Commun. math. Phys. 50, 97—102 (1976)

Communications in Mathematical Physics

A Possible Constructive Approach to φ_4^4 III

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Abstract. We continue our investigation into a constructive approach for φ_d^4 and prove under the hypotheses of our previous work that on a lattice and in the single phase region the theory is uniquely determined by the intermediate renormalization conditions. Then the Gaussian theory and the Ising model are the two extremal cases for all φ_d^4 theories.

In the first of two papers devoted to a constructive approach to φ_4^4 in the single phase region ([4, 5]) we presented an ansatz based on an implicit function problem and suggested by multiplicative renormalization. The mapping is (within the Euclidean approach on a fixed lattice) from the set of the three renormalization constants to the set of the three (intermediate) normalization constants. We will call this map the renormalization map. In Ref. [4] we made the assumption that this C^∞-map is everywhere locally injective (i.e. of maximal rank). Using this assumption and a certain assumption about the image at infinity (see below), we will prove that this map is actually globally injective, i.e. one-to-one. Thus the normalization constants uniquely determine the theory in the lattice approximation.

The strategy for the proof will be as follows: First we show that a certain set which we showed, in Ref. [4], to be in the image is actually all of the image. In particular the image is simply connected. This result also allows one to view the Ising model as one extremal case of a φ_d^4 theory on a lattice: It is the case where $\varphi^2 = \text{const.}$ or equivalently where the bare coupling constant tends to infinity. This conforms with conventional wisdom, see e.g. Ref. [1, 2, 7]. Secondly we show that the renormalization map is proper, thus making the set of renormalization constants a covering space of the set of normalization constants. The global injectivity then follows from a standard theorem in homotopy theory.

To fix the notation, we start by reviewing the results of Ref. [4].

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* On leave of absence from the Freie Universität Berlin

** Supported in part by the Deutsche Forschungsgemeinschaft and by the National Science Foundation under Grant MPS 75-21212

Let $\mathscr{T}$ be a discrete torus in d dimensions, i.e. a set of the form

$$\mathscr{T}=\mathbb{Z}_{n_1}\times\ldots\times\mathbb{Z}_{n_d} \quad (n_d\geqq 3).$$

For given $\alpha=(\alpha_1,\alpha_2,\alpha_3)$ or $\delta=(\delta_1,\delta_2,\delta_3)$ both in $\mathbb{R}^+\times\mathbb{R}^+\times\mathbb{R}$ and related by a C^∞ diffeomorphism

$$\begin{aligned}\alpha_1&=\delta_1\\ \alpha_2&=2\delta_2\\ \alpha_3&=2\delta_1\delta_3-2\mathrm{d}\delta_2\end{aligned} \tag{1}$$

we define a probability measure on $\mathbb{R}^{|\mathscr{T}|}$ by

$$\begin{aligned}d\mu(\{x_j\}_{j\in\mathscr{T}})&=N^{-1}\exp\left\{-\alpha_1\sum_{j\in\mathscr{T}}x_j^4+\alpha_2\sum_{i,jN\cdot N}x_ix_j+\alpha_3\sum_{j\in\mathscr{T}}x_j^2\right\}\prod_{j\in\mathscr{T}}dx_j\\ &=N^{-1}\exp\left\{-\delta_1\sum_{j\in\mathscr{T}}(x_j^2-\delta_3)^2-\delta_2\sum_{i,jN\cdot N}(x_i-x_j)^2\right\}\prod_{j\in\mathscr{T}}dx_j\end{aligned} \tag{2}$$

