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Generalized K-Flows

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Abstract. The classical concept of K-flow is generalized to cover situations encountered in non-equilibrium quantum statistical mechanics. The ergodic properties of generalized K-flows are discussed. Several non-isomorphic examples are constructed, which differ already in the type (II_1, III_λ, and III_1) of the factor on which they are defined. In particular, generalized factor K-flows with dynamical entropy either zero (singular K-flows) or infinite (special non-abelian K-flows) are constructed.

Introduction

The *motivation* for this paper stems from the following schematic description of the *purpose* of non-equilibrium statistical mechanics.

Given a *dissipative, thermodynamical* system $\{\mathfrak{N}_S, \phi_S, \gamma(\mathbb{R}^+)\}$, devise: (i) a thermal bath $\{\mathfrak{N}_R, \phi_R\}$, and (ii) an interaction between $\mathfrak{N}_S$ and $\mathfrak{N}_R$, in such a manner that the following conditions be satisfied. Firstly, the composite dynamical system $\{\bar{\mathfrak{N}} = \mathfrak{N}_S \otimes \mathfrak{N}_R, \bar{\phi} = \phi_S \otimes \phi_R, \bar{\alpha}(\mathbb{R})\}$ should be *conservative*, and understandable from the laws of hamiltonian mechanics. Secondly, $\gamma(\mathbb{R}^+)$ should appear as the *restriction*, to the system $\mathfrak{N}_S$ of interest, of the total evolution $\bar{\alpha}(\mathbb{R})$; namely, for every (normal) state ψ on $\mathfrak{N}_S$, every observable N in $\mathfrak{N}_S$, and all positive times t, one should have:

$$\langle \psi \otimes \phi_R ; \bar{\alpha}(t)[N \otimes I] \rangle = \langle \psi ; \gamma(t)[N] \rangle . \tag{1}$$

To be specific, we shall assume that ϕ_S and ϕ_R are thermal equilibrium states, respectively for the von Neumann algebras $\mathfrak{N}_S$ and $\mathfrak{N}_R$. In line with the ideas of non-equilibrium thermodynamics, we shall further assume that $\gamma(\mathbb{R}^+)$ is a semi-group of positive, linear maps of $\mathfrak{N}_S$ into itself such that $\phi_S \circ \gamma(t) = \phi_S$ for every $t \in \mathbb{R}^+$, and that $\langle \psi ; \gamma(t)[N] \rangle$ approaches $\langle \phi_S ; N \rangle$ when t tends to $+\infty$, for every

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(normal) state ψ on $\mathfrak{N}_S$, and every observable N in $\mathfrak{N}_S$; we also assume $\gamma(t)[I]=I$ for all $t\in\mathbb{R}^+$ and $\gamma(0)=\mathrm{id}$; in particular we note that the adjoint $\gamma(t)^*$ transforms states into states. Finally, in line with the ideas of *non-equilibrium statistical mechanics*, we shall require that $\bar{\alpha}(\mathbb{R})$ is a group of automorphisms of $\bar{\mathfrak{N}}$ with $\bar{\phi}\circ\bar{\alpha}(t)=\bar{\phi}$ for every $t\in\mathbb{R}$. We admit that several limiting procedures might be necessary to pass from ordinary hamiltonian mechanics to the automorphism group $\bar{\alpha}(\mathbb{R})$. Amongst these we are quite willing to accept the "thermodynamical" and the "long-time, weak-coupling" limits, provided that these limits be mathematically controlled.

In agreement with [26] we will, throughout this paper, mean by "thermal equilibrium" state ϕ_S (resp. ϕ_R or $\bar{\phi}$) a faithful normal state on $\mathfrak{N}_S$ (resp. $\mathfrak{N}_R$ or $\bar{\mathfrak{N}}$). We denote by $\sigma_S(\mathbb{R})$ and $\sigma_R(\mathbb{R})$ the modular automorphism groups corresponding respectively (via [24]) to ϕ_R and ϕ_S. We note that $\bar{\phi}$ then satisfies the Kubo-Martin-Schwinger boundary condition with respect to the automorphism group $\bar{\sigma}(\mathbb{R})$ defined on $\bar{\mathfrak{N}}$ by $\bar{\sigma}(t)=\sigma_S(t)\otimes\sigma_R(t)$. We will refer to $\bar{\sigma}(\mathbb{R})$ as the evolution of the composite system when the interaction between $\mathfrak{N}_S$ and $\mathfrak{N}_R$ is switched *off*. An interpretation of the fact that $\bar{\phi}$ should indeed satisfy the KMS condition with respect to the "free" evolution $\bar{\sigma}(\mathbb{R})\neq\bar{\alpha}(\mathbb{R})$ is proposed in our discussion of the concrete model of Example III.2 below.

We next remark that the understanding of the mechanism of the passage from the conservative evolution $\bar{\alpha}(\mathbb{R})$ to the dissipative evolution $\gamma(\mathbb{R}^+)$ only involves the restriction $\alpha(\mathbb{R})$ of $\bar{\alpha}(\mathbb{R})$ to the $\bar{\alpha}(\mathbb{R})$-stable von Neumann algebra:

$$\mathfrak{N}=\{\bar{\alpha}(t)[N]\,|\,t\in\mathbb{R}, N\in\mathfrak{N}_S\}''\,. \tag{2}$$

Since $\bar{\sigma}(\mathbb{R})$ commutes with $\bar{\alpha}(\mathbb{R})$ and since $\mathfrak{N}_S$ is $\bar{\sigma}(\mathbb{R})$-stable, $\mathfrak{N}$ is $\bar{\sigma}(\mathbb{R})$-stable as well. Upon defining ϕ and $\sigma(\mathbb{R})$ as the respective restrictions of $\bar{\phi}$ and $\bar{\sigma}(\mathbb{R})$ to $\mathfrak{N}$, we see that ϕ is an equilibrium state on $\mathfrak{N}$, satisfying the KMS condition w.r.t. $\sigma(\mathbb{R})$. Furthermore, there exist then conditional expectations $\mathscr{E}_0$ from $\bar{\mathfrak{N}}$ onto $\mathfrak{N}$, $\bar{\mathscr{E}}$ from $\bar{\mathfrak{N}}$ onto $\mathfrak{N}_S$, and $\mathscr{E}$ from $\mathfrak{N}$ onto $\mathfrak{N}_S$ with $\bar{\mathscr{E}}=\mathscr{E}\circ\mathscr{E}_0$, determined uniquely by the conditions:

$$\bar{\phi}\circ\mathscr{E}_0=\bar{\phi}\,;\;\bar{\phi}\circ\bar{\mathscr{E}}=\bar{\phi}\,;\;\phi\circ\mathscr{E}=\phi\,. \tag{3}$$

Our condition (1) can thus be rewritten as:

$$\bar{\mathscr{E}}\circ\bar{\alpha}(t)\circ\bar{\mathscr{E}}=\gamma(t)\circ\bar{\mathscr{E}}\quad\text{for all}\quad t\in\mathbb{R}^+ \tag{4}$$

or if we restrict our attention to $\mathfrak{N}$, as:

$$\mathscr{E}\circ\alpha(t)\circ\mathscr{E}=\gamma(t)\circ\mathscr{E}\quad\text{for all}\quad t\in\mathbb{R}^+\,. \tag{5}$$

The relations (4) or (5) precisely express, in the von Neumann algebraic language, that the reduced evolution $\gamma(\mathbb{R}^+)$ is obtained from the conservative evolutions $\bar{\alpha}(\mathbb{R})$ or $\alpha(\mathbb{R})$ by a "projection technique". As a consequence of (4) it is easily seen that $\gamma(t)$ are not only positive, as we explicitely assumed in the beginning but are in fact completely positive, faithful maps. This implies that one can actually reconstruct canonically from $\{\mathfrak{N}_S, \gamma(\mathbb{R}^+)\}$: a von Neumann algebra $\mathfrak{N}$, a group $\alpha(\mathbb{R})$ of automorphisms of $\mathfrak{N}$, and a conditional expectation $\mathscr{E}$ from $\mathfrak{N}$ onto $\mathfrak{N}_S$ such that (5) and (2) are satisfied.

We finally remark that the "backward trajectory"

$$\mathscr{A}_t = \{\alpha(s)[N] \mid s \leqq t, N \in \mathfrak{N}_S\}''$$

of $\mathfrak{N}_S$ in $\mathfrak{N}$ is $\sigma(\mathbb{R})$-stable, increasing in t, and that $\{\mathscr{A}_t \mid t \in \mathbb{R}\}$ generates $\mathfrak{N}$.

These remarks motivate the principal aim of this paper, namely: to isolate those properties which extend to the aggregate $\{\mathfrak{N}, \phi, \alpha(\mathbb{R}), \mathscr{A}_t\}$ the essential features encountered in the classical theory of K-flows. These properties, once identified, will be verified to hold in physical situations of relevance for non-equilibrium statistical mechanics. We shall show that this extension carries over to the quantum mechanical realm, the ergodic and other spectral properties of classical K-flows.

In Section I we give the basic definitions for what we call *generalized K-flows;* we then prove, in that section, some of their general ergodic and hereditary properties. Section II is devoted to the study of some consequences of an additional assumption which we term "*weak-reversibility*". Non-isomorphic *examples* of generalized K-flows are constructed in Section III, showing explicitly that this concept leads to a genuine generalization of the classical K-flow theory. We also indicate in this section how one of these examples is intimately linked with a statistical mechanical description of the thermodynamical system corresponding to the diffusion of a quantum particle in a harmonic well. In Section IV we discuss a generalization of the concept of *dynamical entropy;* we show that the resulting entropy is strictly positive on every non-singular generalized K-flow; we compute it for various special non-abelian factor K-flows where it happens to be infinite; as in the case of the classical K-flow associated to Brownian motion, the proof proceeds by embedding Bernoulli shifts of arbitrary large entropy in the quantum K-flows considered.

We might finally mention to close this "introduction" that some of the ideas to be developed in the following pages have been approached with lesser generality, in previous publications [9]; most of the proofs then presented are now superseded by those given in the present paper which is self-contained, and can thus be read independently of [9].

I. Generalized *K*-Flows

1. Definition. A *generalized K-flow* is an aggregate $\{\mathfrak{N}, \phi, \alpha(\mathbb{R}), \mathscr{A}\}$ where $\mathfrak{N}$ is a von Neumann algebra acting on a separable Hilbert space $\mathfrak{H}$; ϕ is a faithful normal state on $\mathfrak{N}$; $\alpha(\mathbb{R})$ is a group of automorphisms of $\mathfrak{N}$ such that: (a) for each $N \in \mathfrak{N}$ the function $t \in \mathbb{R} \mapsto \alpha(t)[N] \in \mathfrak{N}$ is str.-op. continuous, (b) $\phi \circ \alpha(t) = \phi$ for all $t \in \mathbb{R}$; and $\mathscr{A}$ is a von Neumann subalgebra of $\mathfrak{N}$ satisfying the following four conditions: (i) $\mathscr{A} \subseteqq \alpha(t)[\mathscr{A}]$ for $t \in \mathbb{R}^+$; (ii) the von Neumann algebra generated by $\{\alpha(t)[\mathscr{A}] \mid t \in \mathbb{R}\}$ coincides with $\mathfrak{N}$; (iii) $\mathbb{C}I$ is the largest von Neumann algebra contained in all $\alpha(t)[\mathscr{A}]$, with t running over $\mathbb{R}$; (iv) $\mathscr{A}$ is stable under the modular group $\sigma(\mathbb{R})$ canonically associated to ϕ.

We recall (for details, see § 13 in [24a]) that, given a faithful normal state ϕ on a von Neumann algebra $\mathfrak{N}$, there exists a unique continuous one-parameter group $\sigma(\mathbb{R})$ of automorphisms of $\mathfrak{N}$ with respect to which ϕ satisfies the Kubo-

Martin-Schwinger (KMS) boundary condition. It is this group which is called the modular group canonically associated to ϕ. The set $\mathfrak{N}_\phi$ of fixed points of $\mathfrak{N}$ under $\sigma(\mathbb{R})$ is a von Neumann subalgebra of $\mathfrak{N}$ which we refer to as the centralizer of $\mathfrak{N}$ with respect to ϕ; this nomenclature is justified by the fact that $\mathfrak{N}_\phi = \{N \in \mathfrak{N} | \langle \phi; [N, M] \rangle = 0 \text{ for all } M \in \mathfrak{N}\}$. We can assume without loss of generality that there exists in $\mathfrak{H}$ a vector Φ cyclic and separating for $\mathfrak{N}$ such that $\langle \phi; N \rangle = (\Phi, N\Phi)$ for all $N \in \mathfrak{N}$.

2. Definition. A generalized K-flow $\{\mathfrak{N}, \phi, \alpha(\mathbb{R}), \mathscr{A}\}$ is said to be *regular* if every maximal abelian subalgebra of the centralizer $\mathfrak{N}_\phi$ of $\mathfrak{N}$ w.r.t. ϕ, is already maximal abelian in $\mathfrak{N}$. At the opposite extreme, $\{\mathfrak{N}, \phi, \alpha(\mathbb{R}), \mathscr{A}\}$ is said to be *singular* if $\mathfrak{N}_\phi = \mathbb{C}I$.

3. Remarks. (i) We will explicitly construct, in Section III, a regular, as well as a singular generalized K-flow where $\mathfrak{N}$ is a factor of type III_1 in the classification of Connes [4], thus proving the existence of non-isomorphic generalized K-flows.

(ii) The distinction between $\alpha(\mathbb{R})$ and $\sigma(\mathbb{R})$ should be kept in mind throughout this paper. In line with our motivation, as presented in the introduction, we shall occasionally refer to $\alpha(\mathbb{R})$ as the "true evolution" and to $\sigma(\mathbb{R})$ as the "free evolution", although this nomenclature should not be given too much of a metaphysical meaning. The point however is to distinguish them. Indeed $\alpha(t_1) = \sigma(t_2)$ for any $t_1, t_2 \neq 0$ occurs exactly when $\mathfrak{N} = \mathbb{C}I$, a trivial situation in which we are clearly not interested. Besides this trivial case, the dimension of the Hilbert space $\mathfrak{H}$ on which $\mathfrak{N}$ acts must be infinite (as will follow immediately from Theorem 5 below). Actually, $\alpha(\mathbb{R})$ should be regarded as a rather drastic perturbation of $\sigma(\mathbb{R})$ in a sense which we are now going to make precise.

Since $\phi \circ \alpha(t) = \phi = \phi \circ \sigma(t)$ for all $t \in \mathbb{R}$, both $\alpha(\mathbb{R})$ and $\sigma(\mathbb{R})$ are unitarily implementable; the condition $U(t)\Phi = \Phi = U^\sigma(t)\Phi$ for all $t \in \mathbb{R}$ determines uniquely the corresponding unitary groups. Let H and H^σ be their respective generators. Form now $V = H - H^\sigma$ and $H_\lambda = H^\sigma + \lambda V$ with $\lambda \in \mathbb{R}$ strictly positive. We further remark after [24], that the uniqueness of $\sigma(\mathbb{R})$, and the fact that $\phi \circ \alpha(t) = \phi$ for all $t \in \mathbb{R}$, imply that $\alpha(\mathbb{R})$ and $\sigma(\mathbb{R})$ commute. Consequently $\alpha_\lambda(\mathbb{R})$ defined by $\alpha_\lambda(t) = \alpha(\lambda t)\sigma((1-\lambda)t)$ is a continuous group of automorphisms of $\mathfrak{N}$ with $\phi \circ \alpha_\lambda(t) = \phi$ for all $t \in \mathbb{R}$. The generator of the corresponding unitary group is precisely H_λ. Clearly the aggregate $\{\mathfrak{N}, \phi, \alpha_\lambda(\mathbb{R}), \mathscr{A}\}$ is again a generalized K-flow, however small $\lambda > 0$ might be chosen. Hence the K-flow property is *stable* under the perturbation $V \to \lambda V (\lambda > 0)$; which is to say that H [resp. $\alpha(\mathbb{R})$] is a drastic perturbation of H^σ [resp. $\sigma(\mathbb{R})$]. This will be further emphasized, in the case of regular generalized K-flows by the comparison of the respective ergodic properties of $\alpha(\mathbb{R})$ and $\sigma(\mathbb{R})$.

