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Correlation Inequalities for Multicomponent Rotators

François Dunlop

Institut des Hautes Etudes Scientifiques, F-91440 Bures-sur-Yvette, France

Abstract. A recent approach to G.H.S. and Lebowitz inequalities is used to prove Griffiths' second inequality for 3 and 4 component models (e.g. Classical Heisenberg model, $|\varphi|^4$ Euclidean fields). Applications include monotonicity of the mass gap in the external field, and two-sided inequalities between "parallel" and "transverse" correlations.

1. Introduction

The inequalities proven by Ginibre for ferromagnetic plane rotators are very powerful [1, 2]. Their proof, however, depends essentially on the commutative structure of the circle group, through the use of characters. Elementary trigonometry and a very special symmetry were also used in [1], but we show in Section 4 that these can be avoided if one is only interested in positive correlations of vectors, e.g.

$$\langle \boldsymbol{S}_i \cdot \boldsymbol{S}_j \rangle \geqq \langle \boldsymbol{S}_i \rangle \cdot \langle \boldsymbol{S}_j \rangle \,. \tag{1.1}$$

Namely, for D-dimensional classical ferromagnets, we reduce (1.1) to a *first* Griffiths inequality for a similar system with interactions of the form

$$(\hat{x}, U(i)^{-1} U(j)\hat{x}) + (\hat{y}, U(i)^{-1} U(j)\hat{y}) \tag{1.2}$$

where $\hat{x}$ and $\hat{y}$ are fixed orthogonal unit vectors in $\mathbb{R}^D$ and $U(j)$, for each site j, is to be integrated over the rotation group acting in $\mathbb{R}^D$ with the Haar measure. In the commutative case ($D=2$), the functions (1.2) are positive definite on the product group (over the sites), which is more than what we need. As a by-product we prove that the parallel mass gap is larger than the transverse mass gap (see 1.3).

We do not know how to deal with non commutative rotations. This article is mainly devoted to 3 and 4 component models, where each spin may be considered as a plane rotator plus an Ising (continuous) spin, or as two plane rotators, negatively correlated. Of course we'll never have in this way more than two components on the same footing in the correlations, and therefore no scalar product, but all the inequalities known for plane rotators will be generalized to any two among the three or four components.

The necessary tools come from the clarification work of Mehta, Schrader [3], and Ellis, Monroe, Newman [4], around G.H.S. and Lebowitz inequalities. They are described in Section 2 together with our general theorem.

Section 3 contains the applications. In particular, if $S_j^{\parallel}$ and $S_j^{\perp}$ are components of the j'th spin respectively parallel and perpendicular to the external field, then

$$\langle S_j^{\perp} S_k^{\perp} \rangle \geqq \langle S_j^{\parallel} S_k^{\parallel} \rangle - \langle S_j^{\parallel} \rangle \langle S_k^{\parallel} \rangle , \tag{1.3}$$

$$\langle S_j^{\perp} S_k^{\perp} \rangle^2 \leqq \langle S_j^{\parallel} S_k^{\parallel} \rangle^2 - \langle S_j^{\parallel} \rangle^2 \langle S_k^{\parallel} \rangle^2 \tag{1.4}$$

$$\langle S_j^{\perp} S_k^{\perp} S_l^{\parallel} \rangle - \langle S_j^{\perp} S_k^{\perp} \rangle \langle S_l^{\parallel} \rangle \leqq 0 \tag{1.5}$$

(1.3) is new also for plane rotators, and is proven again for that case in Section 4 as an application of the formalism outlined at the beginning of this introduction.

Positive correlations between *scalar products*, such as

$$\langle (\boldsymbol{S}_i \cdot \boldsymbol{S}_j)(\boldsymbol{S}_k \cdot \boldsymbol{S}_l) \rangle \geqq \langle \boldsymbol{S}_i \cdot \boldsymbol{S}_j \rangle \langle \boldsymbol{S}_k \cdot \boldsymbol{S}_l \rangle \tag{1.6}$$

seem to be a difficult and open problem for $D>2$ (except for Gaussian spins: see the end of Section 4). Fortunately these inequalities can now be bypassed in proving the infinite volume limit for general couplings, provided the Lee-Yang theorem and a large external field expansion hold [5]. Therefore our inequalities apply to the infinite volume Classical Heisenberg and 3-component $(|\boldsymbol{\varphi}|^4)_2$ and $(|\boldsymbol{\varphi}|^4)_3$ theories, in the presence of an external field.