Here N is a normalization constant, making the measure in question a probability measure. $N\cdot N$ stands for nearest neighbors. Let $\langle\,\rangle$ denote the expectation w.r.t. this measure. For fixed $\mathscr{T}$ and fixed lattice spacing $\mathrm{a}>0$, the renormalization map is then given by

$$y=(y_1,y_2,y_3)=T(\alpha)=S(\delta)\in(\mathbb{R}^+)^3$$

with

$$\begin{aligned}y_1&=|\mathscr{T}|^{-1}\mathrm{a}^2\sum_{i,j\in\mathscr{T}}\langle x_ix_j\rangle\\ y_2&=|\mathscr{T}|^{-1}\mathrm{a}^4\sum_{i,j\in\mathscr{T}}(i-j)^2\langle x_ix_j\rangle\\ y_3&=-|\mathscr{T}|^{-1}\mathrm{a}^{4+d}\sum_{i,j,k,l}\langle x_i;x_j;x_k;x_l\rangle.\end{aligned} \tag{3}$$

Here $(i-j)^2$ is the (translation invariant) square distance on $\mathscr{T}$, $|\mathscr{T}|$ is the number of elements in $\mathscr{T}$ and $\langle A_1;\ldots;A_n\rangle$ denotes the truncated expectation of the random variables $A_1,\ldots,A_n$.

Since we use the untruncated expectation values in the definition of y_1 and y_2, we will automatically restrict ourselves to the single phase region. We have the estimates

$$\begin{aligned}y_2&\leqq \mathrm{a}^2\max_{i,j\in\mathscr{T}}(i-j)^2y_1\\ y_3&\leqq 2\mathrm{a}^{\mathrm{d}}|\mathscr{T}|y_1^2\end{aligned} \tag{4}$$

due to Griffiths first and second inequalities and the Lebowitz inequality (see e.g. [6]).

In Ref. [4] we considered the sets

$$\begin{aligned}\mathscr{M}_1&=\{y=T(\alpha)|\alpha_2\to 0\}\\ \mathscr{M}_2&=\{y=T(\alpha)|\alpha_1\to 0,\ -\alpha_3>\mathrm{d}\alpha_2\geqq 0\}\\ \mathscr{M}_3&=\lim_{t\to\infty}\{y=T(\alpha)|\alpha_2=t\}\\ \mathscr{M}_4&=\lim_{t\to\infty}\{y=T(\alpha)|\alpha_1=t\},\end{aligned}$$

$\mathcal{M}_3$ was shown to take the form

$$y \in \mathcal{M}_3 \Leftrightarrow y_1 > 0\,, \quad y_2 = |\mathcal{T}|^{-1} \mathrm{a}^2 \sum_{i,j \in \mathcal{T}} (i-j)^2 y_1$$
$$0 < y_3 < 2\mathrm{a}^{\mathrm{d}} |\mathcal{T}| y_1^d$$

and $\mathcal{M}_4$ was shown to take the form

$$\begin{aligned} &y \in \mathcal{M}_4 \Leftrightarrow \\ &y_1 = |\mathcal{T}|^{-1} \mathrm{a}^2 \gamma \sum_{i,j \in \mathcal{T}} \langle \sigma_i \sigma_j \rangle (J) \\ &y_2 = |\mathcal{T}|^{-1} \mathrm{a}^4 \gamma \sum_{i,j \in \mathcal{T}} (i-j)^2 \langle \sigma_i \sigma_j \rangle (J) \\ &y_3 = -|\mathcal{T}|^{-1} \mathrm{a}^{4+\mathrm{d}} \gamma^2 \sum_{i,j,k,l \in \mathcal{T}} \langle \sigma_i ; \sigma_j ; \sigma_k ; \sigma_l \rangle (J) \end{aligned} \tag{5}$$

for some $\gamma > 0$, $J > 0$. Here $\sigma_i = \pm 1$ are Ising model variables of an Ising model on $\mathcal{T}$ with coupling strength J and expectations $\langle \cdot \rangle (J)$. In Ref. [4] we made the conjecture that $\mathcal{M}_4$ is a two-dimensional submanifold (with boundary), which lies schlicht over $\mathcal{M}_2$ under the natural projection $(y_1, y_2, y_3) \mapsto (y_1, y_2, 0)$. Using this conjecture and the conjecture that the renormalization map T has maximal rank everywhere, we proved that the open submanifold $K \subset (\mathbb{R}^+)^3$ defined to be the open set with boundary $\partial K = \overline{\bigcup_{i=1}^{4} \mathcal{M}_i}$ is in the image of T.