(iii) The condition that $\mathscr{A}$ be $\sigma(\mathbb{R})$-stable, and the commutativity of $\alpha(\mathbb{R})$ and $\sigma(\mathbb{R})$, clearly imply that $\mathscr{A}_t = \alpha(t)[\mathscr{A}]$ is also $\sigma(\mathbb{R})$-stable, for every $t \in \mathbb{R}$. This is known [24] to be equivalent to the existence, for every $t \in \mathbb{R}$, of a unique faithful normal conditional expectation $\mathscr{E}_t$ from $\mathfrak{N}$ onto $\mathscr{A}_t$ such that $\phi \circ \mathscr{E}_t = \phi$, a fact which we shall use repeatedly in the sequel.

(iv) In previous papers [9] we imposed, instead of condition (I.1.iii) above, the apparently stronger condition $\bigcap_t [\mathscr{A}_t \Phi] = \mathbb{C}\Phi$, where $[\mathscr{A}_t \Phi]$ denotes the closure of the linear manifold $\mathscr{A}_t \Phi = \{A\Phi | A \in \mathscr{A}_t\}$. Since Φ is separating for $\mathfrak{N}$, this condition clearly implies that $\bigcap_t \mathscr{A}_t = \mathbb{C}I$, which is our present condition

(I.1.iii). The converse is however also true, as we shall presently see, thus solving a question left open in [9a, Remark 3.2.i].

4. Lemma. *For every generalized K-flow:* $\bigcap_t [\mathscr{A}_t \Phi] = \mathbb{C}\Phi$.

We derive this lemma as an immediate consequence of part (iii) of the following strong-martingale theorem.

Sublemma. *Let* ϕ *be a faithful normal state on a von Neumann algebra* $\mathfrak{N}$; $\{\mathscr{A}_t | t \in \mathbb{R}\}$ *be a collection of von Neumann subalgebras of* $\mathfrak{N}$ *such that: (i) for every* $t \in \mathbb{R}$, $\mathscr{A}_t$ *is stable under the modular group* $\sigma(\mathbb{R})$ *canonically associated to* ϕ, *and (ii)* $s \leqq t$ *implies* $\mathscr{A}_s \subseteq \mathscr{A}_t$; *let further, for each* $t \in \mathbb{R}$, $\mathscr{E}(\,.\,|\mathscr{A}_t)$ *be the faithful normal conditional expectation from* $\mathfrak{N}$ *onto* $\mathscr{A}_t$ *such that* $\phi \circ \mathscr{E}(\,.\,|\mathscr{A}_t) = \phi$. *Then: (i) there exists a unique faithful normal conditional expectation* $\mathscr{E}(\,.\,|\bigcap_t \mathscr{A}_t)$ *from* $\mathfrak{N}$ *onto* $\bigcap_t \mathscr{A}_t$ *such that* $\phi \circ \mathscr{E}(\,.\,|\bigcap_t \mathscr{A}_t) = \phi$; *(ii) for every* $N \in \mathfrak{N}$, $\mathscr{E}(N|\bigcap_t \mathscr{A}_t) = \text{s-lim}_{t \to -\infty} \mathscr{E}(N|\mathscr{A}_t)$; *(iii) with* Φ *denoting the vector in* $\mathfrak{H}$, *cyclic and separating for* $\mathfrak{N}$, *associated to* ϕ: $\bigcap_t [\mathscr{A}_t \Phi] = [(\bigcap_t \mathscr{A}_t)\Phi]$.

Proof. For every $s, t \in \mathbb{R}$ with $s \leqq t$ we have:

$$E_s \equiv [\mathscr{A}_s \Phi] \subseteq [\mathscr{A}_t \Phi] \equiv E_t\,.$$

Upon denoting by the same symbol a closed subspace of $\mathfrak{H}$ and the corresponding orthogonal projection, we have thus:

$$\bigcap_t [\mathscr{A}_t \Phi] \equiv E = \text{s-lim}_{t \to -\infty} E_t\,.$$

At fixed $N \in \mathfrak{N}$, fixed $\hat{\Phi} \in \mathfrak{H}$, and fixed $\varepsilon > 0$, we can thus find, since Φ is separating for $\mathfrak{N}$, an $X \in \mathfrak{N}'$ and a $T \in \mathbb{R}$ such that:

$$\|\hat{\Phi} - X\Phi\| \leqq \varepsilon/3\,\|N\|$$

and

$$\|(E_t - E_s)N\Phi\| \leqq \varepsilon/3\,\|X\| \quad \text{for all} \quad s, t \leqq T.$$

From [24] we know that, for each $t \in \mathbb{R}$, $\mathscr{E}(\cdot|\mathscr{A}_t)$ is a projection of norm 1, and $\mathscr{E}(N|\mathscr{A}_t)\Phi = E_t N\Phi$ for all $N \in \mathfrak{N}$. We have thus for all $s, t \leqq T$:

$$\begin{aligned} &\|\mathscr{E}(N|\mathscr{A}_t)\hat{\Phi} - \mathscr{E}(N|\mathscr{A}_s)\hat{\Phi}\| \\ &\leqq \|\mathscr{E}(N|\mathscr{A}_t)(\hat{\Phi} - X\Phi)\| + \|\mathscr{E}(N|\mathscr{A}_s)(\hat{\Phi} - X\hat{\Phi})\| \\ &\quad + \|\mathscr{E}(N|\mathscr{A}_t)X\Phi - \mathscr{E}(N|\mathscr{A}_s)X\Phi\| \\ &\leqq 2\|N\|\varepsilon/3\|N\| + \|X\|\varepsilon/3\|X\| = \varepsilon \end{aligned}$$

i.e. $\bar{N} \equiv \lim_{t \to -\infty} \mathscr{E}(N|\mathscr{A}_t)$ exists strongly on $\mathfrak{H}$ for every $N \in \mathfrak{N}$. Since $\mathscr{E}(N|\mathscr{A}_s) \in \mathscr{A}_s \subseteq \mathscr{A}_t$ for all $s \leqq t$, we have that $\bar{N} \in \mathscr{A}_t$ for all $t \in \mathbb{R}$ and thus $\bar{N} \in \bigcap_t \mathscr{A}_t$. On the other hand:

$$\bar{N}\Phi = \text{s-lim}_{t \to -\infty} E_t N\Phi = EN\Phi\,,$$

so that $E\mathfrak{N}\Phi \subseteq (\bigcap_t \mathscr{A}_t)\Phi$, and thus, since Φ is cyclic in $\mathfrak{H}$ for $\mathfrak{N}$:

$$\bigcap_t [\mathscr{A}_t \Phi] \equiv E\mathfrak{H} \subseteq [(\bigcap_t \mathscr{A}_t)\Phi] \subseteq \bigcap_t [\mathscr{A}_t \Phi]$$

which proves (iii). Now our condition $\{\mathscr{A}_t$ is $\sigma(\mathbb{R})$-stable for every $t\in\mathbb{R}\}$ implies that $(\bigcap_t\mathscr{A}_t)$ is $\sigma(\mathbb{R})$-stable. There exists consequently [24] a unique faithful normal conditional expectation $\mathscr{E}(.|\bigcap_t\mathscr{A}_t)$ from $\mathfrak{N}$ onto $(\bigcap_t\mathscr{A}_t)$ such that $\phi\circ\mathscr{E}(.|\bigcap_t\mathscr{A}_t)=\phi$, thus proving (i). Furthermore [24] this conditional expectation is determined by the relation $\mathscr{E}(N|\bigcap_t\mathscr{A}_t)\Phi=FN\Phi$ where F is the projector onto $[(\bigcap_t\mathscr{A}_t)\Phi]$, which, by (iii), is E. We have thus for every $N\in\mathfrak{N}$:

$$\mathscr{E}(N|\textstyle\bigcap_t\mathscr{A}_t)\Phi=EN\Phi=\lim_{t\to-\infty}E_tN\Phi=\bar{N}\Phi\,.$$

Since Φ is separating for $\mathfrak{N}$ we have thus, see the proof of (iii),

$$\mathscr{E}(N|\textstyle\bigcap_t\mathscr{A}_t)=\bar{N}=\text{s-lim}_{t\to-\infty}\mathscr{E}(N|\mathscr{A}_t)$$

which proves (ii). q.e.d.

Remark. We see that $\mathscr{E}(\,.\,|\mathscr{A}_{-t})$ is a decreasing martingale in the sense of Arveson [3] so that, from (i) and the obvious consistency relations between conditional expectations on a refining collection of $\sigma(\mathbb{R})$-stable von Neumann subalgebras, (ii) can be obtained as a consequence of Theorem 6.1.7 in [3]. Hence our argument contains in particular, an alternate proof of Arveson's result, for the special martingales considered here. This path however would not noticeably shorten the proof of (iii) (which is the result we are actually interested in).

5. Theorem. *Let $\{\mathfrak{N},\phi,\alpha(\mathbb{R}),\mathscr{A}\}$ be a generalized K-flow, and H be the generator of the strongly continuous, one-parameter, unitary group $U(\mathbb{R})$ implementing $\alpha(\mathbb{R})$, with $U(t)\Phi=\Phi$ for all $t\in\mathbb{R}$. Then: (i) $\mathrm{Sp}_d(H)=\{0\}$; (ii) $\Psi\in\mathfrak{H}$ and $H\Psi=0$ imply $\Psi=\lambda\Phi$ with $\lambda\in\mathbb{C}$; (iii) H has homogeneous Lebesgue spectrum on the orthocomplement $\mathfrak{H}^\perp$ of $\mathbb{C}\Phi$ in $\mathfrak{H}$.*

Proof. By construction of $U(\mathbb{R})$, $H\Phi=0$; it is thus sufficient to consider the restriction $U^\perp(\mathbb{R})$ of $U(\mathbb{R})$ to $\mathfrak{H}^\perp$. For every $s\in\mathbb{R}$ let now $E_s^\perp=[\mathscr{A}_s\Phi]\ominus\mathbb{C}\Phi$. One then checks easily from Definition 1 and Lemma 4, that $\{U^\perp(t),E_s^\perp|s,t\in\mathbb{R}\}$ defines on $\mathfrak{H}^\perp$ a system of imprimitivity based on $\mathbb{R}$. Since $\mathfrak{H}$ (and thus $\mathfrak{H}^\perp$) is assumed to be separable, von Neumann uniqueness theorem [15] is applicable to the situation considered here (see, for instance, Theorems III.1.5 and 6 in [8]); we thus have $\mathfrak{H}^\perp=\oplus\mathfrak{H}_n$ with $\mathfrak{H}_n$ $U(\mathbb{R})$-stable, and the restriction $U_n(\mathbb{R})$ of $U(\mathbb{R})$ to $\mathfrak{H}_n$ is unitarily equivalent to $V(\mathbb{R})$ defined on $\mathscr{L}^2(\mathbb{R},dx)$ by $(V(t)\Psi)(x)=\Psi(x-t)$. q.e.d.

Remark. This theorem extends thus to the generalized K-flows of Definition 1 an important property of classical K-flows proven first by Sinai [22], and already generalized to some special non-abelian K-flows in [9a]. It should however be pointed out here that no assertion is made yet on the multiplicity of the absolutely continuous part of the spectrum of H. As pointed out in [9a], the spectral properties of H stated in the above theorem already imply strong ergodic properties, which we now state for generalized K-flows. These are listed below in an order which make their proof follow immediately from the theorem and from general results on quantum dynamical systems collected in pp. 181–187 of [8].

Corollary. *For every generalized K-flow $\{\mathfrak{N},\phi,\alpha(\mathbb{R}),\mathscr{A}\}$ we have: (i) ϕ is extremal $\alpha(\mathbb{R})^*$-invariant; (ii) ϕ is strongly mixing i.e. for every $N,M\in\mathfrak{N}$: $\lim_{|t|\to\infty}\langle\phi;N\alpha(t)[M]\rangle=\langle\phi;N\rangle\langle\phi;M\rangle$; (iii) for every invariant mean η on $\mathbb{R}$, any $\Psi_1,\Psi_2\in\mathfrak{H}$*

and any $N \in \mathfrak{N}: \eta(\Psi_1, \alpha(.)[N]\Psi_2) = (\Psi_1, \eta[N]\Psi_2)$ *with* $\eta[N] = \langle\phi; N\rangle I$; *(iv)* $\alpha(\mathbb{R})$ *acts in a* η*-abelian manner on* ϕ, *i.e. for every* $N_1, N_2, N_3, N_4 \in \mathfrak{N}\eta < \phi;$ $N_1[\alpha(.)[N_2], N_3]N_4\rangle = 0$; *(v)* ϕ *is the only normal,* $\alpha(\mathbb{R})^*$*-invariant state on* $\mathfrak{N}$, *(vi) the algebra of fixed points of* $\mathfrak{N}$ *under* $\alpha(\mathbb{R})$ *reduces to* $\mathbb{C}I$.

Remarks. (i) An asymptotic abelianness property stronger that (iv) above will be obtained in Section II, under the additional assumption of weak-reversibility. (ii) Properties (v) and (vi) of the above corollary reinforce our Remarks (I.3.ii) on the structural differences between the "true evolution" $\alpha(\mathbb{R})$ and the "free evolution" $\sigma(\mathbb{R})$. Indeed, whereas ϕ is KMS for $\sigma(\mathbb{R})$, (v) above implies that $\mathfrak{N}$ admits no faithful normal state with respect to which $\alpha(\mathbb{R})$ would satisfy the KMS condition. Moreover, even to assume (vi) to hold for $\sigma(\mathbb{R})$ as well, amounts exactly to imposing that the flow be singular; in other words, for any non-singular generalized K-flow, the perturbation V from $\sigma(\mathbb{R})$ to $\alpha(\mathbb{R})$ destroys all the non-trivial invariants of the "free-motion", and this remains true if V is replaced by λV, however small one might choose $\lambda > 0$.

6. Theorem. *Let* $\{\mathfrak{N}, \phi, \alpha(\mathbb{R}), \mathscr{A}\}$ *be a generalized K-flow, and* H^σ *be the generator of the strongly continuous, one-parameter, unitary group* $U^\sigma(\mathbb{R})$ *implementing* $\sigma(\mathbb{R})$, *with* $U^\sigma(t)\Phi = \Phi$ *for all* $t \in \mathbb{R}$. *Then:* (i) $\mathrm{Sp}_d(H^\sigma)$ *is a subgroup of the additive group* $\mathbb{R}$; (ii) *the following conditions are equivalent:* (a) 0 *is a non-degenerate eigenvalue of* H^σ, (b) $\{\mathfrak{N}, \phi, \alpha(\mathbb{R}), \mathscr{A}\}$ *is* **singular**, (c) $\mathrm{Sp}_d(H^\sigma) = \{0\}$ *with multiplicity 1*, (d) ϕ *is extremal* $\sigma(\mathbb{R})^*$*-invariant*, (e) $\mathfrak{N}$ *is a factor and* $\sigma(\mathbb{R})$ *acts in a* η*-abelian manner on* ϕ; (iii) *if* $\dim \mathfrak{H} \geqq 2$, *any of the above conditions* (a)–(e) *implies* $\mathrm{Sp}(H^\sigma) = \mathbb{R}$ *and* $\mathfrak{N}$ *is a type* III_1*-factor;* (iv) *if* 0 *is an isolated point in* $\mathrm{Sp}(H^\sigma)$, *then* $\{\mathfrak{N}, \phi, \alpha(\mathbb{R}), \mathscr{A}\}$ *is* **regular**; (v) *if* $\mathrm{Sp}(H^\sigma)$ *is discrete, then* $\{\mathfrak{N}, \phi, \alpha(\mathbb{R}), \mathscr{A}\}$ *is* **regular**.