Finally we remark that (1.3)–(1.5) for general D would be a consequence of (1.6) type inequalities for $[D/2]$ and $[(D+1)/2]$ components.

2. The General Result

$\{\boldsymbol{S}_j = (S_j^1 \dots S_j^4) : j = 1, \dots, N\}$ will be a family of random vectors in $\mathbb{R}^4$. In order to combine the newly discovered [6, 7], and clarified [3, 4], negative correlations with plane rotators, we shall use the following variables:

$$\begin{aligned} S_j^1 &= \varrho_j \cos\varphi_j \qquad S_j^3 = \tau_j \cos\psi_j \\ &\qquad\qquad\qquad\qquad j = 1, \dots, N \\ S_j^2 &= \varrho_j \sin\varphi_j \qquad S_j^4 = \tau_j \sin\psi_j \end{aligned} \tag{2.1}$$

with

$$\begin{aligned} &\varrho_j \in [0, \infty) \\ &\varphi_j \in [0, 2\pi) \quad \text{or} \quad \varphi_j \in \{2\pi k/p_j : k = 0, \dots, p_j - 1\} \\ &\tau_j \in [0, \infty) \\ &\psi_j \in [0, 2\pi) \quad \text{or} \quad \psi_j \in \{2\pi l/q_j : l = 0, \dots, q_j - 1\}. \end{aligned} \tag{2.2}$$

The measure at each site will be of the form

$$d\mu_j(\boldsymbol{S}_j) = d\nu_j(\varrho_j, \tau_j) d\varphi_j d\psi_j \tag{2.3}$$

where $d\varphi_j$, $d\psi_j$ are the invariant measures on the jth configuration groups (circle or cyclic group Z_{p_j}, Z_{q_j}), and $d\nu_j$ is a finite positive measure on $[0,\infty)\times[0,\infty)$, satisfying the

Hypothesis of negative correlation[1]:

$$\int \prod_{\substack{p\in P\\ q\in Q}} (f_p(\varrho)\pm f_p(\varrho'))\,(f_q(\tau')\pm f_q(\tau))d\nu_j(\varrho,\tau)d\nu_j(\varrho',\tau')\geqq 0 \tag{2.4}$$

for any finite family $\{f_n : n\in P\cup Q\}$ of positive increasing bounded functions on $[0,\infty)$ and for any sequence of plus or minus signs.

Example 1. Percus system

$$\begin{aligned} &\varphi_j\in\{0,\pi\}\\ &\psi_j\in\{0,\pi\}\\ &d\nu_j(\varrho,\tau)=(\delta(\varrho)\delta(\tau-1)+\delta(\varrho-1)\delta(\tau))d\varrho d\tau\,. \end{aligned} \tag{2.5}$$

This is a duplicated Ising spin, introduced by Percus [8] to extend some second Griffiths inequalities to arbitrary signs of the magnetic field.

Example 2. Plane Rotator

$$\begin{aligned} &\varphi_j\in\{0,\pi\}\\ &\psi_j\in\{0,\pi\}\\ &d\nu_j(\varrho,\tau)=\delta(\varrho^2+\tau^2-1)d\varrho d\tau\,. \end{aligned} \tag{2.6}$$

Example 3. Classical Heisenberg spin

$$\begin{aligned} &\varphi_j\in[0,2\pi)\\ &\psi_j\in\{0,\pi\}\\ &d\nu_j(\varrho,\tau)=\delta(\varrho^2+\tau^2-1)d\varrho^2 d\tau\,. \end{aligned} \tag{2.7}$$

Example 4. $|\boldsymbol{\Phi}|^4$ lattice euclidean field

$$d\nu_j(\varrho,\tau)=\exp[-a(\varrho^2+\tau^2)^2+b(\varrho^2+\tau^2)]\varrho^\alpha d\varrho\tau^\alpha d\tau \tag{2.8}$$

where $a>0$, and α, β depend on the choice of φ, ψ groups.