Using the same assumptions, we will prove

Lemma 1. *K is the image of T;*

$$K = \operatorname{Im} T\,.$$

In particular $\operatorname{Im} T$ *is simply connected.*

Since $\mathcal{M}_3$ tends to infinity in the thermodynamic limit $\mathrm{a}^{\mathrm{d}} |\mathcal{T}| \to \infty$, $\mathrm{a} \to 0$, and since $\mathcal{M}_2$ describes a Gaussian theory, Lemma 1 allows one to say that the Ising model and the Gaussian theory are the extremal cases of a φ_{d}^4 theory on a lattice.

Similarly we will prove

Lemma 2. *The map T from* $(\mathbb{R}^+)^2 \times \mathbb{R}$ *onto K is proper, i.e. the preimage of any compact set is compact. In particular T makes* $(\mathbb{R}^+)^2 \times \mathbb{R}$ *a covering space of K.*

The following theorem is then an immediate consequence of these lemmas and a standard theorem on homotopies of covering spaces (see e.g. [3], p. 251).

Theorem. *Under the assumptions stated above, the map T defines a diffeomorphism of* $(\mathbb{R}^+)^2 \times \mathbb{R}$ *on* $K = \operatorname{Im} T$.

The remainder of this article is devoted to a proof of the two lemmas. We start with a

Proof of Lemma1. Assume K is not the entire image of T. Hence there is $y^0 \in \operatorname{Im} T$ which is not in K. We shall show that this assumption leads to a contradiction.

We may also assume that $y^0 \notin \partial K$, for otherwise, by local injectivity, there will be another y^1 in $\operatorname{Im} T$ close to y^0 with $y^1 \notin \partial K \cup K$. There are two possibilities:

(i) $y_2^0 > |\mathcal{T}|^{-1} a^2 \sum_{i,j \in \mathcal{T}} (i-j)^2 y_1^0$

(ii) $y_3^0 > \tilde{y}_3$ where $\tilde{y}_3$ is such that $(y_1^0, y_2^0, \tilde{y}_3) \in \bar{\mathcal{M}}_4$.

In case (i), let $y^\infty = (y_1^\infty, y_2^\infty, y_3^\infty)$ be defined by

$$y_1^\infty = y_1^0,\ y_3^\infty = y_3^0$$

and

$$y_2^\infty = \sup_{(y_1^0, y_2^1, y_3^0) \in \operatorname{Im} T} y_2^1 .$$

Note that by (4) $y_2^\infty < \infty$. In case (ii), let $y^\infty = (y_1^\infty, y_2^\infty, y_3^\infty)$ be defined by

$$y_1^\infty = y_1^0,\ y_2^\infty = y_2^0,\ y_3^\infty = \sup_{(y_1^0, y_2^0, y_3^1) \in \operatorname{Im} T} y_3^1 .$$

Again by (4) $y_3^\infty < \infty$. By construction, $y^\infty \notin K \cup \partial K$ and furthermore $y^\infty \notin \operatorname{Im} T$, since $y^\infty \in \operatorname{Im} T$ would contradict the construction of y^∞ and the local injectivity of T. Let α^l and δ^l be sequences in $(\mathbb{R}^+)^2 \times \mathbb{R}$ related by (1) such that the sequence

