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The Existence of Maximal Slicings in Asymptotically Flat Spacetimes

M. Cantor*, A. Fischer** ***, J. Marsden+ ***, N. Ō Murchadha++, and J. York++

Department of Physics and Astronomy, University of North Carolina, Chapel Hill, North Carolina 27514, USA

Abstract. We consider Cauchy data (g, π) on $\mathbb{R}^3$ that are asymptotically Euclidean and that satisfy the vacuum constraint equations of general relativity. Only those (g, π) are treated that can be joined by a curve of "sufficiently bounded" initial data to the trivial data $(\delta, 0)$. It is shown that in the Cauchy developments of such data, the maximal slicing condition $\mathrm{tr}\,\pi = 0$ can always be satisfied. The proof uses the recently introduced "weighted Sobolev spaces" of Nirenberg, Walker, and Cantor.

Consider the set $\mathscr{C}$ of spacetimes which are the Cauchy developments of initial data (g, π) on $\mathbb{R}^3$ which are asymptotically Euclidean and which satisfy the constraint equations [see (3) and (4) below] in the dynamical formulation of general relativity [1]. In 1968, Brill and Deser [2] conjectured that one can maximally slice any such spacetime, i.e. one can find spacelike hypersurfaces on which $\mathrm{tr}\,\pi = 0$. In a Hamiltonian analysis of general relativity $\mathrm{tr}\,\pi$ assumes the role of a gauge variable (see for example [12]) and so one would expect that the $\mathrm{tr}\,\pi = 0$ condition can be met in any such spacetime. Here we prove that the Brill-Deser conjecture is true.

We consider only those (g, π) which can be joined by a curve of "sufficiently bounded" initial data (to be explained later) to flat space $(\delta, 0)$. Thus we are considering the component $\mathscr{C}_0$ of $(\delta, 0)$ in the set of asymptotically Euclidean solutions of the constraint equations. $\mathscr{C}_0$ is restricted to those 3-metrics which are derived from Lorentz metrics on $\mathbb{R}^4$ that are near the "background" Minowski metric. The set $\mathscr{C}_0$ is discussed in [7–11].

* Present address: Department of Mathematics, Duke University, Durham, North Carolina 27706, USA

** Present address: Department of Mathematics, University of California, Santa Cruz, California 95064, USA

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+ Present address: Department of Mathematics, University of California, Berkeley, California 94720, USA

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In her note [6], Choquet-Bruhat proves a theorem for spacetimes with compact spacelike hypersurfaces which is similar to our step 2 below. She also notes her proof extends to yield the local result for spacetimes with noncompact spacelike hypersurfaces. The authors became aware of [6] after the present work was completed.

We shall prove:

Theorem. *Let $(g,\pi)\in\mathscr{C}_0$. Then in the Cauchy development of (g,π) there is a slice on which the trace of the second fundamental form is zero. (Recall that this entails* $\operatorname{tr}\pi=0$*).*

There is a similar theorem for the component of $\mathscr{C}$ containing any given (g,π) with $\operatorname{tr}\pi=0$ or in the case of compact hypersurfaces, $\operatorname{tr}\pi/\mu_g=$ constant (see [6] and [12]). The constant depends on the hypersurface. This theorem is proven similarly to the one in this paper.

The proof requires the use of the weighted Sobolev spaces $M^p_{s.\delta}$ introduced in [3]. For compact hypersurfaces, the usual Sobolev spaces $W^{s.p}$ will do, as in [9].

Definition. Let $\sigma(x)=(1+|x|^2)^{1/2}$. For $1\leqq p\leqq\infty$, s a nonnegative integer, and $\delta\in\mathbb{R}$, let $M^p_{s.\delta}(\mathbb{R}^n,\mathbb{R}^q)$ be the completion of $C^\infty_0(\mathbb{R}^n,\mathbb{R}^q)$ with respect to the norm

$$|f|_{p.s.\delta}=\sum_{|\alpha|\leqq s}(|(D^\alpha f)\sigma^{\delta+|\alpha|}|_{L^p}).$$

The elementary properties of these spaces are discussed in [3, 4].