To prove (2.4) for this last case, we write

$$\begin{aligned} \exp-2a(\varrho^2\tau^2+\varrho'^2\tau'^2)&=\exp-a(\varrho^2+\varrho'^2)(\tau^2+\tau'^2)\\ &\quad\cdot\exp a(\varrho^2-\varrho'^2)(\tau'^2-\tau^2) \end{aligned} \tag{2.9}$$

and expand the second factor. The rest of the measure is completely symmetric under $\varrho\leftrightarrow\varrho'$ and $\tau\leftrightarrow\tau'$.

Example 5. Duplicated 1-component φ^4 field.

$$\exp(-\varphi^4-\varphi'^4)=\exp[-(\varrho^4+\tau^4)/8-6\varrho^2\tau^2/8] \tag{2.10}$$

where

$$\begin{aligned} &\varrho=|\varphi+\varphi'|\\ &\tau=|\varphi-\varphi'|\,. \end{aligned}$$

[1] This hypothesis has now been fully investigated by Ellis and Newman

Next we specify the interaction between sites: it should be ferromagnetic in the system of φ-rotators or ψ-rotators, but not mix the two. More precisely let Q_1 be the multiplicative positive cone generated by the real parts of the characters on the product of φ-configuration groups, together with the positive increasing functions of one variable $\varrho_j : j=1, \ldots, N$. Define Q_2 similarly with respect to (τ, ψ). Then the interaction should be

$$-(h_1+h_2) \in -(Q_1+Q_2).$$

Rather than impose bounds on our functions, we decide to perform integrals first over angles, possibly using uniform convergence (on a compact) and then over the modules, possibly using monotone convergence:

Lemma. *Let $h_1 \in Q_1$, $h_2 \in Q_2$ and $\{d\mu_j : j=1, \ldots, N\}$ be as* (2.3), (2.4) *above and such that*

$$Z = \int \exp(h_1+h_2) \prod_{j=1}^{N} d\mu_j(S_j) < \infty. \tag{2.11}$$

Then

$$\langle f \rangle = Z^{-1} \int \prod_{j=1}^{N} d\nu_j(\varrho_j, \tau_j) \int \prod_{j=1}^{N} d\varphi_j d\psi_j f \exp(h_1+h_2) \tag{2.12}$$

is well defined for $f \in Q_1 \cdot Q_2$ and satisfies the first Griffiths inequality

$$0 \leqq \langle f \rangle \leqq \infty. \tag{2.13}$$

Proof. The series expansion of the exponential converges uniformly in φ, ψ. The angle integral is then a series in positive functions of ϱ, τ, because characters have positive integrals. Therefore term by term integration in ϱ, τ is allowed by the monotone convergence theorem.

Theorem 1. *Let $\{S_j : j=1, \ldots, N\}$ be a family of random vectors in $\mathbb{R}^4$ with joint probability distribution*

$$\exp(h_1+h_2) \prod_{j=1}^{N} d\mu_j(S_j)$$

as in the lemma. Then

$$\forall f_1, g_1 \in Q_1; \quad \forall f_2, g_2 \in Q_2$$

$$\langle f_1 g_1 \rangle \geqq \langle f_1 \rangle \langle g_1 \rangle \tag{2.14}$$

$$\langle f_2 g_2 \rangle \geqq \langle f_2 \rangle \langle g_2 \rangle,$$

$$\langle f_1 g_2 \rangle \leqq \langle f_1 \rangle \langle g_2 \rangle. \tag{2.15}$$