$$y^l = T(\alpha^l) = S(\delta^l)$$

converges to y^∞. By passing to a subsequence if necessary we may assume both sequences α^l and δ^l converge to α^∞ and δ^∞ respectively. (We allow the values $\alpha_1^\infty = \delta_1^\infty = 0$ or ∞, $\alpha_2^\infty = 2\delta_2^\infty = 0$ or ∞ and α_3^∞, $\delta_3^\infty = \pm\infty$.) We will also assume $\alpha_4^l = \alpha_3^l + d\alpha_2^l$ to be convergent to α_4^∞ (possibly equal to $\pm\infty$). Now neither α^∞ nor δ^∞ are in $(\mathbb{R}^+)^3$ since, otherwise, by continuity

$$\begin{aligned} y^\infty &= \lim_{l\to\infty} T(\alpha^l) (= \lim_{l\to\infty} S(\delta^l)) \\ &= T(\alpha^\infty) (= S(\delta^\infty)) \end{aligned}$$

would be in the image of T (or S respectively), which is a contradiction. We now show that this last property implies that either y^l converges to a point on ∂K or tends to infinity, contradicting the fact that the (finite) limiting point y^∞ is not in ∂K. We distinguish three different cases:

Case 1. $\alpha_1^\infty = \delta_1^\infty = 0$.

We write $\mu = \mu(\alpha)$ as

$$\begin{aligned} d\mu(\{x_j\}_{j\in\mathcal{T}}) = N^{-1} \exp\left[-\alpha_1 \sum_{j\in\mathcal{T}} x_j^4 - \frac{\alpha_2}{2} \sum_{i,jN\cdot N} (x_i - x_j)^2 \right] \\ \exp(\alpha_3 + d\alpha_2) \sum_{j\in\mathcal{T}} x_j^2 \prod_{j\in\mathcal{T}} dx_j . \end{aligned} \tag{6}$$

If α_2^∞ is finite and $-\infty < \alpha_4^\infty < 0$, then the y^l converge to a point on $\mathcal{M}_2 \subset \partial K$. If α_2^∞ is finite and $\alpha_4^\infty = -\infty$, then the y^l converge to 0. If α_2^∞ is finite and $\alpha_4^\infty \geqq 0$, then the y_1^l tend to ∞. If $\alpha_2^\infty = +\infty$ and $\alpha_4^\infty = -\infty$, then $y^l \to 0 \in \partial K$. If $\alpha_2^\infty = +\infty$ and $-\infty < \alpha_4^\infty < 0$, then the y^l converge to a point on $\bar{\mathcal{M}}_2 \cap \bar{\mathcal{M}}_3 \subset \partial K$. Finally, if $\alpha_2^\infty = \infty$ and $0 \leqq \alpha_4^\infty \leqq \infty$, then $y_1^l \to \infty$, since the only effect of $\alpha_2^l \to \infty$ is to concentrate the measure on the set $x_i = x_j$ (all i, j).

Case 2. $0<\alpha_1^\infty=\delta_1^\infty<\infty$.

If $\alpha_2^\infty=0$ and $|\alpha_3^\infty|<\infty$, then y^l converges to a point on $\mathcal{M}_1$. If $\alpha_2^\infty=0$, and $\alpha_3^\infty=-\infty$ or $+\infty$, then the y_1^l go to 0 or ∞ respectively. If $0<\alpha_2^\infty<\infty$, then there are only the possibilities $\alpha_3^\infty=-\infty$ or $+\infty$ and again the y_1^l tend to 0 or $+\infty$ respectively. Finally, let $\alpha_2^\infty=+\infty$. We consider subcases depending on the value of α_4^∞. First we note that for fixed α_1 and α_2, y_1 (and y_2) is a monotonic function of $\alpha_4=\alpha_3+d\alpha_2$ due to Griffiths second inequality (see e.g. [6]). Hence if $|\alpha_4^\infty|<\infty$, the sequences y^l converges to a point on $\mathcal{M}_3\subset\partial K$. If $\alpha_4^\infty=-\infty$, the y^l converge to $0\in\partial K$ and if $\alpha_4^\infty=+\infty$, then the y_1^l (and y_2^l) tend to ∞.

