)$ by defining

$$Q_T(A)=Q_T(\mathrm{Re}\,A)+iQ_T(\mathrm{Im}\,A) \quad \text{on} \quad \mathscr{D}_T\,, \tag{4.7}$$

we find that (4.3) generalizes to

$$[Q_T(A)^*, Q_T(B)]=Q_T([A^*,B])+\gamma_T(A,B)\cdot 1\,, \tag{4.8}$$

on $\mathscr{D}_T$ for all $A, B \in O_T(\mathscr{H})+iO_T(\mathscr{H})$, and we furthermore have

$$Q_T(A)^* \supset Q_T(A^*)\,. \tag{4.9}$$

We note that $\gamma_T(A^*, B^*)=-\gamma_T(B,A)$.

Remark 4.4. Consider complex subspace $\mathscr{V}$ of $O_T(\mathscr{H})+iO_T(\mathscr{H})$ with the property that all operators in $\mathscr{V}$ are commuting and $\mathscr{V}$ is invariant under adjoint operation, then (4.8) gives on $\mathscr{D}_T$

$$[Q_T(A)^*, Q_T(B)]=\gamma_T(A,B)1\,, \tag{4.10}$$

for all $A, B \in \mathscr{V}$. These are just the commutation relations of the self-dual *CCR* algebra considered by Araki and Shiraishi [7], and Araki [8].

Remark 4.5. Araki [9] has discussed factorizable representations of commutation relations similar to (4.3).

Conclusions

We have considered the fermion C^*-algebra $\mathfrak{A}(\mathscr{H})$ over $\mathscr{H}$ together with certain one-parameter groups of *-automorphisms of $\mathfrak{A}(\mathscr{H})$. Let A be a self-adjoint (s.a.) operator on $\mathscr{H}$. There exists a unique strongly continuous one-parameter *-automorphism group α_s of $\mathfrak{A}(\mathscr{H})$ with the property that $\alpha_s(a(f))=a(e^{isA}f)$ $f\in\mathscr{H}$.

Let ω_T be the gauge-invariant quasi-free state of $\mathfrak{A}(\mathscr{H})$ associated with T, $0\leqq T\leqq I$ and let $\mathscr{H}_T$, π_T, and Ω_T be the Hilbert-space the representation and the cyclic vector associated with the GNS construction.

Let furthermore $O_T(\mathscr{H})$ denote the real vectorspace consisting of bounded s.a. operators A on $\mathscr{H}$ with $\mathrm{tr}(TA(1-T)A)<\infty$.

For $A\in O_T(\mathscr{H})$ the automorphism α_s of $\mathfrak{A}(\mathscr{H})$ extends to an inner automorphism of $\pi_T(\mathfrak{A}(\mathscr{H}))''$. Let $W_T(sA)$ denote the implementing strongly continuous group. $W_T(sA)$ is unique up to a phase $e^{i\varphi_s}$. Let $Q_T(A)$ denote the s.a. generator i.e. $W_T(sA)=e^{isQ_T(A)}$. It is shown that $\Omega_T\in\mathscr{D}(Q_T(A))$ and the phase is then chosen such that $(\Omega_T, Q_T(A)\Omega_T)=0$.

For $A, B\in O_T(\mathscr{H})$ one gets $W_T(A)W_T(B)W_T(A)^{-1}=W_T(e^{iA}Be^{-iA})e^{ib_T(A,B)}$ and there exists a dense domain $\mathscr{D}_T\subset\mathscr{H}_T$ such that $Q_T(A)\mathscr{D}_T\subset\mathscr{D}_T$ for all $A\in O_T(\mathscr{H})$ and the restriction of $Q_T(A)$ to $\mathscr{D}_T$ is essentially s.a.

The *-algebras generated by the map $A\to W_T(A)$ are called observable algebras. Applications to quantum field theory will be considered in a second paper.

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