Proof. We introduce an identically distributed system, denote it by primes, and recall the identities

$$\begin{aligned} xy + x'y' &= \tfrac{1}{2}[(x+x')(y+y') + (x-x')(y-y')] \\ &= \tfrac{1}{2}[(x+x')(y+y') + (x'-x)(y'-y)], \end{aligned} \tag{2.16}$$

$$xy - x'y' = \tfrac{1}{2}[(x+x')(y-y') + (x-x')(y+y')]. \tag{2.17}$$

The arbitrariness in the factorization (2.16) is raised by looking at the inequalities to be proven:

$$\begin{aligned}&\langle (f_1-f_1')(g_1-g_1')\rangle \geqq 0\\&\langle (f_2'-f_2)(g_2'-g_2)\rangle \geqq 0\\&\langle (f_1-f_1')(g_2'-g_2)\rangle \geqq 0\end{aligned}\tag{2.18}$$

and at the hypothesis of negative correlation (2.4): We are led to make different choices in (2.16) for the functions from Q_1 and for those from Q_2.

By iteration, h_1+h_1' will be written as a finite multinomial, with positive coefficients, in sums and differences:

$\chi(\{\varphi\})\pm\chi(\{\varphi'\})$: χ the real part of a character.

$\alpha(\varrho_j)\pm\alpha(\varrho_j')$: $j=1, \ldots, N$. α positive increasing.

Similarly h_2+h_2' will give rise to

$\varkappa(\{\psi'\})\pm\varkappa(\{\psi\})$: $\varkappa$ the real part of a character,

$\beta(\tau_j')\pm\beta(\tau_j)$: $j=1, \ldots, N$. β positive increasing.

Applying the same procedure to f_i and g_i, and expanding the exponential, we obtain a series in such multinomials, converging uniformly in the angles (φ, ψ) and pointwise in the modules (ϱ, τ).

The angle integrals are positive, as shown by Ginibre ([1] or Section 4). Every term in the resulting series then has a positive ϱ, τ integral, by the hypothesis (2.4). If the bigger side of the inequality is infinite, there is nothing to prove (by convention $+\infty\geqq+\infty$). If not, it's angle average is an absolutely integrable series which can be subtracted to the other side. Indeed a positive series plus (or minus) an absolutely integrable series is term by term integrable in any order.

Remark. For a two-component system, or duplicated one component system, Theorem 1 and it's proof are essentially contained in [3, 4][2]. Special cases were first proven by Lebowitz [6], using F.K.G. inequalities, and by Monroe [7], using Gaussian integrals.

3. Applications

We leave aside the problem of possible limits as a lattice spacing goes to zero or a volume to infinity. Statements about mass gaps, for example, are meant to apply to theories where corresponding unique limits have been or will be controlled in some way.

For the applications related to examples 1 and 2 of the preceding section, we refer to the original papers [6, 7, 3, 4], and now turn to examples 3 and 4. As all the results are already contained in Theorem 1, we shall only describe a classical Heisenberg model, in the presence of an external field. Results can easily be translated for 2 or 4 components, or for a lattice field theory.

[2] A different proof has been given by Sylvester in the duplicated one component case, and by Bricmont in the two component case

Theorem 2. *Let* $\{S_j=(S_j^1, S_j^2, S_j^3):j=1, \ldots, N\}$ *be a family of random unit vectors in* $\mathbb{R}^3$ *with joint probability distribution*

$$Z^{-1}\exp\left[\sum_{j=1}^{N} a_j\cdot S_j+\sum_{i,j=1}^{N} J_{ij}S_i\cdot S_j\right]\prod_{j=1}^{N} d\omega_j$$

where $d\omega_j$ *is the invariant measure on the sphere and* Z *is the partition function.*