Proof. $\phi \circ \alpha(t) = \phi$ for all $t \in \mathbb{R}$ implies that $\sigma(\mathbb{R})$ commutes with $\alpha(\mathbb{R})$ and thus $U^\sigma(\mathbb{R})$ commutes with $U(\mathbb{R})$, the unitary group implementing $\alpha(\mathbb{R})$. From part (vi) of the corollary to Theorem 5 above, we see that all the conditions of Theorem 3.2 in [12] now hold for every generalized K-flow, so that (i) is thus established. To prove (ii), we form $\mathfrak{H}_\lambda = \{\Psi \in \mathfrak{H} \mid U^\sigma(t)\Psi = \exp(-i\lambda t)\Psi$ for all $t \in \mathbb{R}\}$ and $\mathfrak{N}_\lambda = \{N \in \mathfrak{N} \mid \sigma(t)[N] = \exp(-i\lambda t)N$ for all $t \in \mathbb{R}\}$. From [10] we know that $\mathfrak{H}_\lambda = [\mathfrak{N}_\lambda \Phi]$ and thus in particular $\mathfrak{H}_0 = [\mathfrak{N}_\phi \Phi]$. Since Φ is separating for $\mathfrak{N}$, this proves the equivalence of (a) and (b). Let us now see that these conditions imply (c). Let $\lambda \in \mathrm{Sp}_d(H^\sigma)$ and $X \in \mathfrak{N}_\lambda$. We have then $X^* \in \mathfrak{N}_{-\lambda}$ and $X^*X, XX^* \in \mathfrak{N}_\phi$. Our condition (b) now implies $X^*X = x^2 I$ and $XX^* = y^2 I$ with $x, y \in \mathbb{R}^+$. From $\|X^*X\| = \|XX^*\|$ we conclude that $x = y$ and thus, in particular, $\langle\phi; X^*X\rangle = \langle\phi; XX^*\rangle$. On the other hand ϕ KMS w.r.t. $\sigma(\mathbb{R})$ and $X \in \mathfrak{N}_\lambda$ imply $\langle\phi; XN\rangle = e^{-\lambda}\langle\phi; NX\rangle$ for every $N \in \mathfrak{N}$. These two equalities together give $(1 - e^{-\lambda})\langle\phi; X^*X\rangle = 0$, i.e. either $\lambda = 0$, or $\|X\Phi\| = 0$ for all $X \in \mathfrak{N}_\lambda$ and thus $\mathfrak{H}_\lambda = [\mathfrak{N}_\lambda \Phi] = \{0\}$ which is to say that $\mathrm{Sp}_d(H^\sigma) = \{0\}$. The multiplicity statement in (c) follows then from (a). Conversely (c) trivially implies (a). We now remark (see for instance Theorem II.2.8 in [8]) that (a) implies (d); and (see for instance Corollary 2 pp. 206–207 in [8]) that (d) is equivalent to (e). Furthermore (see again Theorem II.2.8 in [8]) (e) and (d) imply (a). This concludes the proof of (ii). To prove (iii) we use Corollary 3.2.3 and Corollary 3.2.7 in [4] to see that $\mathrm{Sp}(H^\sigma)$ is either $\{0\}$, $\omega\mathbb{Z}$ or $\mathbb{R}$. The first case would imply $\mathfrak{N}_\phi = \mathfrak{N}$ which is ruled out by the conjunction of (b) and $\dim \mathfrak{H} \geqq 2$.

If the second case were realized, part (vi) of the corollary to Theorem 5 would imply that ϕ is homogeneous and periodic in the sense of Takesaki [25] whose Proposition 1.7 would in turn imply that $\{\mathfrak{N}, \phi, \alpha(\mathbb{R}), \mathscr{A}\}$ is regular; this again is ruled out by the conjunction of (b) and $\dim \mathfrak{H} \geqq 2$. Hence (b) only allows $\mathrm{Sp}(H^\sigma) = \mathbb{R}$. Again from Corollary 3.2.7 of [4], we conclude that $S(\mathfrak{N}) = [0, \infty[$, i.e. $\mathfrak{N}$ is of Type III_1. Part (iii) is thus proven. Part (iv) follows immediately from Lemma 4.2.3 in [4]. Part (v) follows finally from the remark that if one replaces in [25] the average over one period of $\sigma(\mathbb{R})$ by an arbitrary invariant mean over $\mathbb{R}$ itself, then Takesaki's proof of his Proposition 1.7 extends from the case $\sigma(\mathbb{R})$ periodic to the case $\sigma(\mathbb{R})$ almost periodic i.e. $\mathrm{Sp}(H^\sigma)$ discrete, which is the assumption in (v). q.e.d.

Remark. We shall construct, in Section III, a regular generalized K-flow with $\mathfrak{N}$ a type III_1 factor, and $\mathrm{Sp}_d(H^\sigma)$ dense in $\mathbb{R}$, thus ruling out the possibility of proving the converse of the implications (iii) and (iv) of the above theorem.

7. Theorem. *Let $\{\mathfrak{N}, \phi, \alpha(\mathbb{R}), \mathscr{A}\}$ be a generalized K-flow with ϕ not a trace on $\mathfrak{N}$; and let H be defined as in Theorem 5. Then H has homogeneous Lebesgue spectrum with infinite multiplicity on the orthocomplement $\mathfrak{H}^\perp$ of $\mathbb{C}\Phi$ in $\mathfrak{H}$.*

Proof. From Theorem 5, we already know the "homogeneous Lebesgue spectrum" part of the present theorem; we thus only have to prove infinite multiplicity. Let $\mathfrak{M}$ be the von Neumann algebra generated on $\mathfrak{H}^\perp$ by the system of imprimitivity $\{U^\perp(t), E_s^\perp | s, t \in \mathbb{R}\}$ defined in the proof of Theorem 5. Let further $U^\sigma(\mathbb{R})$ be defined as in Theorem 6, and $V(\mathbb{R})$ be its restriction to $\mathfrak{H}^\perp$. Notice now that $V(\mathbb{R})$ commutes with $U^\perp(\mathbb{R})$ follows from $\sigma(\mathbb{R})$ commutes with $\alpha(\mathbb{R})$; the latter property, together with the condition that $\mathscr{A}$ be $\sigma(\mathbb{R})$-stable, implies also that $\alpha(s)[\mathscr{A}]$ is $\sigma(\mathbb{R})$-stable; this in turn implies that $V(\mathbb{R})$ commutes with $E_s^\perp$ for every $s \in \mathbb{R}$. Consequently, $V(\mathbb{R}) \subseteqq \mathfrak{M}'$. We now proceed by contradiction. Suppose indeed that H were to have homogeneous Lebesgue spectrum on $\mathfrak{H}^\perp$ with multiplicity $n < \infty$. We could then write $\mathfrak{H}^\perp = \mathfrak{H}_0 \otimes \mathbb{C}^n$ with $\mathfrak{H}_0 = \mathscr{L}^2(\mathbb{R}, dx)$; and $\mathfrak{M} = B(\mathfrak{H}_0) \otimes \mathbb{C}I$. $V(\mathbb{R}) \subseteqq \mathfrak{M}'$ would then imply $V(t) = I \otimes v(t)$ with $v: \mathbb{R} \to \mathscr{U}(\mathbb{C}^n)$ a finite-dimensional, continuous unitary representation of $\mathbb{R}$. Consequently, $v(\mathbb{R})$ and thus $V(\mathbb{R})$ would have discrete spectrum, with at most finitely many different eigenvalues; so would then have H^σ. Because of Theorem 6.i, this however would imply that $H^\sigma = 0$, and thus $\sigma(\mathbb{R}) = \mathrm{id}$, that is to say ϕ would be a trace on $\mathfrak{N}$. q.e.d.

Remark. This theorem extends the result of Sinai [22] to a large class of generalized K-flows of possible interest to quantum statistical mechanics (see for instance Section III).

8. Theorem. *Let $\{\mathfrak{N}, \phi, \alpha(\mathbb{R}), \mathscr{A}\}$ be a generalized K-flow; $\mathfrak{N}_\phi$ be the centralizer of $\mathfrak{N}$ with respect to ϕ; $\alpha_\phi(\mathbb{R})$ (resp. ϕ) be the restriction of $\alpha(\mathbb{R})$ (resp. ϕ) to $\mathfrak{N}_\phi$; and $\mathscr{A}_\phi$ be the von Neumann algebra $\mathfrak{N}_\phi \cap \mathscr{A}$. Then $\{\mathfrak{N}_\phi, \phi, \alpha_\phi(\mathbb{R}), \mathscr{A}_\phi\}$ is a regular generalized K-flow.*

Proof. We first should note that $\phi \circ \alpha(t) = \phi$ for all $t \in \mathbb{R}$ implies that $\mathfrak{N}_\phi$ is stable under $\alpha(\mathbb{R})$. Hence $\alpha_\phi(\mathbb{R})$ is indeed a continuous group of automorphisms of $\mathfrak{N}_\phi$ with $\phi \circ \alpha_\phi(t) = \phi$ for all $t \in \mathbb{R}$. Furthermore $\mathscr{A}_t^\phi \equiv \alpha_\phi(t)[\mathscr{A}_\phi] = \mathfrak{N}_\phi \cap \mathscr{A}_t$ and the following properties are thus immediately inherited from the corresponding conditions on $\{\mathfrak{N}, \phi, \alpha(\mathbb{R}), \mathscr{A}\}$: $\mathscr{A}_\phi \subseteq \mathscr{A}_t^\phi$ for all $t \in \mathbb{R}^+$ and $\bigcap_t \mathscr{A}_t^\phi = \mathbb{C}I$. Since $\mathfrak{N}_\phi$ is the centralizer of $\mathfrak{N}$ w.r.t. ϕ, the modular group $\sigma_\phi(\mathbb{R})$ of $\mathfrak{N}_\phi$ w.r.t. ϕ is the identity map, so that we trivially have $\mathscr{A}_\phi$ is $\sigma_\phi(\mathbb{R})$-stable, and $(\mathfrak{N}_\phi)_\phi = \mathfrak{N}_\phi$. The proof of the theorem will consequently be complete when we get $\bigvee_t \mathscr{A}_t^\phi = \mathfrak{N}_\phi$ which we shall now proceed to prove. Since ϕ is faithful and normal and $\phi \circ \sigma(t) = \phi$ for all $t \in \mathbb{R}$, $\mathfrak{N}$ is $\sigma(\mathbb{R})$-finite in the sense of Kovács and Szücs [14]. We therefore know that the unique normal faithful conditional expectation from $\mathfrak{N}$ onto $\mathfrak{N}_\phi$ satisfying $\phi \circ \mathscr{E}(\,.\,|\mathfrak{N}_\phi) = \phi$ is given, for every $N \in \mathfrak{N}$, by $\mathscr{E}(N|\mathfrak{N}_\phi) = \mathfrak{N}_\phi \cap \mathscr{K}(N, \sigma(\mathbb{R}))$, where $\mathscr{K}(N, \sigma(\mathbb{R}))$ intersects $\mathfrak{N}_\phi$ at exactly one point and is defined as the weak closure of the convex hull of the orbit of N under $\sigma(\mathbb{R})$. Our condition that $\mathscr{A}$ be $\sigma(\mathbb{R})$-stable thus implies, for every $t \in \mathbb{R}$, $\mathscr{E}(\mathscr{A}_t|\mathfrak{N}_\phi) \subseteq \mathfrak{N}_\phi \cap \mathscr{A}_t$; since on the other hand $\mathscr{A}_t^\phi = \mathscr{E}(\mathscr{A}_t^\phi|\mathfrak{N}_\phi) \subseteq \mathscr{E}(\mathscr{A}_t|\mathfrak{N}_\phi)$, we have: $\mathscr{A}_t^\phi = \mathscr{E}(\mathscr{A}_t|\mathfrak{N}_\phi)$. Our condition $\mathfrak{N} = \bigvee_t \mathscr{A}_t$ means that for every $N \in \mathfrak{N}$, there exists $\{A_t \in \mathscr{A}_t$ with $t \in \mathbb{R}\}$ such that $N = \text{u.w.-lim}\, A_t$. In particular for every $N \in \mathfrak{N}_\phi$ the u.w.-continuity of the projection $\mathscr{E}(\,.\,|\mathfrak{N}_\phi)$ implies: $N = \mathscr{E}(N|\mathfrak{N}_\phi) = \mathscr{E}(\text{u.w.-lim}\, A_t|\mathfrak{N}_\phi) = \text{u.w.-lim}\, \mathscr{E}(A_t|\mathfrak{N}_\phi)$. Hence for every $N \in \mathfrak{N}_\phi$ there exists $\{B_t \in \mathscr{A}_t^\phi$ with $t \in \mathbb{R}\}$ such that $N = \text{u.w.-lim}\, B_t$, which is to say that $\mathfrak{N}_\phi \subseteqq \bigvee_t \mathscr{A}_t^\phi$; since $\mathscr{A}_t^\phi \subseteqq \mathfrak{N}_\phi$ for every $t \in \mathbb{R}$, we get $\mathfrak{N}_\phi = \bigvee_t \mathscr{A}_t^\phi$. q.e.d.

Remarks. (i) This theorem will play a central role in the computation of the dynamical entropy of generalized K-flows (see Section IV); (ii) As a consequence of this theorem, we see that every non-singular generalized K-flow contains at least one sub-flow which is a regular generalized K-flow; (iii) The conclusion of the theorem had already been obtained [9c] under the additional condition that $\sigma(\mathbb{R})$ be periodic. The present extension is motivated by the existence (see Section III) of regular generalized K-flows for which $\sigma(\mathbb{R})$ is not periodic. (iv) Except possibly for the regularity of the resulting K-flow, the present proof extends moreover immediately to the case where $\mathfrak{N}_\phi$ is replaced by $\mathfrak{N}^G$, the algebra of fixed points of $\mathfrak{N}$ under a group $G = \{g\}$ of automorphisms of $\mathfrak{N}$ satisfying the following conditions: (a) $\phi \circ g = \phi$ for all $g \in G$, (b) G commutes with $\alpha(\mathbb{R})$, and (c) $\mathscr{A}$ is G-stable; notice, in particular, that condition (a) doesn't need to be imposed separately in case $\mathfrak{N}$ is a finite factor. (v) The gist of the proof is to show that, for certain subalgebras $\mathfrak{X}$ of $\mathfrak{N}$, $\mathscr{E}(\mathscr{A}_t|\mathfrak{X})$ is a von Neumann algebra, namely $\mathscr{A}_t \cap \mathfrak{X}$, so that the u.w.-continuity of the conditional expectation $\mathscr{E}(\,.\,|\mathfrak{X})$ can be used to prove the distributive law $\mathfrak{X} \cap (\bigvee_t \mathscr{A}_t) = \bigvee_t (\mathfrak{X} \cap \mathscr{A}_t)$. (vi) When, however, $\mathfrak{N}$ is abelian this distributivity holds unrestrictedly; hence the first part of the proof of the theorem shows that whenever $\mathfrak{N}$ is abelian, every $\alpha(\mathbb{R})$-stable subalgebra $\mathfrak{X}$ of $\mathfrak{N}$ inherits the (regular!) K-flow structure of $\{\mathfrak{N}, \phi, \alpha(\mathbb{R}), \mathscr{A}\}$. Another type of hereditary behaviour is exemplified by the following two results.