Case 3. $\alpha_1^\infty=\delta_1^\infty=\infty$.

We work with the δ parameters. We first assume that δ_2^∞ and δ_3^∞ are finite. Then the y^l converge to a point on $\overline{\mathcal{M}}_4\in\partial K$. If $\delta_2^\infty=\infty$ and δ_3^∞ is finite, then the y^l converge to a point in $\overline{\mathcal{M}}_4\cap\overline{\mathcal{M}}_3\subset\partial K$. Similarly, if δ_2^∞ is finite and $\delta_3^\infty=-\infty$ or $+\infty$, then $y_1^l\to 0$ or $+\infty$ respectively. Finally, if $\delta_2^\infty=+\infty$ and $\delta_3^\infty=-\infty$ or $+\infty$, then the y^l tend to $0\in\partial K$ or the y_1^l tend to ∞ respectively, since the only effect of large δ_2^l is to localize the measure on the set $x_i=x_j$ (all i,j).

By the preceding arguments this concludes the proof of Lemma 1, and we turn to a

Proof of Lemma 2. Assume that T is not proper. Then there is a compact set K_0 in $K=\operatorname{Im}T$, such that $T^{-1}(K_0)$ is not compact in $\mathbb{R}^+\times\mathbb{R}^+\times\mathbb{R}$. By the Heine-Borel theorem, there is a sequence α^l which does not converge in $(\mathbb{R}^+)^2\times\mathbb{R}$ but for which $y^l \underset{\text{def}}{=} T(\alpha^l)$ is convergent to a point y^∞ in $\operatorname{Im}T$. By going to a subsequence, we may assume that the α_i^l $(i=1,2,3,4)$ and the δ_i^l $(i=1,2,3)$ are convergent (allowing for the values 0 or $\pm\infty$). We may now repeat the arguments used in the proof of Lemma 1 to show that the y^l converge either to a point in ∂K or tend to infinity, contradicting the fact that $y^\infty\in\operatorname{Im}T=K$ is the limit. Hence T is proper. In particular the preimage of a point is a finite set. The last claim of the lemma then follows easily (see e.g. [3]) using the conjectured local injectivity.

Acknowledgements. It is a pleasure to thank A. Jaffe and K. Osterwalder for their hospitality at Harvard University, and R. Bott, R. Hermann, and E. Seiler for helpful discussions.

Note Added in Proof

1. Computer calculations carried out by D. Marcheson (N.Y.U) indicate that Conjecture 2 of Ref. [4] is valid for $d=1$.
2. $C(d)$ given in Appendix B of Ref. [4] is universal if and only if the two-scale-factor universality hypothesis of D. Staufer, M. Ferer and M. Wortis [Phys. Rev. Letters **29**, 345 (1975)] holds.
3. There are nontrivial ϕ_3^4 theories obtainable through lattice approximations [J. Feldman and K. Osterwalder, Ann. Phys. **97**, 80 (1976); J. Magnen and R. Seneor, to appear in Ann. Inst. H. Poinc.]. Hence if Conjectures 1 and 2 of Ref. [4] are valid, we must have $C(d=3)>0$. In particular this gives the scaling law $dv=2\Delta_4-\gamma$ for $d=3$ [R. Schrader, Phys. Rev. B **14**, 172—173 (1976)].
4. Without using Conjecture 1 and 2 of Ref. [4] it may be shown directly that case (i) in the proof of Lemma 1 may not occur (R. Schrader, New correlation inequalities for Ising Models and $P(\phi)$ Theories, Berlin preprint).
5. $\mathcal{M}_4$ lies schlicht over $\mathcal{M}_2$ if the correlation length of the Ising model on $\mathcal{T}$ (defined by the second moment of the spin-spin correlation) is strictly nonincreasing in J for all J.

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Communicated by A. Jaffe

Received May 13, 1976