Assume

$$a_j=(0,0,a_j),\ a_j\geqq 0\ \forall_j$$

$$J_{ij}\geqq 0\ \forall_{i,j}.$$

Then

$$\left\langle\prod_{i\in I} S_i^a\prod_{j\in J} S_j^a\right\rangle \geqq \left\langle\prod_{i\in I} S_i^a\right\rangle\left\langle\prod_{j\in J} S_j^a\right\rangle \qquad a=1,2,3 \tag{3.1}$$

$$\left\langle\prod_{i\in I} S_i^a\prod_{j\in J} S_j^3\right\rangle \leqq \left\langle\prod_{i\in I} S_i^a\right\rangle\left\langle\prod_{j\in J} S_j^3\right\rangle \qquad a=1,2 \tag{3.2}$$

$$\langle S_i^aS_j^a\rangle\geqq\langle S_i^3S_j^3\rangle-\langle S_i^3\rangle\langle S_j^3\rangle \qquad a=1,2 \tag{3.3}$$

$$\langle S_i^aS_j^a\rangle^2\leqq\langle S_i^3S_j^3\rangle^2-\langle S_i^3\rangle^2\langle S_j^3\rangle^2 \qquad a=1,2 \tag{3.4}$$

Proof. (3.1) and (3.2) are straightforward applications of Theorems 1, with different labellings of the components.

(3.3) can be written

$$\langle(S_i^3-S_i^2)(S_j^3+S_j^2)\rangle\leqq\langle S_i^3-S_i^2\rangle\langle S_j^3+S_j^2\rangle \tag{3.5}$$

which is nothing but (3.2) applied to

$$\{S_j=(S_j^1, S_j^3+S_j^2, S_j^3-S_j^2):j=1,\ldots,N\}.$$

Notice that the combination of (3.2) and (3.3) shows that the mass gap increases with the external field.

Finally the proof of (3.4), given Theorem 1, is the same as for plane rotators [2]. Again it can be used to show that the susceptibility is infinite in case of spontaneous magnetization [9].

The next theorem will generalize this last inequality to higher order correlation functions.

Theorem 3. *Let* $\{S_j:j=1,\ldots,N\}$ *be as in Theorem 2, and* $S_I^1\equiv\prod_{i\in I} S_i^1$, *etc. Then*

(i)

$$|\langle S_I^aS_J^bS_K^c\rangle-\langle S_I^a\rangle\langle S_J^bS_K^c\rangle|\leqq 2(\langle S_I^3S_J^3S_K^3\rangle-\langle S_I^3\rangle\langle S_J^3S_K^3\rangle)\ \forall a,b,c, \tag{3.6}$$

(ii) *if the index set J is even*

$$|\langle S_I^3S_J^aS_K^3S_L^a\rangle-\langle S_I^3S_J^a\rangle\langle S_K^3S_L^a\rangle|$$

$$\leqq 3(\langle S_I^3S_J^3S_K^3S_L^3\rangle-\langle S_I^3S_J^3\rangle\langle S_K^3S_L^3\rangle)\ \forall a, \tag{3.7}$$

(iii) *if the index set J is odd*

$$\langle S_I^3S_J^aS_K^3S_L^a\rangle^2\leqq 3(\langle S_I^3S_J^3S_K^3S_L^3\rangle^2-\langle S_I^3S_J^3\rangle^2\langle S_K^3S_L^3\rangle^2) \qquad a=1,2. \tag{3.8}$$

The analogous result for $D=2$ shows that if the component in the external field obeys clustering, then all correlations do. When $D>2$, we don't reach all the correlations because our method can deal with at most two components on the same footing.

Proof (i). We take inequalities from Theorems 1 or 2 and reexpress them in the next line:

$$\begin{aligned} &1)\quad \langle S_I^a(S_J^a+S_J^b)\rangle \geqq \langle S_I^a\rangle \langle S_J^a+S_J^b\rangle \\ &\quad\ \langle S_I^a S_J^a\rangle - \langle S_I^a\rangle\langle S_J^a\rangle \geqq \langle S_I^a\rangle\langle S_J^b\rangle - \langle S_I^a S_J^b\rangle \geqq 0 \\ &\quad\ a \neq b\,, \qquad b=1,2\,, \end{aligned} \tag{3.9}$$

$$\begin{aligned} &2)\quad \langle (S_I^3+S_I^a)(S_J^3-S_J^a)\rangle \geqq \langle S_I^3+S_I^a\rangle\langle S_J^3-S_J^a\rangle \\ &\quad\ \langle S_I^3 S_J^3\rangle - \langle S_I^3\rangle\langle S_J^3\rangle + \langle S_I^3\rangle\langle S_J^a\rangle - \langle S_I^3 S_J^a\rangle \\ &\quad\ \geqq \langle S_I^a S_J^a\rangle - \langle S_I^a\rangle\langle S_J^a\rangle \qquad a=1,2\,. \end{aligned} \tag{3.10}$$