Scholium A. *Let $\{\mathfrak{N}, \phi, \alpha(\mathbb{R}), \mathscr{A}\}$ be a generalized K-flow with $\mathscr{A} \cap \mathscr{A}' \subseteqq \mathfrak{N} \cap \mathfrak{N}'$. Then the aggregate $\{\mathfrak{Z}, \alpha_z(\mathbb{R}), \phi, \mathscr{A}_z\}$ is a regular generalized K-flow, where $\mathfrak{Z} = \mathfrak{N} \cap \mathfrak{N}'$, $\alpha_z(\mathbb{R})$ (resp. ϕ) is the restriction of $\alpha(\mathbb{R})$ (resp. ϕ) to $\mathfrak{Z}$, and $\mathscr{A}_z = \mathscr{A} \cap \mathfrak{Z}$.*

Proof. For any $A \in \mathscr{A}$ and any $Z \in \mathfrak{Z}$ we have $A\mathscr{E}(Z|\mathscr{A}) = \mathscr{E}(AZ|\mathscr{A}) = \mathscr{E}(Z|\mathscr{A})A$; hence $\mathscr{E}(\mathfrak{Z}|\mathscr{A}) \subseteqq \mathscr{A} \cap \mathscr{A}'$. On the other hand our condition $\mathscr{A} \cap \mathscr{A}' \subseteqq \mathfrak{Z}$ implies $\mathscr{A} \cap \mathscr{A}' = \mathscr{E}(\mathscr{A} \cap \mathscr{A}'|\mathscr{A}) \subseteqq \mathscr{E}(\mathfrak{Z}|\mathscr{A})$. Hence $\mathscr{E}(\mathfrak{Z}|\mathscr{A}) = \mathscr{A} \cap \mathscr{A}'$. Moreover $\mathscr{A} \cap \mathfrak{Z} \supseteqq \mathscr{A} \cap \mathscr{A}' \supseteqq \mathscr{A} \cap \mathfrak{N}' = \mathscr{A} \cap \mathfrak{Z}$, so that we have $\mathscr{A}_z = \mathscr{E}(\mathfrak{Z}|\mathscr{A})$. Since $\mathfrak{Z}$ is $\alpha(\mathbb{R})$-stable: $\mathfrak{Z} = \alpha(t)[\mathfrak{Z}] \supseteqq \alpha(t)\,[\mathscr{A} \cap \mathscr{A}'] = \mathscr{A}_t \cap \mathscr{A}_t'$, and the preceeding reasoning shows as well that $\alpha_z(t)[\mathscr{A}_z]=\mathscr{A}_t\cap\mathfrak{Z}=\mathscr{E}(\mathfrak{Z}|\mathscr{A}_t)$. Instead of appealing to the u.w.-continuity of the conditional expectation, as we did in the proof of the preceeding theorem, we now use the fact that $\{\mathscr{E}(\,.\,|\mathscr{A}_t)|t\in\mathbb{R}\}$ is an increasing martingale with $\bigvee_t\mathscr{A}_t=\mathfrak{N}$. A slight modification of the sublemma to Lemma 4 (or alternatively of Arveson's Proposition 6.1.9 in [3]) shows that, for every $N\in\mathfrak{N}$, $N=s\text{-}\lim_{t\to\infty}\mathscr{E}(N|\mathscr{A}_t)$. In particular every element Z in $\mathfrak{Z}$ can thus be seen as a s-lim of elements $\mathscr{E}(Z|\mathscr{A}_t)$ in $\alpha_z(t)[\mathscr{A}_z]$ as $t\to\infty$. Consequently $\mathfrak{Z}\subseteqq\bigvee_t\alpha_z(t)[\mathscr{A}_z]$; since $\mathscr{A}_z\subseteqq\mathfrak{Z}$ we have thus $\mathfrak{Z}=\bigvee_t\alpha_z(t)[\mathscr{A}_z]$. The remainder of the scholium follows by the same argument as that used in the first part of the proof of the theorem, and from the fact that $\mathfrak{Z}$ is abelian. q.e.d.

Remark. For any generalized K-flow $\{\mathfrak{N},\phi,\alpha(\mathbb{R}),\mathscr{A}\}$, this scholium confirms that: $\mathscr{A}$ (resp. $\mathscr{A}_\phi$) be a factor implies $\mathfrak{N}$ (resp. $\mathfrak{N}_\phi$) is a factor; it is in agreement with: $\mathfrak{Z}$ (resp. $\mathfrak{Z}_\phi=\mathfrak{N}_\phi\cap\mathfrak{N}'_\phi$) is either equal to $\mathbb{C}I$, or is infinite dimensional.

Scholium B. *Let* $\{\mathfrak{N},\phi,\alpha(\mathbb{R}),\mathscr{A}\}$ *be a generalized K-flow;* $\mathfrak{N}_d$ *be the von Neumann subalgebra of* $\mathfrak{N}$ *generated by* $\{\mathfrak{N}_\lambda|\lambda\in\mathrm{Sp}_d(H^\sigma)\}$ *with* $\mathfrak{N}_\lambda=\{N\in\mathfrak{N}|\sigma(t)[N]=\exp(-i\lambda t)N$ *for all* $t\in\mathbb{R}\}$*;* $\alpha_d(\mathbb{R})$ *(resp.* ϕ*) be the restriction of* $\alpha(\mathbb{R})$ *(resp.* ϕ*) to* $\mathfrak{N}_d$*;* $\mathscr{A}_d=\mathfrak{N}_d\cap\mathscr{A}$*. Then* $\{\mathfrak{N}_d,\phi,\alpha_d(\mathbb{R}),\mathscr{A}_d\}$ *is a regular generalized K-flow.*

Proof. $\alpha(\mathbb{R})$ commutes with $\sigma(\mathbb{R})$ implies that each $\mathfrak{N}_\lambda$, and thus $\mathfrak{N}_d$ itself, is stable under $\alpha(\mathbb{R})$; hence $\alpha_d(\mathbb{R})$ is indeed a continuous group of automorphisms of $\mathfrak{N}_d$. $\mathscr{A}_t$ being $\sigma(\mathbb{R})$-stable we further have $\mathscr{E}(\mathfrak{N}_\lambda|\mathscr{A}_t)\subseteqq\mathfrak{N}_\lambda\cap\mathscr{A}_t$ and thus $\mathscr{E}(\mathfrak{N}_\lambda|\mathscr{A}_t)=\mathfrak{N}_\lambda\cap\mathscr{A}_t$. From Theorem 6(i) we see that the algebraic sum $\sum_\lambda\mathfrak{N}_\lambda$ is already u.w.-dense in $\mathfrak{N}_d$; hence the u.w.-continuity of $\mathscr{E}(\,.\,|\mathscr{A}_t)$ allows to pass from the above relation to $\mathscr{E}(\mathfrak{N}_d|\mathscr{A}_t)=\mathfrak{N}_d\cap\mathscr{A}_t=\alpha_d(t)[\mathscr{A}_d]$. We can therefore appeal, as in Scholium A, to the fact that $\{\mathscr{E}(\,.\,|\mathscr{A}_t)|t\in\mathbb{R}\}$ is an increasing martingale with $\bigvee_t\mathscr{A}_t=\mathfrak{N}$, and thus conclude that $\bigvee_t\alpha(t)[\mathscr{A}_d]=\mathfrak{N}_d$. Now $\mathfrak{N}_d$ is clearly $\sigma(\mathbb{R})$-stable so that the modular automorphism group of $\mathfrak{N}_d$ associated to ϕ is simply the restriction $\sigma_d(\mathbb{R})$ of $\sigma(\mathbb{R})$ to $\mathfrak{N}_d$; hence $\mathscr{A}_d$ is $\sigma_d(\mathbb{R})$-stable. We further have $\bigcap_t\alpha(t)[\mathscr{A}_d]=\mathfrak{N}_d\bigcap(\bigcap_t\mathscr{A}_t)=\mathbb{C}I$, and finally $\mathscr{A}_d\subseteqq\alpha(t)[\mathscr{A}_d]$ for all $t\in\mathbb{R}^+$. Hence $\{\mathfrak{N}_d,\phi,\alpha_d(\mathbb{R}),\mathscr{A}_d\}$ is indeed a generalized K-flow. Its regularity follows from Theorem 6(v) since the restriction of H^σ to $[\mathfrak{N}_d\Phi]=\sum_\lambda[\mathfrak{N}_\lambda\Phi]$ has clearly discrete spectrum. q.e.d.

Remarks. (i) This scholium can be looked upon as a strengthening of Rem (ii) under Theorem 8. (ii) The above proof would go through as well if one were to replace $\mathfrak{N}_d$ by $\mathfrak{N}_\varrho$ where ϱ is any subgroup of $\mathrm{Sp}_d(H^\sigma)$; in particular, we could have taken $\varrho=\{0\}$, in which case we would have been back in the situation covered by Theorem 8; or, if $\mathrm{Sp}_d(H^\sigma)\neq\{0\}$, we could have taken for any $\lambda\in\mathrm{Sp}_d(H^\sigma)$ with $\lambda\neq0$, $\varrho=\lambda\mathbb{Z}$ in which case ϕ would have been a periodic homogeneous state on $\mathfrak{N}_\varrho$.

II. Weak Reversibility

1. Definition. A generalized K-flow $\{\mathfrak{N},\phi,\alpha(\mathbb{R}),\mathscr{A}\}$ is said to be *reversible*, if there exists a von Neumann subalgebra $\hat{\mathscr{A}}$ of $\mathfrak{N}$ such that $\{\mathfrak{N},\phi,\hat{\alpha}(\mathbb{R}),\hat{\mathscr{A}}\}$ is a generalized K-flow with $\hat{\alpha}(\mathbb{R})$ defined by $\hat{\alpha}(t)\equiv\alpha(-t)$ for every $t\in\mathbb{R}$. A generalized K-flow is said to be *weakly reversible* if $\mathfrak{N}=\bigvee_t\mathscr{A}_t^c$ with $\mathscr{A}_t^c\equiv\mathfrak{N}\cap\mathscr{A}'_t$.

Remarks. (i) We shall construct, in Section III, generalized K-flows which are reversible and weakly reversible. (ii) If $\mathfrak{N}$ is abelian, then the flow is trivially weakly reversible. The question as to whether these "classical" K-flows are all reversible seems to have been a long-standing problem (for some elaboration on this point see Ex. III.1 below). (iii) On the opposite extreme, every weakly reversible generalized K-flow for which $\mathfrak{N}$ is a factor is also reversible. (iv) This is in particular true if we replace in (iii) the condition $\mathfrak{N}$ be a factor by the condition $\mathscr{A}$ be a factor (see for instance the remark under Scholium I.8.A). (v) If, in addition to the condition of remark (iv), we impose that the von Neumann algebra generated by $\mathscr{A}$ and $\mathscr{A}^c$ be stable under $\alpha(\mathbb{R})$, then $\mathfrak{N}$ is isomorphic (see [24c]), for each $t\in\mathbb{R}$, to $\mathscr{A}_t\otimes\mathscr{A}_t^c$; we shall construct, in Section III, an example of such a flow, whose structure is very much reminiscent of that of a Bernoulli flow. (vi) Coming back to the general meaning of the weak reversibility condition, we note that, whereas $\bigvee_t\mathscr{A}_t=\mathfrak{N}$ is to be seen as a condition on the "distant future", the condition $\bigvee_t\mathscr{A}_t^c=\mathfrak{N}$ appears as one involving the "remote past". In fact, in the same way as $\bigvee_t\mathscr{A}_t=\mathfrak{N}$ implies $\bigcap_t\mathscr{A}_t^c\subseteq\mathfrak{N}\cap\mathfrak{N}'$, the condition $\bigvee_t\mathscr{A}_t^c=\mathfrak{N}$ implies in turn that $\bigcap_t\mathscr{A}_t\subseteq\mathfrak{N}\cap\mathfrak{N}'$; hence one of our defining conditions for a generalized K-flow, namely $\bigcap_t A_t=\mathbb{C}I$, actually follows, when $\mathfrak{N}$ (or $\mathscr{A}$) is a factor from the weak reversibility condition $\bigvee_t\mathscr{A}_t^c=\mathfrak{N}$. This symmetry between the roles of $\mathscr{A}$ and $\mathscr{A}^c$ is further emphasized by the following remark. (vii) The condition that $s\geqq t$ imply $\mathscr{A}_s\supseteq\mathscr{A}_t$ imposes that $\mathscr{A}_s\cap\mathscr{A}_t'=\mathscr{E}(\mathscr{A}_t^c|\mathscr{A}_s)$ for all $s\geqq t$. From a martingale argument similar to that used repeatedly at the end of Section I, we see that $\mathscr{A}_t^c=\bigvee_s(\mathscr{A}_s\cap\mathscr{A}_t')$. It is then easily seen that the symmetric relation, namely $\mathscr{A}_t=\bigvee_s(\mathscr{A}_t\cap\mathscr{A}_s')$, implies the weak reversibility property $\bigvee_t\mathscr{A}_t^c=\mathfrak{N}$.

2. Theorem. *Let* $\{\mathfrak{N},\phi,\alpha(\mathbb{R}),\mathscr{A}\}$ *be a weakly reversible generalized K-flow;* $\mathfrak{A}$ *(resp.* $\mathfrak{C}$*) be the C^*-subalgebra of* $\mathfrak{N}$ *generated by* $\{\mathscr{A}_t|t\in\mathbb{R}\}$ *(resp. by* $\{\mathscr{A}_t^c|t\in\mathbb{R}\}$*). Then:* (i) *both* $\mathfrak{A}$ *and* $\mathfrak{C}$ *are strongly dense in* $\mathfrak{N}$; (ii) *for each* $A\in\mathfrak{A}$ *and each* $C\in\mathfrak{C}$: $\lim_{t\to\infty}\|[A,\alpha(t)[C]]\|=0$; (iii) *for every faithful normal state* ψ *on* $\mathfrak{N}$: $\mathrm{Sp}(H_\psi^\sigma)\supseteq\mathrm{Sp}(H^\sigma)$ *where* H_ψ^σ *is the generator of the continuous, one parameter, unitary group* $U_\psi^\sigma(\mathbb{R})$ *implementing the modular automorphism group* $\sigma_\psi(\mathbb{R})$ *canonically associated to* ψ.

Proof. Denoting by $\bigcup$ the set-theoretical union, we clearly have $\bigcup_t\mathscr{A}_t\subseteq\mathfrak{A}\subseteq\mathfrak{N}$, and $\bigcup_t\mathscr{A}_t^c\subseteq\mathfrak{C}\subseteq\mathfrak{N}$. $\bigcup_t\mathscr{A}_t$ is strongly dense in $\mathfrak{N}$ by definition of a generalized K-flow, and $\bigcup_t\mathscr{A}_t^c$ is strongly dense in $\mathfrak{N}$ by the condition of weak-reversibility. This proves (i). To prove (ii), we first note that for arbitrary, but fixed, $A\in\mathfrak{A}$, $C\in\mathfrak{C}$ and $\varepsilon>0$, one can find finite $x,y\in\mathbb{R}$ and $A_x\in\mathscr{A}_x$, $C_y\in\mathscr{A}_y^c$ such that $[A_x,\alpha(t)[C_y]]=0$ for all $t\geqq x-y$, $\|A-A_x\|\leqq\varepsilon/4\|C\|$, and $\|C-C_y\|\leqq\varepsilon/L$ with $L=4\|A\|+\varepsilon/\|C\|$. We have then for every $t\geqq x-y$: $\|[A,\alpha(t)[C]]\|\leqq\|[A-A_x,\alpha(t)[C]]\|+\|[A_x,\alpha(t)[C-C_y]]\|+\|[A_x,\alpha(t)[C_y]]\|\leqq 2(\varepsilon/4\|C\|)\times\|C\|+2(\|A\|+\varepsilon/4\|C\|)\times\varepsilon/L=\varepsilon$. Hence for every $A\in\mathfrak{A}$, $C\in\mathfrak{C}$ and $\varepsilon>0$, one can find a finite $T\in\mathbb{R}$ such that $\|[A,\alpha(t)[C]]\|\leqq\varepsilon$ for all $t\geqq T$. This proves (ii). To prove (iii) we draw from (i) and (ii) that $\alpha(\mathbb{R})$, with $\mathbb{R}$ equipped with its natural order, is a net of automorphisms of $\mathfrak{N}$ satisfying the conditions: (a) $\phi=\phi\circ\alpha(t)$ for all $t\in\mathbb{R}$, with ϕ faithful normal state on $\mathfrak{N}$; and (b) there exists a weakly dense sub*-algebra of $\mathfrak{N}$, namely $\mathfrak{C}$, which is strongly $\alpha(\mathbb{R})$-central, i.e. for every $C\in\mathfrak{C}$ there exists a weakly total self-adjoint subset of $\mathfrak{N}$, namely $\mathfrak{A}$, such that $[A,\alpha(t)[C]]\to 0$

strongly for every $A \in \mathfrak{A}$. The assumptions of Theorem 1 in [1] are thus satisfied, the conclusion of which is precisely part (iii) of the present theorem. q.e.d.

Remarks. The first two conclusions of the theorem strengthen considerably the η-abelianness property (iv) of the corollary to Theorem 5. Indeed a statement on averaged time behaviour, in the weak-operator topology, is now replaced by the convergence, in the norm topology, of point-wise limits in t. (ii) The third conclusion of the theorem shows in particular that when $\mathfrak{N}$ is a factor, its type as determined by Connes' invariant $S(\mathfrak{N})$, can be obtained directly from the spectrum of H^σ itself (see Corollary 3.2.5.d in [4]).

III. Examples

1. Classical Flow of Brownian Motion

A K-flow is defined, in classical probability theory (see for instance [2]) as an aggregate $(\Omega, \mu, T(\mathbb{R}), \zeta)$ where (Ω, μ) is a non-atomic Lebesgue space (i.e. is isomorphic to [0, 1] with Lebesgue measure); $T(\mathbb{R})$ is a group of automorphisms of (Ω, μ) such that for each measurable subset $X \subseteq \Omega$, the subset $\{(\omega, t) | T(t)[\omega] \in X\}$ is measurable in $\Omega \times \mathbb{R}$; and ζ is a σ-algebra of measurable subsets of Ω satisfying the following three conditions: (i) $\zeta \subseteq T(t)[\zeta]$ for every $t \in \mathbb{R}^+$; (ii) the σ-algebra generated by $\{T(t)[\zeta] | t \in \mathbb{R}\}$ coincides with the σ-algebra of all measurable subsets of Ω; and (iii) $\{\emptyset, \Omega\}$ is the largest σ-algebra, of measurable subsets of Ω, contained in all $T(t)[\zeta]$ with t running over $\mathbb{R}$.

With these ingredients we construct the following objects: $\mathfrak{H} = \mathscr{L}^2(\Omega, \mu)$; $\mathfrak{N}$ the image of $\mathscr{L}^\infty(\Omega, \mu)$ under $\pi : f \in \mathscr{L}^\infty(\Omega, \mu) \mapsto \pi(f) \in \mathfrak{B}(\mathfrak{H})$ defined by $\pi(f)\Psi(\omega) = f(\omega)\Psi(\omega)$ for every $\Psi \in \mathfrak{H}$; $\phi : \pi(f) \in \mathfrak{N} \mapsto \langle \phi; \pi(f) \rangle = \mu(f) = \int f(\omega) d\mu(\omega) \in \mathbb{C}$; for each $t \in \mathbb{R}$, $\alpha(t) : \pi(f) \in \mathfrak{N} \mapsto \alpha(t)[\pi(f)] = \pi(f \circ T(t)) \in \mathfrak{N}$; $\mathscr{A} = \{\pi(\chi_\xi) | \xi \in \zeta\}''$ where χ_ξ denotes the indicator function of the measurable subset ξ.

The aggregate $\{\mathfrak{N}, \phi, \alpha(\mathbb{R}), \mathscr{A}\}$ just constructed is clearly a regular generalized K-flow. This route can be treaded in the opposite direction starting from any generalized K-flow $\{\mathfrak{N}, \phi, \alpha(\mathbb{R}), \mathscr{A}\}$ under the necessary and sufficient condition that $\mathscr{A}$ be abelian (recall that throughout the paper $\mathfrak{H}$ is assumed to be separable!); we therefore refer to these particular dynamical systems as *classical K-flows.*

With the classical notation in hand, we now want to comment briefly on the reversibility question for classical K-flows of finite entropy (for the latter concept, see [2, 17] or Section IV below). For an arbitrary, but fixed $t_0 \in \mathbb{R}^+$, and with n running over $\mathbb{Z}$, we write $T_0(n)$ for $T(nt_0)$. From the K-flow properties of $\{\Omega, \mu, T(\mathbb{R}), \zeta\}$ we conclude immediately that: (i) $\zeta \subseteq T_0(n)[\zeta]$ for every $n \in \mathbb{Z}^+$; (ii) under $T_0(\mathbb{Z})$, ζ generates the σ-algebra of μ-measurable subsets of Ω; (iii) $\{\emptyset, \Omega\}$ is the largest σ-algebra of μ-measurable subsets of Ω, contained in all $T_0(n)[\zeta]$ with n running over $\mathbb{Z}$. This is to say that ζ induces on the discrete-time, dynamical system $\{\Omega, \mu, T_0(\mathbb{Z})\}$ the structure of a K-system. By Corollary 3 to Theorem 2 in [19] this system is reversible; i.e. there exists a partition $\tilde{\zeta}$, into μ-measurable subsets of Ω, which induces on $\{\Omega, \mu, \tilde{T}_0(\mathbb{Z})\}$ the structure of a K-system, with $\tilde{T}_0(n) = T_0(-n)$ for every $n \in \mathbb{Z}$. We next remark that the properties of the original K-flow imply that the "reversed" flow $\{\Omega, \mu, \hat{T}(\mathbb{R})\}$, with $\hat{T}(t) = T(-t)$ for every $t \in \mathbb{R}$, is ergodic, measurable and of finite entropy. The combination of the above two remarks with the results recently proven in [20] (see in particular Section IV there; some statements to the same end-effect were announced, without proof, in [18], see in particular Theorem 2 and Corollary to Theorem 3 there) points to the existence of a partition $\hat{\zeta}$ into μ-measurable subsets of Ω such that $\{\Omega, \mu, \hat{T}(\mathbb{R}), \hat{\zeta}\}$ becomes a K-flow; this supports thus the conjecture that every classical K-flow of finite entropy is reversible. The finiteness of the entropy actually is not necessary for reversibility, as can be seen, for instance from the following example. The question of whether there might still exist classical K-flows, with infinite entropy, which are not reversible is however not yet settled.

An inspiring example of a classical K-flow is provided by the flow of Brownian motion [11]. Let indeed $\Omega = \mathscr{S}(\mathbb{R})^*$ be the dual of the Schwartz space $\mathscr{S}(\mathbb{R})$ of real-valued, rapidely decreasing functions on $\mathbb{R}$; Z be the σ-algebra of subsets of Ω generated by the cylinder sets $\xi(f_1, \ldots, f_n; B) = \{\omega \in \Omega | (\langle \omega; f_1 \rangle, \ldots, \langle \omega; f_n \rangle) \in B\}$, where n runs over $\mathbb{Z}^+$, $\{f_1, \ldots, f_n\}$ runs over the collection of n-tuples of elements $f_i \in \mathscr{S}(\mathbb{R})$, and B runs over the Borel subsets of $\mathbb{R}^n$. Let further $C: f \in \mathscr{S}(\mathbb{R}) \mapsto \exp\{-\Theta \|f\|^2/4\} \in \mathbb{R}$ be the characteristic function of Brownian motion, where Θ is an arbitrary, but fixed, element of $\mathbb{R}$, with $\Theta \geqq 1$, and $\|.\|$ denotes the $\mathscr{L}^2$-norm on $\mathscr{S}(\mathbb{R})$. From Bochner-Minlos' theorem, we know that $\{C(f) = \int_\Omega \exp(-i\langle \omega; f \rangle) d\mu(\omega) | f \in \mathscr{S}(\mathbb{R})\}$ defines a unique measure μ on (Ω, Z). We next define, for every $t \in \mathbb{R}$, the mapping $S(t): \mathscr{S}(\mathbb{R}) \to \mathscr{S}(\mathbb{R})$ by: $S(t)[f](x) = f(x-t)$; and from this the group $T(\mathbb{R})$ of automorphisms of (Ω, μ) by: $\langle T(t)[\omega]; f \rangle = \langle \omega; S(t)[f] \rangle$. We finally define $\mathscr{S}_0 = \{f \in \mathscr{S}(\mathbb{R}) | \operatorname{supp}[f] \subset (-\infty, 0]\}$, and ζ the σ-subalgebra of Z generated by the cylinder sets $\xi(f_1, \ldots, f_n; B)$ with $f_i \in \mathscr{S}_0$. It is then easy to check that $\{\Omega, \mu, T(\mathbb{R}), \zeta\}$ is a K-flow, and the corresponding classical K-flow is reversible, and evidently weakly reversible. From our point of view, one of the most stricking interests of this classical example is that so much of its essential structure persists, mutatis mutandis, in the next examples, where $\mathfrak{N}$ will be the "opposite" of an abelian von Neumann algebra, namely a factor.

2. *Regular Generalized K-Flows, with $\mathfrak{N}$ Type III_λ-Factors*

For every $\lambda \in]0, 1[$, we now construct a regular generalized K-flow $\{\mathfrak{N}, \phi, \alpha(\mathbb{R}), \mathscr{A}\}$ for which $\mathfrak{N}$ is a factor of type III_λ. To this effect, we consider the functional $\hat{\phi}: f \in \mathscr{T} \mapsto \exp\{-\Theta \|f\|^2/4\} \in \mathbb{R}$ where $\mathscr{T} = \mathscr{L}^2_{\mathbb{C}}(\mathbb{R}, dx)$ and $\Theta = (1+\lambda)/(1-\lambda)$. We know (see for instance Theorem III.1.7 in [8]) that $\hat{\phi}$ determines uniquely, up to unitary equivalence: (a) a separable Hilbert space $\mathfrak{H}$; (b) a mapping $W: f \in \mathscr{T} \mapsto W(f) \in \mathscr{U}(\mathfrak{H})$ with $W(f)W(g) = W(f+g)\exp\{i\,\mathrm{Im}(f, g)/2\}$ and $W(\lambda f)$ weakly continuous in $\lambda \in \mathbb{R}$ for every $f \in \mathscr{T}$; (c) a vector $\Phi \in \mathfrak{H}$ such that $\hat{\phi}(f) = (\Phi, W(f)\Phi)$ for all $f \in \mathscr{T}$ and $\operatorname{Span}\{W(f) | f \in \mathscr{T}\}$ dense in $\mathfrak{H}$. Let $\mathfrak{N}$ be the von Neumann algebra on $\mathfrak{H}$ generated by $\{W(f) | f \in \mathscr{T}\}$, and ϕ be the state on $\mathfrak{N}$ defined by: $\langle \phi; N \rangle = (\Phi, N\Phi)$ for all $N \in \mathfrak{N}$. We next introduce the group $\alpha(\mathbb{R})$ of automorphisms of $\mathfrak{N}$ by defining, for each $t \in \mathbb{R}$, $\alpha(t)$ as the extension to $\mathfrak{N}$ of $\alpha(t)[W(f)] = W(u_t f)$ with $u_t \in \mathscr{U}(\mathscr{T})$ defined by $(u_t f)(x) = \exp(-ixt) f(x)$ for every $f \in \mathscr{T}$. For an arbitrary, but fixed $a \in \mathbb{R}^+$, we single out the vector $f_a \in \mathscr{T}: f_a(x) = a^{1/2}\,[(a^2+x^2)\pi]^{-1/2}$; we finally define $\mathscr{T}_0 = Cl.\ \operatorname{Span}\{u_s f_a | s \leqq 0\}$ and $\mathscr{A} = \{W(f) | f \in \mathscr{T}_0\}''$. To check that $\{\mathfrak{N}, \phi, \alpha(\mathbb{R}), \mathscr{A}\}$ is a generalized K-flow, and to obtain easily the special properties of this aggregate, it is convenient to remark that an explicite realization can be obtained as follows. Let $\{\mathfrak{H}_F, W_F, \Phi_F\}$ be the (irreducible!) Fock representation of the CCR on $\mathscr{T}$; and introduce $\xi_+=[1/(1-\lambda)]^{1/2}$, $\xi_-=[\lambda/(1-\lambda)]^{1/2}$. We can now identify $\mathfrak{H}$ with $\mathfrak{H}_F \otimes \mathfrak{H}_F$, Φ with $\Phi_F \otimes \Phi_F$ and $W(f)$ with $W_F(\xi_+ f)\otimes W_F(\xi_- f^*)$ where f^* is defined by $\tilde{f}^*(k)=\tilde{f}(k)^*$ ($\tilde{f}$ standing for the Fourier transform of f in $\mathscr{T}$).

Upon noticing now that ϕ satisfies the KMS condition w.r.t. the group $\sigma(\mathbb{R})$ of automorphisms of $\mathfrak{N}$ defined by $\sigma(t)[W(f)]=W(\lambda^{+it}f)$, we conclude that $\{\mathfrak{N}, \phi, \alpha(\mathbb{R}), \mathscr{A}\}$ is a generalized K-flow, that $\mathfrak{N}$ is a factor and that $\mathrm{Sp}(H^\sigma)=(\ln\lambda)\mathbb{Z}$. One further checks easily that $\mathscr{A}^c=\{W(f)\,|\,f\in\mathscr{T}_0^\perp\}''$, that $\mathscr{A}$ is a factor, that $\mathfrak{N}$ can be seen as $\mathscr{A}_t\otimes\mathscr{A}_t^c$, and that $\mathfrak{N}=\bigvee_t \alpha(t)[\mathscr{A}^c]$. From Remark (iii) under Definition II.1, one has thus that $\{\mathfrak{N}, \phi, \alpha(\mathbb{R}), \mathscr{A}\}$ is not only weakly-reversible, but also is reversible. By Remark (ii) under Theorem II.2, we have $S(\mathfrak{N})=\{\lambda^n|n\in\mathbb{Z}\}^-$, i.e. $\mathfrak{N}$ is a type III_λ-factor. Consequently ϕ is periodic and homogeneous (see Corollary under Theorem I.5) in the sense of Takesaki, so that (see Proposition I.7 in [25]) every maximal abelian von Neumann subalgebra of the centralizer $\mathfrak{N}_\phi$ of $\mathfrak{N}$ (w.r.t. ϕ) is already maximal abelian in $\mathfrak{N}$. Hence the generalized K-flow just constructed is regular.

From a mathematical point of view, this example shows first of all that the concept of generalized K-flows covers more than classical K-flows. This example was already noticed in [9b]; what is new here is that we now have noted its weak-reversibility, proved its reversibility, determined its factor type, and shown its regularity. This example thus establishes explicitly that the additional conditions of weak-reversibility, reversibility and regularity can be superimposed without contradictions to the structure of generalized K-flows, even when $\mathfrak{N}$ is a factor (i.e. the opposite of an abelian von Neumann algebra!).

From a physical point of view, the interest of this example is that it brings into contact the general scheme for non-equilibrium statistical mechanics outlined in the introduction and the theory of generalized K-flows. Specifically, let $\mathfrak{N}_S=\{W(zf_a)|z\in\mathbb{C}\}''$ and $\mathfrak{N}_R=\{W(f)|(f,f_a)=0\}''$. Then $\mathfrak{N}\cong\mathfrak{N}_S\otimes\mathfrak{N}_R$ and $\phi\cong\phi_S\otimes\phi_R$ with ϕ_S (resp. ϕ_R) the restriction of ϕ to $\mathfrak{N}_S$ (resp. $\mathfrak{N}_R$). Furthermore $\mathfrak{N}=\{\alpha(t)[\mathfrak{N}_S]|t\in\mathbb{R}\}''$. Finally, with $\mathscr{E}$ denoting the unique faithful, normal conditional expectation from $\mathfrak{N}$ onto $\mathfrak{N}_S$ with $\phi\circ\mathscr{E}=\phi:\mathscr{E}\alpha(t)\mathscr{E}=\gamma(t)\mathscr{E}$, $t\in\mathbb{R}^+$, defines a continuous semi-group $\gamma(\mathbb{R}^+)$ of completely positive, faithful maps from $\mathfrak{N}_S$ into itself with $\phi\circ\gamma(t)=\phi$ for all $t\in\mathbb{R}^+$. To be completely specific, for each $t\in\mathbb{R}^+$, $\gamma(t)$ is determined by:

$$\gamma(t)[W(zf_a)]=W(e^{-at}zf_a)\exp\{-\Theta|z|^2(1-e^{-2at})/4\}.$$

One thus sees that, at fixed $z\in\mathbb{C}$, each one of the abelian von Neumann subalgebras $\mathfrak{N}_z=\{W(\lambda zf_a)|\lambda\in\mathbb{R}\}''\subset\mathfrak{N}_S$ is stable under $\gamma(\mathbb{R}^+)$. Transposing then $\gamma(\mathbb{R}^+)$ to the predual $(\mathfrak{N}_z)_*$ of $\mathfrak{N}_z$, one finally checks that $\gamma(\mathbb{R}^+)_*$, restricted to $(\mathfrak{N}_z)_*$, is the integral solution of a diffusion equation in a harmonic well:

$$\{\partial_t - D_z[\partial_\xi^2 + V_z'(\xi)\partial_\xi + V_z''(\xi)]\}\psi_z(\xi,t)=0$$

with $\psi_z(.,t)\in\mathscr{L}^1(\mathbb{R},d\xi)$ defined by:

$$\langle\psi;\gamma(t)[W(\lambda zf_a)]\rangle=\int\psi_z(\xi,t)e^{-i\lambda\xi}d\xi$$

and

$$V_z(\xi)=\tfrac{1}{2}\Omega_z^2\xi^2\,;\,\Omega_z^2=2|z|^{-2}\Theta^{-1}\,;\,D_z=\tfrac{1}{2}a|z|^2\Theta\,.$$

(In these expressions Plank's constant $h/2\pi$ and the natural temperature $\beta=1/kT$ have been normalized to 1 for notational convenience.)