If further I is a translate of J in a translation invariant theory, then

$$\langle S_I^3 S_J^3\rangle - \langle S_I^3\rangle\langle S_J^3\rangle \geqq \langle S_I^a S_J^a\rangle - \langle S_I^a\rangle\langle S_J^a\rangle\,, \tag{3.11}$$

$$\begin{aligned} &3)\quad \langle S_I^a S_J^b(S_K^b \pm S_K^c)\rangle \leqq \langle S_I^a\rangle\langle S_J^b(S_K^b\pm S_K^c)\rangle \\ &\quad\ \langle S_I^a\rangle\langle S_J^b S_K^b\rangle - \langle S_I^a S_J^b S_K^b\rangle \geqq \pm(\langle S_I^a S_J^b S_K^c\rangle - \langle S_I^a\rangle\langle S_J^b S_K^c\rangle) \\ &\quad\ a \neq b \neq c\,, \qquad c=1,2\,. \end{aligned} \tag{3.12}$$

Proof (ii).

$$\begin{aligned} &\langle S_I^3(S_J^3\pm S_J^a)S_K^3\rangle \geqq \langle S_I^3(S_J^3\pm S_J^a)\rangle\langle S_K^3\rangle \\ &\langle S_I^3 S_J^3 S_K^3\rangle - \langle S_I^3 S_J^3\rangle\langle S_K^3\rangle \geqq \pm(\langle S_I^3 S_J^a S_K^3\rangle - \langle S_I^3 S_j^a\rangle\langle S_K^3\rangle)\,. \end{aligned} \tag{3.13}$$

When the index set L is non empty, we apply the same procedure to it and use the result (3.13).

Proof (iii). When J is odd, $S_J^3 \pm S_J^a$ is not expressible in terms of characters. We introduce an independent copy and apply (ii) to $S_J^3 S_J'^3 \pm S_J^a S_J'^a$.

4. A Look at General Multicomponent Rotators

We shall now describe what may be a first step towards Griffiths' inequalities for classical ferromagnets with any number of components. The method also sheds some light on plane rotators, and has given our first proof of (1.3) for that case.

Unit rotators and $|\varphi|^4$ rotators will be considered separately, although each model is a limit of the other [2].

Let $\{\boldsymbol{S}_j\cdot : j=1, \ldots, N\}$ be a family of random unit vectors in $\mathbb{R}^D (D \geqq 2)$ with joint probability distribution

$$Z^{-1} \exp\left(\sum_{j=1}^{N} \boldsymbol{a}_j\cdot\boldsymbol{S}_j + \sum_{i,j=1}^{N} J_{ij}\boldsymbol{S}_i\cdot\boldsymbol{S}_j\right) \prod_{j=1}^{N} d\omega_j \tag{4.1}$$

where Z is the normalization factor and $d\omega_j$ is the invariant measure on the unit sphere in $\mathbb{R}^D$. We duplicate the system, and define the Percus variables

$$\begin{aligned} \boldsymbol{t}_j &= \boldsymbol{S}_j + \boldsymbol{S}'_j \\ \boldsymbol{q}_j &= \boldsymbol{S}_j - \boldsymbol{S}'_j \end{aligned} \qquad j=1, \ldots, N. \tag{4.2}$$

They satisfy

$$\begin{aligned} \boldsymbol{t}_j^2 + \boldsymbol{q}_j^2 &= 2 \\ \boldsymbol{t}_j \cdot \boldsymbol{q}_j &= 0 \end{aligned} \tag{4.3}$$

or

$$\begin{aligned} \boldsymbol{t}_j &= \cos\alpha_j U(j)\hat{\boldsymbol{x}} \\ \boldsymbol{q}_j &= \sin\alpha_j U(j)\hat{\boldsymbol{y}} \end{aligned} \tag{4.4}$$

where $\alpha_j \in [0, 2\pi)$, $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{y}}$ are fixed orthogonal vectors, and $U(j)$ is a rotation in $\mathbb{R}^D$. We shall integrate at each site over the product of a circle group and a rotation group in $\mathbb{R}^D$ with the Haar measure. Indeed the original measure on the spheres were rotation invariant, and α_j is half the angle between $\boldsymbol{S}_j$ and $\boldsymbol{S}'_j$ (modulo π). Of course this description is redundant: the configuration space at each site is covered twice when $D=2$, four times when $D=3$, infinitely many times when $D>3$.

The joint probability distribution of the duplicated system can then be written

$$\begin{aligned} Z^{-1}Z'^{-1} \exp\Bigg\{ \sum_{j=1}^{N} \cos\alpha_j(\boldsymbol{a}_j, U(j)\hat{\boldsymbol{x}}) & \\ + \tfrac{1}{2} \sum_{i,j=1}^{N} J_{ij}[\cos\alpha_i \cos\alpha_j(\hat{\boldsymbol{x}}, U(i)^{-1}U(j)\hat{\boldsymbol{x}}) & \\ + \sin\alpha_i \sin\alpha_j(\hat{\boldsymbol{y}}, U(i)^{-1}U(j)\hat{\boldsymbol{y}})]\Bigg\} \prod_{j=1}^{N} d\alpha_j d\Omega_j & \end{aligned} \tag{4.5}$$

where $d\Omega_j$ is the Haar measure on the rotation group acting in $\mathbb{R}^D$.

Assuming

$$\begin{aligned} &\boldsymbol{a}_j = a_j\hat{\boldsymbol{x}}, \qquad a_j \geqq 0 \ \forall j \\ &J_{ij} \geqq 0 \ \forall i, j \end{aligned} \tag{4.6}$$

the simplest correlation to investigate is

$$\langle \boldsymbol{S}_i \cdot \boldsymbol{S}_j \rangle - \langle \boldsymbol{S}_i \rangle \langle \boldsymbol{S}_j \rangle = \tfrac{1}{2} \langle \sin\alpha_i \sin\alpha_j(\hat{\boldsymbol{y}}, U(i)^{-1}U(j)\hat{\boldsymbol{y}}) \rangle. \tag{4.7}$$

We conjecture that every term in the expansion of (4.5) in (4.7) gives a positive contribution to the average.

The commutative case belongs to example 2 of Ginibre [1]: only then $U(i)^{-1}U(j)$ is a representation of the product group (product over the sites) and therefore $(\xi, U(i)^{-1}U(j)\xi)$ is a positive definite function for any fixed vector ξ. This property is stable under multiplication and implies that the integral of the function is positive.

Thus the above conjecture holds when $D=2$, and more generally, we have the

Theorem 4. *Let* $\{S_j=(S_j^1, S_j^2): j=1, \ldots, N\}$ *be a system of plane rotators with joint probability distribution* (4.1). *Then, introducing a duplicate system*

$$\Big\langle \prod_{i\in I}(S_i^1+S_i'^1) \prod_{(i,j)\in K}\{(S_i^1+S_i'^1)(S_j^1+S_j'^1)\pm(S_i^2+S_i'^2)(S_j^2+S_j'^2)\}$$
$$\cdot \prod_{(i,j)\in L}\{(S_i^2-S_i'^2)(S_j^2-S_j'^2)\pm(S_i^1-S_i'^1)(S_j^1-S_j'^1)\}\Big\rangle \geqq 0$$

for any index sets I, K, L and any sequence of plus or minus signs. In particular

$$\langle S_i^2 S_j^2\rangle \geqq \langle S_i^1 S_j^1\rangle - \langle S_i^1\rangle \langle S_j^1\rangle \ \forall i,j\,.$$