Hence $\{\mathfrak{N}_s, \phi_s, \gamma(\mathbb{R}^+)\}$ is a bona fide dissipative thermodynamical system the evolution of which is governed by recognizable transport equations. $\{\mathfrak{N}_R, \phi_R\}$ serves as a thermal bath for this thermodynamical system, in the sense described in the introduction.

Since all the details necessary to a complete physical interpretation of this model can be found in [7, 9b, f], let it only be said here that $\{\mathfrak{N}, \phi, \alpha(\mathbb{R})\}$ is obtained, in the interaction picture, as the thermodynamical limit, followed by the long-time weak-coupling limit of an infinite chain $\mathbb{Z}$ of one-dimensional harmonic oscillators; $\mathfrak{N}_S$ corresponds to a single harmonic oscillator in the chain, the remainder of which corresponding to $\mathfrak{N}_R$. More specifically, one starts from a finite assembly of harmonic oscillators interacting through a translation invariant Hamiltonian. The first of the above-mentioned limits extrapolates from the Hamiltonian mechanics of this finite system to that of the chain $\mathbb{Z}$. The reason for this limit is to remove to infinity the recurrences proper to finite systems. The second limit consists in taking the combined limits $\lambda\to 0$, $\tau\to\infty$, with $\lambda^2\tau=t$ fixed. It is in this rescaled time t that our evolution $\alpha(\mathbb{R})$ is expressed. The reason for this limit is to compress to $t=0$ the intermediary regimes developing on the microscopic time-scale; and thus to isolate the asymptotic character of the thermodynamical equations, in a time-scale adjusted to the coupling constant λ. Consequently, the resulting $\alpha(\mathbb{R})$ contains the cumulative long-time effects of the evolution, and it is thus different from the free evolution $\sigma(\mathbb{R})$. Consistency requires that the same limiting procedure be simultaneously applied to the equilibrium state ϕ_λ of the system; this state evidently only feels the "$\lambda\to 0$" part of this limit. This explains why ϕ happens indeed to be KMS with respect to the free evolution $\sigma(\mathbb{R})\neq\alpha(\mathbb{R})$.

3. A Regular Generalized K-Flow, with $\mathfrak{N}$ a Type II_1-Factor

Let $\{\mathfrak{N}, \phi, \alpha(\mathbb{R}), \mathscr{A}\}$ be any one of the reversible regular generalized K-flow constructed in (2) above. Since $\mathfrak{N}$ is a type III_λ factor, and since the flow is regular, the centralizer $\mathfrak{N}_\phi$ of $\mathfrak{N}$, with respect to ϕ, is non-trivial. From the physical point of view, we recall that $\mathfrak{N}_\phi$ is the algebra of the constants of the motion under the "free" evolution $\sigma(\mathbb{R})$. Let $\alpha_\phi(\mathbb{R})$ (resp. ϕ) be the restriction of $\alpha(\mathbb{R})$ (resp. ϕ) to $\mathfrak{N}_\phi$, and $\mathscr{A}_\phi$ be the von Neumann algebra $\mathscr{A}\cap\mathfrak{N}_\phi$. From Theorem I.8 we know that $\{\mathfrak{N}_\phi, \phi, \alpha_\phi(\mathbb{R}), \mathscr{A}_\phi\}$ is a regular generalized K-flow. From Theorem I.5, we conclude that the Hilbert space $[\mathfrak{N}_\phi\Phi]$ is infinite-dimensional. From Theorem II.2.iii, we next conclude (via Corollary 12 in [5] or Theorems 2.4.1 or 4.2.6 in [4], or Takesaki's analysis [25]) that $\mathfrak{N}_\phi$ is a factor, of type II_1 since ϕ is a faithful normal finite trace on $\mathfrak{N}_\phi$, and $[\mathfrak{N}_\phi\Phi]$ is infinite dimensional. We have thus indeed obtained a regular generalized K-flow with $\mathfrak{N}$ a type II_1-factor. We furthermore remark that $\{\mathfrak{N}_\phi, \phi, \alpha_\phi(\mathbb{R}), \mathscr{A}_\phi\}$ inherits, from our initial flow $\{\mathfrak{N}, \phi, \alpha(\mathbb{R}), \mathscr{A}\}$, the property of being weakly reversible and thus of being reversible since $\mathfrak{N}_\phi$ is a factor.

4. Regular Generalized K-Flows, with $\mathfrak{N}$ Type III_1-Factors

For every $\omega \in \mathbb{R}^+$ irrational we construct a regular generalized K-flow with $\mathfrak{N}$ a type III_1-factor, depending on ω. To this effect, let $\{\mathfrak{N}_i, \phi_i, \alpha_i(\mathbb{R}), \mathscr{A}_i\}$ $(i=1,2)$ be two regular generalized K-flow of the type constructed in (2) above, with $\omega_i = -\ln \lambda_i$ $(i=1,2)$ and $\omega_1/\omega_2 = \omega$. Form now the von Neumann algebra $\mathfrak{N} = \mathfrak{N}_1 \otimes \mathfrak{N}_2$ acting on the separable Hilbert space $\mathfrak{H} = \mathfrak{H}_1 \otimes \mathfrak{H}_2$. Since $\mathfrak{N}_i$ $(i=1,2)$ are factors, so is $\mathfrak{N}$. Let ϕ be the faithful normal state $\phi_1 \otimes \phi_2$ on $\mathfrak{N}$ corresponding to the cyclic and separating vector $\Phi = \Phi_1 \otimes \Phi_2$. Let further $\alpha(\mathbb{R})$ be the continuous group of automorphisms of $\mathfrak{N}$ defined by $\alpha(t) = \alpha_1(t) \otimes \alpha_2(t)$; clearly $\phi \circ \alpha(t) = \phi$ for all $t \in \mathbb{R}$. Let finally $\mathscr{A} = \mathscr{A}_1 \otimes \mathscr{A}_2$. From the weak reversibility of the component K-flows, we have: $\bigvee_t \alpha(t)[\mathscr{A}^c] = \mathfrak{N} = \bigvee_t \alpha(t)[\mathscr{A}]$ and $\bigcap_t \alpha(t)[\mathscr{A}^c] = \mathbb{C}I = \bigcap_t \alpha(t)[\mathscr{A}]$. Notice furthermore that $\sigma(\mathbb{R})$, defined for each $t \in \mathbb{R}$ by $\sigma(t) = \sigma_1(t) \otimes \sigma_2(t)$, is the modular group of automorphisms of $\mathfrak{N}$ for ϕ. We have thus that $\{\mathfrak{N}, \phi, \alpha(\mathbb{R}), \mathscr{A}\}$ is a weakly reversible, generalized K-flow, with $\mathfrak{N}$ a factor. Clearly $H^\sigma = H_1^\sigma \otimes I + I \otimes H_2^\sigma$ so that H^σ is diagonizable, with eigenvalue spectrum $\{k_1\omega_1 + k_2\omega_2 \,|\, k_1, k_2 \in \mathbb{Z}\}$ dense in $\mathbb{R}$ since ω is taken to be irrational. Hence $\mathrm{Sp}(H^\sigma) = \mathbb{R}$. From Theorem II.2 $S(\mathfrak{N}) = \mathbb{R}^+$, which is to say that $\mathfrak{N}$ is of type III_1. Moreover, the diagonalizability of H^σ also implies, by Theorem I.6.v, that the flow is regular. We have thus indeed constructed a weakly reversible, reversible, regular generalized K-flow where $\mathfrak{N}$ is a type III_1 factor, depending on ω.

The physical interpretation of this flow follows along the same lines as in (2) above.

5. A Singular Generalized K-Flow, with $\mathfrak{N}$ a Type III_1-Factor

Let $\mathscr{T} = \mathscr{L}^2(\mathbb{R}^2, dx_1 dx_2)$, and let $u_i(\mathbb{R})$ $(i=1,2)$ be the continuous, one-parameter groups of unitary operators on $\mathscr{T}$ defined respectively by $u_1(t) f(x_1, x_2) = f(x_1 - t, x_2)$ and $u_2(t) f(x_1, x_2) = f(x_1, x_2 - t)$ for all $f \in \mathscr{T}$. One has thus $\lim_{t\to\infty} (f, u_i(t) g) = 0$ for every $f, g \in \mathscr{T}$, and $i = 1, 2$. With h_i denoting the generator of $u_i(\mathbb{R})$ we clearly have:

$$0 \leqq k_i \equiv \exp(-h_i)[1 + \exp(-h_i)]^{-1} \leqq 1 .$$

Let now $\tilde{\mathfrak{A}}$ be the C^*-algebra of the canonical anticommutation relations on $\mathscr{T}$; $\tilde{\alpha}_i(\mathbb{R})$ be the continuous group of automorphisms of $\tilde{\mathfrak{A}}$ defined by $\tilde{\alpha}_i(t)[a(f)] = a(u_i(t) f)$; in the sequel we will write $\tilde{\sigma}(\mathbb{R})$ for $\tilde{\alpha}_1(\mathbb{R})$, and $\tilde{\alpha}(\mathbb{R})$ for $\tilde{\alpha}_2(\mathbb{R})$. Let further $\tilde{\phi}$ be the state on $\tilde{\mathfrak{A}}$ defined by $\langle \tilde{\phi}; I \rangle = 1$ and:

$$\langle \tilde{\phi}; a^*(f_n) \ldots a^*(f_1) a(g_1) \ldots a(g_m) \rangle = \delta_{nm} \operatorname{Det} A$$

where A is the $n \times n$ matrix $A_{ij} = (f_i, k_1 g_j)$.

This is a particular case of the situation studied in [10], and we thus have: $\tilde{\phi}$ is a gauge-invariant, generalized free state on $\tilde{\mathfrak{A}}$; $\tilde{\mathfrak{N}} = \pi_\phi(\tilde{\mathfrak{A}})''$ is a factor; $\tilde{\sigma}(\mathbb{R})$ is the modular automorphism of $\tilde{\mathfrak{N}}$ associated to $\tilde{\phi}$; the centralizer $\tilde{\mathfrak{N}}_\phi$ of $\tilde{\mathfrak{N}}$ is trivial; for $i = 1, 2$, every $\tilde{\psi} \in \tilde{\mathfrak{A}}^*$ and every $A, B \in \tilde{\mathfrak{A}}$, $\lim_{t\to\infty} \langle \tilde{\psi}; [\tilde{\alpha}_i(t)[A], B] \rangle = 0$; and $\tilde{\phi} \circ \tilde{\alpha}(t) = \tilde{\phi}$ for all $t \in \mathbb{R}$.

In order to get a flow involving an algebra of observables rather than fields, we introduce $\tilde{\tau} \in \mathrm{Aut}(\tilde{\mathfrak{A}})$ defined by $\tilde{\tau}[a(f)] = \omega a(f)$, $\omega \in \mathbb{C}_1$, for all $f \in \mathscr{T}$. Let then $\mathfrak{A} = \{A \in \tilde{\mathfrak{A}} \,|\, \tilde{\tau}[A] = A\}$. Since $\tilde{\tau}$ commute with $\tilde{\alpha}_i(\mathbb{R})$, $\tilde{\mathfrak{M}} \equiv \{\pi_\phi(\mathfrak{A})\}''$ is $\tilde{\alpha}_i(\mathbb{R})$-stable.

This implies:

(i) there exists a unique faithful normal conditional expectation $\tilde{\mathscr{E}}$ from $\tilde{\mathfrak{N}}$ onto $\tilde{\mathfrak{M}}$ such that $\tilde{\phi}=\tilde{\phi}\circ\tilde{\mathscr{E}}$; (ii) the restriction of $\tilde{\alpha}_t(\mathbb{R})$ to $\tilde{\mathfrak{M}}$ is again a continuous automorphism group of this algebra. Let now $\mathfrak{N}$ be the restriction of $\tilde{\mathfrak{M}}$ to $\mathfrak{H}=[\tilde{\mathfrak{M}}\Phi]$; we denote by $\alpha(\mathbb{R})$ [resp. $\sigma(R)$ and ϕ] the restriction of $\tilde{\alpha}(\mathbb{R})$ [resp. $\tilde{\sigma}(\mathbb{R})$ and $\tilde{\phi}$] to $\mathfrak{N}$. Clearly ϕ is KMS on $\mathfrak{N}$ w.r.t. $\sigma(\mathbb{R})$, and $\phi\circ\alpha(t)=\phi$ for all $t\in\mathbb{R}$; moreover $\tilde{\mathfrak{N}}_\phi=\mathbb{C}I$ implies $\mathfrak{N}_\phi=\mathbb{C}I$, and thus $\mathfrak{N}$ is a factor. A straightforward computation shows that $\mathfrak{N}=\mathbb{C}I$ would imply $k_1=0$, so that $\mathfrak{N}$ is non-trivial.

To show that $\{\mathfrak{N},\phi,\alpha(\mathbb{R})\}$ supports a generalized K-flow, we still have to identify the refining subalgebra $\mathscr{A}$. To this effect we define $\mathscr{T}_0$ as the closure in $\mathscr{T}$ of $\mathscr{S}_0=\{f\in\mathscr{S}(\mathbb{R}^2)|\operatorname{supp}[f]\subset\mathbb{R}\times(-\infty,0]\}$; and $\tilde{\mathfrak{A}}_0$ the C^*-subalgebra of $\tilde{\mathfrak{A}}$ generated by $\{a(f)|f\in\mathscr{T}_0\}$; we then form $\tilde{\mathfrak{M}}_0=\pi_\phi(\tilde{\mathfrak{A}}_0)''\cap\tilde{\mathfrak{M}}=\tilde{\mathscr{E}}(\pi_\phi(\tilde{\mathfrak{A}}_0)''|\tilde{\mathfrak{M}})$ and we define $\mathscr{A}$ as the restriction of $\tilde{\mathfrak{M}}_0$ to $\mathfrak{H}$. Clearly $\mathscr{A}$ is $\sigma(\mathbb{R})$-stable, $\mathscr{A}\subseteqq\alpha(t)[\mathscr{A}]$ for all $t\in\mathbb{R}^+$, and $\bigvee_t\alpha(t)[\mathscr{A}]=\mathfrak{N}$.

We still have to prove that $\bigcap_t\alpha(t)[\mathscr{A}]=\mathbb{C}I$. This is done as follows. Let $\mathscr{F}$ be the set of all regions of $\mathbb{R}^2$ of the form $\Omega=\mathbb{R}\times\Omega_1$ with $\Omega_1\subset\mathbb{R}$ compact; $\mathscr{F}$ becomes a directed set under the usual inclusion. For every $\Omega\in\mathscr{F}$ we now form the CAR C^*-algebra $\tilde{\mathfrak{A}}(\Omega)$ on $\mathscr{L}^2(\Omega,dx_1dx_2)\subset\mathscr{T}$; and the local algebra of observables $\mathfrak{A}(\Omega)=\tilde{\mathfrak{A}}(\Omega)\cap\mathfrak{A}$. Clearly $\mathfrak{A}$ is the C^*-inductive limit of $\{\mathfrak{A}(\Omega)|\Omega\in\mathscr{F}\}$. We next consider, for every $\Omega\in\mathscr{F}$:

$$\mathfrak{A}(\Omega')={}^n\overline{\bigcup\mathfrak{A}(\Omega'')}$$

where the union $\bigcup$ is taken over all $\Omega''\in\mathscr{F}$ disjoint from Ω. Define now $\mathfrak{B}(\Omega)=\pi_\phi(\mathfrak{A}(\Omega'))''|\mathfrak{H}$, and

$$\mathfrak{B}=\bigcap_{\Omega\in\mathscr{F}}\mathfrak{B}(\Omega).$$

Clearly $\bigcap_t\alpha(t)[\mathscr{A}]\subseteqq\mathfrak{B}\subseteqq\mathfrak{N}\cap\mathfrak{N}'$, the second inclusion steming directly from the local commutativity in $\mathfrak{A}$. Since $\mathfrak{N}$ is a factor $\bigcap_t\alpha(t)[\mathscr{A}]=\mathbb{C}I$. Hence $\{\mathfrak{N},\phi,\alpha(\mathbb{R}),\mathscr{A}\}$ is a generalized K-flow. It is singular, since $\mathfrak{N}_\phi=\mathbb{C}I$; and, see for instance Theorem I.6, $\mathfrak{N}$ is a type III_1-factor.