To recover Ginibre's inequalities, one must then use the symmetry between α_j and $\Omega_j \equiv \beta_j$. Indeed (4.5) becomes

$$Z^{-1}Z'^{-1}\exp\left\{\sum_{j=1}^{N} a_j\cos\alpha_j\cos\beta_j + \tfrac{1}{2}\sum_{i,j=1}^{N} J_{ij}\cos(\alpha_i-\alpha_j)\cos(\beta_i-\beta_j)\right\}\prod_{j=1}^{N} d\alpha_j d\beta_j$$

so that

$$\langle f(\{\alpha\})f(\{\beta\})\rangle \geqq 0 \ \forall f\,.$$

We now turn to $|\boldsymbol{\varphi}|^4$ rotators, for which the individual measures $d\omega_j$ in (4.1) are replaced by

$$\exp(-|\boldsymbol{S}|^4)\, d^D\boldsymbol{S}\,.$$

The shape of the potential will really depend on the sign of J_{ii}, irrelevant here. As for the Percus variables, now unrestricted, we set

$$\begin{aligned} \boldsymbol{t}_j &= t_j U(j)\hat{\boldsymbol{x}} \\ \boldsymbol{q}_j &= q_j U(j) U_0(\psi_j)\hat{\boldsymbol{y}} \end{aligned} \tag{4.8}$$

where t_j and q_j are positive, $U(j)$ is a general rotation in $\mathbb{R}^D$, $U_0(\psi_j)$ is a rotation in the $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}})$ plane, and $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{y}}$ will again be chosen orthogonal, because $\boldsymbol{t}_j$ and $\boldsymbol{q}_j$ tend to be that way:

$$\begin{aligned} \exp(-|\boldsymbol{S}_j|^4 - |\boldsymbol{S}_j'|^4 &= \exp(-(|\boldsymbol{t}_j|^2+|\boldsymbol{q}_j|^2 - \tfrac{1}{2}(\boldsymbol{t}_j\cdot\boldsymbol{q}_j)^2)/8 \\ &= \exp[-(t_j^4+6t_j^2\cdot q_j^2+q_j^4)/8 + \tfrac{1}{2}t_j^2q_j^2\cos^2\psi_j]\,. \end{aligned} \tag{4.9}$$

The duplicated joint probability distribution is now

$$\begin{aligned} Z^{-1}Z'^{-1}\exp\Bigg\{ & \sum_{j=1}^{N} a_j t_j(\hat{\boldsymbol{x}}, U(j)\hat{\boldsymbol{x}}) + \sum_{j=1}^{N}\tfrac{1}{2}t_j^2q_j^2(\hat{\boldsymbol{x}}, U_0(\psi_j)\hat{\boldsymbol{x}})^2 \\ & + \tfrac{1}{2}\sum_{i,j=1}^{N} J_{ij}[t_it_j(\hat{\boldsymbol{x}}, U(i)^{-1}U(j)\hat{\boldsymbol{x}}) \\ & + q_iq_j(\hat{\boldsymbol{y}}, U_0(\psi_i)^{-1}U(i)^{-1}U(j)U_0(\psi_j)\hat{\boldsymbol{y}})]\Bigg\} \\ & \cdot \prod_{j=1}^{N} d\psi_j d\Omega_j dv_j(t_j, q_j) \end{aligned} \tag{4.10}$$

where dv_j is a positive measure on $[0, \infty)$.

At this point we could make the same conjecture as for unit rotators, and prove it as well as Theorem 4 in the commutative case. We simply remark that all this becomes trivial in the case of Gaussian spins, where the Percus variables $\boldsymbol{t}_j$ and $\boldsymbol{q}_j$ are independent.

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Note. After sending the present article for publication, we received a reprint of an announcement [10] by Kunz, Pfister, Vuillermot, onto which the referee also kindly draws our attention. The announced theorem is a version of our Theorem 1 in the compact case, and corollaries include (3.1) and (3.2) of our Theorem 2. The proof must be different, as the authors appeal to F.K.G.-Preston inequalities (note however that their Lemma 3, an F.K.G. condition on partial normalization factors, can be proved by our methods without F.K.G. conditions on the couplings $J(A)$).

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