We should remark that this K-flow is also weakly reversible; and thus reversible, since $\mathfrak{N}$ is a factor. Hence this example shows not only that singularity is compatible with the structure of generalized K-flows, but also that it is compatible with the additional conditions of weak reversibility and reversibility.

To conclude this section, we note that all the generalized K-flows we constructed are mutually non-isomorphic.

IV. Dynamical Entropy

1. For a classical system $\{\Omega,\mu,T(\mathbb{R})\}$ the concept of dynamical entropy can be approached in two ways which are conceptually different but nevertheless mathematically equivalent. Whereas in both approaches the dynamical entropy $H(T)$ is defined as the sup of $H(T,\zeta)$ over all finite measurable partitions ζ of Ω, the difference comes in the definition of $H(T,\zeta)$, the entropy of the partition ζ under the flow $T(\mathbb{R})$.

In the first approach, one defines:

$$H(T,\zeta)=\lim_{n\to\infty}H(\zeta\bigvee T[\zeta]\bigvee\ldots\bigvee T^n[\zeta])/(n+1)$$

where T is $T(1)$; $\zeta\bigvee\ldots\bigvee T^n[\zeta]$ is the smallest measurable partition of Ω which refines all $T^k[\zeta]$ $(0\leq k\leq n)$; and for any finite partition ζ of Ω into mutually disjoint measurable subsets ξ_k, $H(\zeta)=\sum_k h[\mu(\xi_k)]$ where h is the continuous function $h: x\in[0,1]\mapsto -x\log x\in\mathbb{R}^+$.

In the second approach, one defines:

$$H(T,\zeta)=\lim_{n\to\infty}H(\zeta|T^{-1}[\zeta]\bigvee\ldots\bigvee T^{-n}[\zeta])$$

where $H(\zeta_1|\zeta_2)$ is the conditional entropy of ζ_1 with respect to ζ_2. Since one can prove (see for instance [2, 17]) that both definitions give the same value for $H(T,\zeta)$, it is only a matter of taste as to which definition one prefers, as long as one deals with a classical dynamical system. Notice also (see for instance [17]) that one can replace in the definition of $H(T)$ the sup over all finite measurable partitions by the sup over all countable measurable partitions ζ with $H(\zeta)<\infty$.

When it comes to generalize the concept of dynamical entropy to a quantum dynamical system $\{\mathfrak{N},\phi,\alpha(\mathbb{R})]$ two difficulties have to be mastered.

The first difficulty is that if ζ denotes an arbitrary (finite) partition of the identity into mutually orthogonal projectors $F_k\in\mathfrak{N}$, then the measurement of ζ can perturb the state ϕ, thus introducing a stochastic element which does not pertain to the time evolution. This effect will be eliminated by restricting the class of admissible partitions ζ, so that the non-vanishing of $H(\alpha)$ will indeed reflect a stochastic behaviour in $\alpha(\mathbb{R})$ itself.

The second difficulty is that ζ and $\alpha(t)[\zeta]$ might not commute, so that the question comes as to what object should take the place of $\bigvee$, the refinement which appears in the above two definitions of $H(T,\zeta)$. Connes and Størmer [6] succeeded in extending the first definition in such a manner that it becomes useful for the classification of Bernoulli shifts on the hyperfinite II_1-factor. In [9c] we were concerned with extending the second definition of $H(T,\zeta)$, which we also consider here as giving a more intuitive feeling of what kind of stochastic behaviour is involved when a quantum dynamical system has strictly positive entropy. We shall compute this entropy for some generalized K-flows at the end of this section.

2. Let $\mathfrak{N}$ be a von Neumann algebra acting on a separable Hilbert space $\mathfrak{H}$; ϕ be a normal state on $\mathfrak{N}$; and ζ be a partition of the identity on $\mathfrak{H}$ into mutually orthogonal projectors $F_k\in\mathfrak{N}$. Following von Neumann [16], we describe the effect of the measurement of ζ as changing ϕ into the state $\zeta[\phi]=\sum_k F_k\phi F_k$, where for every $X\in\mathfrak{N}$ we denote by $X\phi X$ the positive linear mapping $X\phi X$: $N\in\mathfrak{N}\mapsto\langle\phi; X^*NX\rangle\in\mathbb{C}$. If ϕ is faithful, we write $\lambda_k=\langle\phi;F_k\rangle$ and $\phi_k=\lambda_k^{-1}F_k\phi F_k$; we note then that $\lambda_k>0$, $\sum_k\lambda_k=1$, and ϕ_k can be considered as either a normal state on $\mathfrak{N}$, or as a faithful normal state on the reduced von Neumann algebra $F_k\mathfrak{N}F_k=\mathfrak{N}_k$ acting on $F_k\mathfrak{H}$. The effect of the measurement of ζ thus appears as a "channeling" operation which changes ϕ into the mixture $\zeta[\phi]=\sum_k\lambda_k\phi_k$. Clearly in the classical case, where $\mathfrak{N}$ is abelian, $\zeta[\phi]=\phi$ for all ζ. In the non-commutative case, we still have the following easy result which we record here for future reference in the sequel.

Lemma. *Let $\{\mathfrak{N}, \phi, \alpha(\mathbb{R})\}$ be a dynamical system where: $\mathfrak{N}$ is a von Neumann algebra acting on a separable Hilbert space $\mathfrak{H}$; ϕ is a faithful normal state on $\mathfrak{N}$; and $\alpha(\mathbb{R})$ is a group of automorphisms of $\mathfrak{N}$ such that $\phi\circ\alpha(t)=\phi$ for all $t\in\mathbb{R}$. Let further ζ be a partition of the identity on $\mathfrak{H}$ into mutually orthogonal projections $F_k\in\mathfrak{N}$. Let finally $\mathfrak{N}_\phi$ be the von Neumann algebra of fixed points of $\mathfrak{N}$ under the modular group $\sigma(\mathbb{R})$ canonically associated to ϕ. Then the following four conditions are equivalent:* (i) $\zeta[\phi]=\phi$; (ii) $\zeta\subset\mathfrak{N}_\phi$; (iii) $\alpha(t)[\zeta]\subset\mathfrak{N}_\phi$ *for every* $t\in\mathbb{R}$; (iv) $\{\alpha(t)[\zeta]\,|\,t\in\mathbb{R}\}''\subseteqq\mathfrak{N}_\phi$.

Proof. Upon feeding NF_k and F_kN into (i), one checks that (i) implies $\langle\phi;[N,F_k]\rangle=0$, i.e. $F_k\in\mathfrak{N}_\phi$, which is (ii). Since $\phi\circ\alpha(t)=\phi$ for every $t\in\mathbb{R}$, $\alpha(\mathbb{R})$ commutes with $\sigma(\mathbb{R})$ and thus $\mathfrak{N}_\phi$ is $\alpha(\mathbb{R})$-stable; hence (ii) implies (iii). Clearly (iii) and (iv) are equivalent, and (iii) implies (ii) as a particular case. Since $F_k\in\mathfrak{N}_\phi$ implies $\langle F_k\phi F_k;N\rangle=\langle\phi;F_kN\rangle$ for all $N\in\mathfrak{N}$, (ii) implies (i). q.e.d.

Definition. Let $\{\mathfrak{N}, \phi, \alpha(\mathbb{R})\}$ be a dynamical system as in the Lemma. A partition ζ of the identity on $\mathfrak{H}$ into mutual orthogonal projectors $F_k\in\mathfrak{N}$ is said to be *admissible*, if it satisfies any, and thus all, of the four conditions of the lemma. By extension, a von Neumann subalgebra $\mathfrak{M}\subseteqq\mathfrak{N}$ is said to be *admissible* if $\mathfrak{M}\subseteqq\mathfrak{N}_\phi$. We denote by $\boldsymbol{Z}$ the set of all admissible partitions $\zeta\subset\mathfrak{N}$, and by $\boldsymbol{M}$ the set of all admissible von Neumann subalgebras of $\mathfrak{N}$.

3. Let now $\mathfrak{M}$ be a von Neumann algebra acting on a separable Hilbert space $\mathfrak{H}$, and suppose further that $\mathfrak{M}$ admits a normalized, faithful normal trace ϕ. Let ζ be a partition of the identity on $\mathfrak{H}$ into mutually orthogonal projectors $F_k\in\mathfrak{M}$. To define $H_\phi(\zeta)$, the entropy of ζ with respect to ϕ, we can ignore $\mathfrak{M}$ and simply restrict our attention to the abelian von Neumann algebra ζ'' generated by the F_k's; we are thus in a classical situation and can therefore appeal to Khinchin's theorem [13] to conclude that the only reasonable definition of $H_\phi(\zeta)$ is $\sum_k h[\langle\phi;F_k\rangle]$ (compare with IV.1 above).

Suppose now for an instant that $\mathfrak{M}$ is moreover finite-dimensional. For any two partitions ζ_1 and ζ_2 of the identity into minimal projectors of $\mathfrak{M}$, there exists U unitary in $\mathfrak{M}$ such that $\mathrm{Ad}_U[\zeta_1]=\zeta_2$. Since moreover $\phi\circ\mathrm{Ad}_U=\phi$, we have $H_\phi(\zeta_1)=H_\phi(\zeta_2)$. We can therefore define the entropy of $\mathfrak{M}$ w.r.t. ϕ as $H_\phi(\mathfrak{M})=H_\phi(\zeta)$ where ζ is any partition of the identity into minimal projectors in $\mathfrak{M}$. Furthermore, since every partition ζ of the identity into mutually orthogonal projectors in $\mathfrak{M}$ can be refined into a partition ζ_m of the identity into minimal projectors in $\mathfrak{M}$, one checks easily that $H_\phi(\mathfrak{M})=H_\phi(\zeta)+\sum_k\lambda_kH_{\phi_k}(\mathfrak{M}_k)$ with λ_k, ϕ_k and $\mathfrak{M}_k$ defined from $\zeta=\{F_k\}$ as in IV.2 above. Hence $\sum_k\lambda_kH_{\phi_k}(\mathfrak{M}_k)$ appears indeed as the residual entropy of $\mathfrak{M}$ w.r.t. ϕ, *after* the measurement of ζ has been performed, i.e. it is the entropy of $\mathfrak{M}$ conditioned by ζ w.r.t. ϕ. This remark might serve as a further motivation for the following definition.

Definition. Let ϕ be a normalized, faithful normal trace on a finite von Neumann algebra $\mathfrak{M}$ acting on a separable Hilbert space $\mathfrak{H}$. Let $\zeta_1=\{F_k\}$ be a partition of the identity on $\mathfrak{H}$ into mutually orthogonal projectors $F_k\in\mathfrak{M}$; and let $\zeta=\{G_j\}$ be similarly defined. We call $H_\phi(\zeta|\zeta_1)=\sum_k\lambda_kH_{\phi_k}(\zeta)$ the *entropy of ζ conditioned by ζ_1 with respect to ϕ.*

Remarks. (i) This definition does evidently not require that ζ and ζ_1 commute. (ii) Upon noticing that $\mathscr{E}(.|\zeta_1): M\in\mathfrak{M}\mapsto\sum_k\langle\phi_k; M\rangle F_k\in\zeta_1''$ is the unique faithful normal conditional expectation from $\mathfrak{M}$ onto ζ_1'' with $\phi\circ\mathscr{E}(.|\zeta_1)=\phi$, we have:

$$H_\phi(\zeta|\zeta_1)=\sum_j\langle\phi; h[\mathscr{E}(G_j|\zeta_1)]\rangle;$$

this expression coincides with the conditional entropy $H_\phi(\zeta|\zeta_1'')$ introduced in [9c]. (iii) In particular, $H_\phi(\zeta|\zeta_1)$ reduces to the classical expression when ζ and ζ_1 do commute. (iv) In [6] a conditional entropy $H_\phi(\mathfrak{X}|\mathfrak{X}_1)$ is defined for $\mathfrak{X}$ and $\mathfrak{X}_1$ finite-dimensional subalgebras of $\mathfrak{M}$, namely:

$$\mathrm{Sup}_{x\in S}\{\sum_k(\langle\phi; h[\mathscr{E}(x_k|\mathfrak{X})]\rangle-\langle\phi; h[\mathscr{E}(x_k|\mathfrak{X}_1)]\rangle)\}$$

where S is the set of all finite families $x=\{x_k\}$ of positive elements of $\mathfrak{M}$ with $\sum_k x_k=I$; one verifies that one has, in case ζ and ζ_1 are finite partitions, $H_\phi(\zeta|\zeta_1)=H_\phi(\zeta''|\zeta_1'')$.

Lemma. *Let ϕ be a normalized, faithful normal trace on a finite von Neumann algebra $\mathfrak{M}$ acting on a separable Hilbert space $\mathfrak{H}$. Let ζ be a partition of the identity on $\mathfrak{H}$ into mutually orthogonal projectors $F_k\in\mathfrak{M}$; and let ζ_0, ζ_1, and ζ_2 be similarly defined. Then:* (i) $H_\phi(\zeta|\zeta_1)\geqq 0$*;* (ii) *a necessary and sufficient condition for $H_\phi(\zeta|\zeta_1)$ to vanish is that $\zeta\subseteqq\zeta_1$;* (iii) $\zeta\subseteqq\zeta_0$ *implies* $H_\phi(\zeta|\zeta_1)\leqq H_\phi(\zeta_0|\zeta_1)$*;* (iv) $\zeta_2\subseteqq\zeta_1$ *implies* $H_\phi(\zeta|\zeta_1)\leqq H_\phi(\zeta|\zeta_2)$.

Proof. (i) follows directly from the definition. To prove the sufficiency in (ii), notice that $\zeta\subseteqq\zeta_1$ means $\zeta''\subseteqq\zeta_1''$ which implies, for every $F_k\in\zeta$, that $\mathscr{E}(F_k|\zeta_1)=F_k$ and thus $h[\mathscr{E}(F_k|\zeta_1)]=0$; by Rem (ii) above, this indeed implies $H_\phi(\zeta|\zeta_1)=0$. Conversely $0=H_\phi(\zeta|\zeta_1)=\sum_{k,j}\langle\phi; F_k\rangle h[\langle\phi_k; G_j\rangle]$ implies, since ϕ is faithful, that $h[\langle\phi_k; G_j\rangle]=0$ for every k, j; consequently $\langle\phi_k; G_j\rangle$ is either 0 or 1. Since ϕ_k is normal and $\sum_j G_j=I$, for each k there exists exactly one j, say $j(k)$, such that $\langle\phi_k; G_j\rangle=1$. This implies $\langle\phi; F_k[I-G_{j(k)}]F_k\rangle=0$; since ϕ is faithful, we thus have $F_k[I-G_{j(k)}]F_k=0$, i.e. $F_k\subseteqq G_{j(k)}$. Since ζ and ζ_1 are partitions of the identity, this implies indeed $\zeta\subseteqq\zeta_1$, thus proving (ii). To prove (iii), it is sufficient to prove that, for every normal state ψ on $\mathfrak{M}$, $\zeta\subseteqq\zeta_0$ implies $H_\psi(\zeta)\leqq H_\psi(\zeta_0)$. Since however $\zeta\subseteqq\zeta_0$ and ζ_0'' abelian, we can restrict ψ to ζ_0'' and be in a classical situation where this result is well-known (see for instance [17]) to follow directly from the concavity of h, thus establishing (iii). Finally (iv) follows from the classical Jensen's inequality (see for instance [17]); by Rem (ii) above we have indeed:

$$\begin{aligned} H_\phi(\zeta|\zeta_1)&=\sum_k\langle\phi; h[\mathscr{E}(F_k|\zeta_1)]\rangle\\ &=\sum_k\langle\phi; \mathscr{E}(h[\mathscr{E}(F_k|\zeta_1)]|\zeta_2)\rangle\\ &\leqq\sum_k\langle\phi; h[\mathscr{E}(\mathscr{E}(F_k|\zeta_1)|\zeta_2)]\rangle\\ &=\sum_k\langle\phi; h[\mathscr{E}(F_k|\zeta_2)]\rangle=H_\phi(\zeta|\zeta_2)\,. \end{aligned}$$

This concludes the proof of the lemma.

Remark. The four conclusions of this lemma confirm, if needed, the interpretation of $H_\phi(\zeta|\zeta_1)$ as a conditional entropy which indeed quantifies the information gained by measuring ζ once ζ_1 has been measured.

4. Keeping in mind that measurements in quantum mechanics are classical operations, and thus are performed via partitions, we propose the following definition.

Definition. Let ϕ be a normalized, faithful normal trace on a finite von Neumann algebra $\mathfrak{M}$ acting on a separable Hilbert space $\mathfrak{H}$; $\mathfrak{A}$ be a von Neumann subalgebra of $\mathfrak{M}$; and ζ be a partition of the identity on $\mathfrak{H}$ into mutually orthogonal projectors in $\mathfrak{M}$. We define the *entropy of ζ conditioned by $\mathfrak{A}$ with respect to ϕ* as:

$$H_\phi(\zeta|\mathfrak{A}) = \mathrm{Inf}_{\zeta_1 \subset \mathfrak{A}} H_\phi(\zeta|\zeta_1).$$

Remarks. (i) The consistency relation $H_\phi(\zeta|\zeta_1) = H_\phi(\zeta|\zeta_1'')$ follows from Lemma IV.3.iii. (ii) If $\mathfrak{A}$ is abelian and $\{\zeta_\alpha|\alpha \in D\}$ is an increasing family of finite partitions of the identity in $\mathfrak{A}$, with $\{\zeta_\alpha|\alpha \in D\}'' = \mathfrak{A}$, then a simple increasing martingale argument shows that $\lim_\alpha H_\phi(\zeta|\zeta_\alpha) = \sum_k \langle \phi; h[\mathscr{E}(F_k|\mathfrak{A})]\rangle$. This establishes in particular the consistency of the present approach with that followed in [9c]. (iii) The operator concavity and continuity of h lead easily to an extension of Jensen's inequality to the non-commutative case, namely $(h \circ \mathscr{E} - \mathscr{E} \circ h)$ is a positive map. This allows to generalize the argument of Lemma IV.3 to give, for a general von Neumann subalgebra $\mathfrak{A}$ of $\mathfrak{M}$: $H_\phi(\zeta|\mathfrak{A}) \geqq \sum_k \langle \phi; h[\mathscr{E}(F_k|\mathfrak{A})]\rangle$. (iv) When $\mathfrak{N}$ is a von Neumann algebra acting in a separable Hilbert space and admitting a faithful normal state ϕ, the above definition extends H_ϕ from $\boldsymbol{Z} \times \boldsymbol{Z}$ to $\boldsymbol{Z} \times \boldsymbol{M}$ with $\boldsymbol{Z}$ and $\boldsymbol{M}$ defined in IV.2. The mapping $H_\phi : (\zeta, \mathfrak{A}) \in \boldsymbol{Z} \times \boldsymbol{M} \mapsto H_\phi(\zeta|\mathfrak{A}) \in \mathbb{R}^+$ so defined satisfies all the conclusions of Lemma IV.3 where we now replace ζ_1 and ζ_2 by arbitrary (i.e. not necessarily abelian!) von Neumann subalgebras $\mathfrak{A}_1$ and $\mathfrak{A}_2$ in $\boldsymbol{M}$, with ζ_0 and ζ now running over $\boldsymbol{Z}$.

5. *Definition.* Let $\mathfrak{N}$ be a von Neumann algebra acting in a separable Hilbert space $\mathfrak{H}$; ϕ be a faithful normal state on $\mathfrak{N}$; $\alpha(\mathbb{R})$ be a continuous group of automorphisms of $\mathfrak{N}$ with $\phi \circ \alpha(t) = \phi$ for all $t \in \mathbb{R}$; ζ be a partition of the identity on $\mathfrak{H}$ into mutually orthogonal projectors F_k in $\mathfrak{N}$; $\mathfrak{A}_n(\zeta)$ be the von Neumann algebra generated by $\{\alpha(k)[\zeta] | k \in \mathbb{Z}, -n \leqq k < 0\}$. If ζ is admissible in the sense of Definition IV.2, we now define the *entropy of ζ under* $\alpha(\mathbb{R})$ as:

$$H_\phi(\zeta, \alpha) = \lim_{n \to \infty} H_\phi(\zeta|\mathfrak{A}_n(\zeta)).$$

We further define the *entropy of the dynamical system* $\{\mathfrak{N}, \phi, \alpha(\mathbb{R})\}$ as:

$$H_\phi(\alpha) = \mathrm{Sup}\, H_\phi(\zeta, \alpha)$$

where the sup is taken over all admissible partitions ζ with $H_\phi(\zeta) < \infty$.

Remarks. (i) By Lemma IV.2, ζ admissible implies that ζ and $\mathfrak{A}_n(\zeta)$ are in $\mathfrak{N}_\phi$, the centralizer of $\mathfrak{N}$ with respect to ϕ. We can therefore restrict our attention to $\mathfrak{M} = \mathfrak{N}_\phi$ and $\phi|\mathfrak{N}_\phi$, and thus use the results of IV.3 and 4 above. (ii) In particular $H_\phi(\zeta|\mathfrak{A}_n(\zeta))$ is a positive, monotonically non-increasing function of $n \in \mathbb{Z}^+$, so that the limit $H_\phi(\zeta, \alpha)$ indeed exists. (iii) The value $+\infty$ is, in principle, allowed by the definition of $H_\phi(\alpha)$; it actually occurs, as we shall see in subsection 7 below, in quite a number of non-isomorphic examples. (iv) Because of Remark (iii) to Definition IV.3, the entropy $H_\phi(\alpha)$ reduces to the classical Kolmogorov dynamical entropy when $\mathfrak{N}$ is abelian. (v) From a rigidly operational point of view, one might object that $\mathfrak{A}_n(\zeta)$ may perhaps contain partitions which are out of reach of laboratory procedure involving only a finite number of physically implementable steps; in this spirit, one should then not be allowed to take, in the definition of $H_\phi(\zeta|\mathfrak{A}_n(\zeta))$, the "Inf" over *all* partitions ζ_1 in $\mathfrak{A}_n(\zeta)$. We should nevertheless remark that the introduction of such restrictions would only make $H_\phi(\alpha)$ larger. This would thus only soften the standards by which we decide that a flow is stochastic. In particular, the validity of our next theorem, which is the principal result of this section, would not be affected, nor would the discussion presented in the subsequent, and last, paragraph of this paper.

6. Theorem. *The entropy of every non-singular generalized K-flow* $\{\mathfrak{N}, \phi, \alpha(\mathbb{R}), \mathscr{A}\}$ *is strictly positive.*

Proof. Let $\mathfrak{N}_\phi$ be the centralizer of $\mathfrak{N}$ with respect to ϕ; P be any projector in the von Neumann algebra $\mathscr{A}_\phi = \mathfrak{N}_\phi \cap \mathscr{A}$; and $\zeta = \{P, I-P\}$ be the corresponding partition of the identity. Clearly ζ is admissible. Since ζ is finite, $H_\phi(\zeta) < \infty$. Hence $H_\phi(\alpha) = 0$ implies in particular $\lim_{n\to\infty} H_\phi(\zeta|\mathfrak{A}_n(\zeta)) = 0$. Since $\mathscr{E}(.|\mathfrak{A}_n(\zeta))$ is an increasing martingale, with $\bigvee_n \mathfrak{A}_n(\zeta) = \mathfrak{A} = \{\alpha(n)[\zeta] | n \in \mathbb{Z},\ n<0\}''$, $H_\phi(\zeta|\mathfrak{A}) = 0$. Consequently, by Remark IV.4.iv, $\zeta \subset \mathfrak{A}$, and thus $\zeta \subset \{\alpha(n)[\mathscr{A}_\phi] | n \in \mathbb{Z},\ n<0\}''$. However by Theorem I.8 $\mathscr{A}_\phi \subseteqq \alpha(t)[\mathscr{A}_\phi]$, i.e. $\alpha(-t)[\mathscr{A}_\phi] \subseteqq \mathscr{A}_\phi$ for every $t \in \mathbb{R}^+$. Hence $\zeta \subset \alpha(-1)[\mathscr{A}_\phi]$, and thus $P \in \alpha(-1)[\mathscr{A}_\phi]$. Since P was chosen arbitrarily in $\mathscr{A}_\phi$, this means $\mathscr{A}_\phi \subseteqq \alpha(-1)[\mathscr{A}_\phi]$. By Theorem I.8 again this implies $\mathscr{A}_\phi = \alpha(-1)[\mathscr{A}_\phi]$ and thus $\mathscr{A}_\phi = \alpha(t)[\mathscr{A}_\phi]$ for all $t \in \mathbb{R}$. On the other hand, still by Theorem I.8, we have $\bigcap_t \alpha(t)[\mathscr{A}_\phi] = \mathbb{C}I$ and $\bigvee_t \alpha(t)[\mathscr{A}_\phi] = \mathfrak{N}_\phi$, which is to say now that $\mathfrak{N}_\phi = \mathbb{C}I$, i.e. $\{\mathfrak{N}, \phi, \alpha(\mathbb{R}), \mathscr{A}\}$ is singular. q.e.d.

7. We now show that the entropy of all the generalized K-flows constructed in Section III can be explicitely computed.

We first examine Example III.2.

For any $f_\omega \in \mathscr{T}$, let $\mathfrak{N}(\omega)$ be the von Neumann subalgebra of $\mathfrak{N}$ generated by $\{W(zf_\omega) | z \in \mathbb{C}\}$; let further $\mathfrak{N}_\phi(\omega) = \mathfrak{N}(\omega) \cap \mathfrak{N}_\phi$. The latter is the algebra of constants of the motion for a single harmonic oscillator; it is therefore abelian, with spectrum isomorphic to $\mathbb{Z}_+$. Furthermore $\phi|\mathfrak{N}_\phi(\omega)$ induces on $\mathbb{Z}_+$ the canonical equilibrium measure ν given by $\nu_\omega(n) = \lambda^{-n}(1-\lambda)$.

For any (not necessarily complete) orthonormal system $\mathscr{F} = \{f_\omega | \omega \in \Omega\}$ in $\mathscr{T}$, we can identify $\mathfrak{N}_\phi(\mathscr{F}) \cong \bigotimes_{\omega\in\Omega} \mathfrak{N}_\phi(\omega)$ as a von Neumann subalgebra of $\mathfrak{N}_\phi$; $\mathfrak{N}_\phi(\mathscr{F})$ is again abelian, with spectrum $\mathbb{Z}_+^\Omega$. The product state $\phi|\mathfrak{N}_\phi(\mathscr{F}) = \bigotimes_{\omega\in\Omega} \phi_\omega$, with $\phi_\omega = \phi|\mathfrak{N}_\phi(\omega)$, induces on $\mathbb{Z}_+^\Omega$ the product measure $\nu = \bigotimes_{\omega\in\Omega} \nu_\omega$.

For any finite positive integer N, we can find $\{f_k | 1 \leqq k \leqq N\}$ with $f_k \in \mathscr{S}(\mathbb{R})$, $\mathrm{supp}(\tilde{f_k}) \subseteqq [0, 1]$, and $(f_k, f_{k'}) = \delta_{kk'}$.

We now take for Ω the set $\{\omega = (k, n) | 1 \leqq k \leqq N, n \in \mathbb{Z}\}$; and for $\mathscr{F}$ the system $\{f_\omega = f_{(k,n)} = u_n f_k | \omega \in \Omega\}$, with $\{f_k | 1 \leqq k \leqq N\}$ chosen as just indicated above; and $u_n = u_{t=n}$ where $\{u_t | t \in \mathbb{R}\}$ is the unitary group on $\mathscr{T}$ which induces on $\mathfrak{N}$ our automorphism group $\alpha(\mathbb{R})$. Notice in particular that we have indeed $(f_\omega, f_{\omega'}) = \delta_{\omega\omega'}$.

This choice of $\mathscr{F}$ has the following remarkable properties. (i) The spectrum X of the abelian von Neumann algebra $\mathfrak{N}_\phi(\mathscr{F})$ is $\mathbb{Z}_+^\Omega = (\mathbb{Z}_+^N)^{\mathbb{Z}}$, i.e. it is the countable product of copies of the countable space $Y = \mathbb{Z}_+^N$. (ii) ϕ induces on Y a probability measure μ_0 with mass p_y at $y=(n_1, \dots, n_N) \in Y$ given by $p_y = \mu_0(n_1, \dots, n_N) = \prod_{k=1}^{N} \nu(n_k)$ with $\nu(m) = \lambda^{-m}(1-\lambda)$, $m = n_k$, $1 \leqq k \leqq N$. The entropy of this distribution is:

$$S(Y, \mu_0) = -\sum_{y \in Y} p_y \log p_y = N S_0(\lambda)$$

with:

$$0 < S_0(\lambda) = -\sum_{n \in \mathbb{Z}} \nu(n) \log \nu(n) < \infty .$$

(iii) ϕ induces on X a probability measure $\mu = \bigotimes_{n \in \mathbb{Z}} \mu_n$ with $\mu_n = \mu_0$ on Y. (iv) $\mathfrak{N}_\phi(\mathscr{F})$ is $\alpha(\mathbb{Z})$-stable. (v) $\alpha(\mathbb{Z})$ induces on X a group $T(\mathbb{Z})$ of μ-measurable, μ-preserving transformations with $T(n) = T^n$ given by:

$$(T[x])_{i,n} = (x)_{i,n-1} .$$

These properties add up to the assertion that $\{X, \mu, T(\mathbb{Z})\}$ is a generalized classical Bernoulli shift (see for instance p. 110 in [21]). Its entropy $H_\mu(T)$ is thus equal to $S(Y, \mu_0)$, which is finite. Therefore [23], it has a finite generator.

Consequently, for each finite positive integer N there is a partition ζ in our original K-flow $\{\mathfrak{N}, \phi, \alpha(\mathbb{R}), \mathscr{A}\}$ which has finite entropy and such that $H_\phi(\zeta, \alpha) = N S_0(\lambda)$. *The entropy of the generalized K-flow* $\{\mathfrak{N}, \phi, \alpha(\mathbb{R}), \mathscr{A}\}$ *constructed in Example III.2 is therefore infinite.*

This proves as well that the generalized K-flow constructed in Example III.3 has infinite entropy, since it is the flow induced by $\{\mathfrak{N}, \phi, \alpha(\mathbb{R}), \mathscr{A}\}$ on its centralizer $\mathfrak{N}_\phi$ where the above argument is actually carried out.

We might also remark that this argument also shows that a dynamical entropy defined along the line followed by Connes and Størmer [6] would also take infinite value on this flow.

Remark (iii) to Scholium I.8.B shows that a generalized K-flow of the type constructed in Example III.2 is always contained as a subflow of any of the generalized K-flows constructed in Example III.4. Consequently the latter have also infinite entropy.

Actually an argument quite similar to that used for the analysis of Example III.2 can be used for the classical case considered in Example III.1. Consequently we have back the classical result that the flow of Brownian motion on $\mathbb{R}$ has infinite entropy.

We should perhaps emphasize that in spite of the common Bernoulli structure emerging in all the generalized K-flows of Examples III.1 to III.4, these flows, although they all have infinite entropy are mutually non-isomorphic.

Finally, the flow of Example III.5 is singular, and hence has no non-trivial admissible partition. Its entropy thus vanishes trivially. This however should be interpreted carefully. Whereas Theorem III.6 shows in effect that one can separate, for every non-singular generalized K-flow, the stochastic elements in the time evolution $\alpha(\mathbb{R})$ from those stochastic elements which might be introduced by the quantum measurement process, such a clean separation is not possible in the case of singular generalized K-flow. Whatever stochastic elements the "true" evolution $\alpha(\mathbb{R})$ of a singular generalized K-flow might have, these can simply not be detected from the constants of motion under the "free" evolution $\sigma(\mathbb{R})$ since these singular flows do not admit any non-trivial such constant.

The point of this section was precisely to show how this separation can be done for every non-singular generalized K-flow; this applies in particular to every regular generalized K-flow. We recall that classical K-flows are regular in the sense of our Definition I.2, and that a regular, non-classical, generalized K-flow can canonically be associated to a quantum transport equation such as that governing the diffusion of a quantum particle in a harmonic well.

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Note Added in Proof. The fact that Rudolph's work [20] supports the conjecture that (finite-entropy) classical K-flows are reversible, also follows from recent work by B. M. Gurevich (private communication, June 1976).