The important technical result for this paper is

Lemma 1. [5]. *Let $n>m$ and $A_\infty=\sum_{|\alpha|=m}\bar{a}_\alpha D^\alpha$ be an elliptic homogeneous operator on $\mathbb{R}^n$. Suppose we have an elliptic operator $A=\sum_{|\alpha|\leqq m}a_\alpha(x)D^\alpha$ on $\mathbb{R}^n$ satisfying for $s\geqq m$, $a_\alpha\in C^{s-m}$ and*

$$\sup|D^\gamma(a_\alpha(x))\cdot\sigma^{m-\alpha+|\gamma|}|<\infty\quad \textit{for}\quad |\alpha|<m$$

and

$$\lim\sup|D^\gamma(a_\alpha(x)-\bar{a}_\alpha)\sigma^{|\gamma|}|<\varepsilon\quad \textit{for}\quad |\alpha|=m$$

and $|\gamma|\leqq s-m$. Then if $p>n/(n-m)$ and $0\leqq\delta<-m+n(p-1)/p$, and ε is sufficiently small, A is an isomorphism between $M^p_{s.\delta}$ and $M^p_{s-m.\delta+m}$.

Remark. The smoothness condition of the a_α may be relaxed by taking completions in the appropriate Banach space of linear operators. This fact is used implicitly below.

We shall apply Lemma 1 where A is the Laplacian with respect to some asymptotically flat metric on $\mathbb{R}^3$. Thus $n=3$ and $m=2$. We assume p and δ are as in the theorem and $s>n/p+2$. The (g,π) we shall consider will be of the form $g=\delta+h$ with $h\in M^p_{s.\delta}$ and $\pi\in M^p_{s-1.\delta+1}$ (see [8]). All norms are taken with respect to the flat background metric. Note that for $g\in\mathscr{C}_0$ these norms are equivalent to those induced by g. Note we may take $\operatorname{tr}\pi\in M^p_{s-1.\delta+2}$. The topology on the space of initial data is given by the $M^p_{s.\delta}$ norms.

The required slicing will be determined by a lapse function $N(\lambda, x) = (-g^{00})^{-1/2}$. Letting the shift vector $g_{0i} = X_i(\lambda, x) = 0$, the Einstein Equations read (here π is a density):

$$\partial g/\partial\lambda = 2N(\pi - \tfrac{1}{2}(\operatorname{tr}\pi)g)(1/\mu_g) = -2Nk \tag{1}$$

$$\begin{aligned}\partial\pi/\partial\lambda = &-N(\operatorname{Ric}(g) - \tfrac{1}{2}R(g)g)\mu_g + \tfrac{1}{2}N(\pi\cdot\pi - \tfrac{1}{2}(\operatorname{tr}\pi)^2)/\mu_g \\ &-2N(\pi\times\pi - \tfrac{1}{2}(\operatorname{tr}\pi)\pi)/\mu_g + (\operatorname{Hess} N - g\nabla^2 N)\mu_g\end{aligned} \tag{2}$$

$$\mathscr{H}(g, \pi) = (\pi\cdot\pi - \tfrac{1}{2}(\operatorname{tr}\pi)^2)/\mu_g - R(g)\mu_g = 0 \tag{3}$$

$$\delta_g\pi = 0 \tag{4}$$

and using $p = \operatorname{tr}\pi/\mu_g = 2\operatorname{tr}k$, we find from the above equations that

$$\partial p/\partial\lambda = 2(k\cdot k - \nabla^2)N\,. \tag{5}$$

Step. 1. If $p=0$ for some λ, we may choose an N such that p is zero for all λ (for which the dynamics is defined).

Proof. Writing $N = 1 + \tilde{N}$ (so that $\tilde{N}$ is close to 0 when N is close to 1), we find

$$\partial p/\partial\lambda = 2k\cdot k + 2(k\cdot k - \nabla^2)\tilde{N}\,.$$

Thus the equation $\partial p/\partial\lambda = 0$ may be solved using Lemma 1 for $\tilde{N}(\lambda) \in M^p_{s+1\ \delta}$ for each λ. Thus for this choice of $N = 1 + \tilde{N}$ in the dynamics the condition $p=0$ will be maintained. □

In what follows we show that whatever p equals at $\lambda = 0$, we may achieve $p=0$ at $\lambda=1$ by choosing a suitable N. Throughout, we shall take $\partial N/\partial\lambda = 0$.

Step 2. (Local Argument). Let $(g_0, \pi_0) \in \mathscr{C}_0$ and suppose $\operatorname{tr}\pi_0 = 0$. Then there is a neighborhood V of (g_0, π_0) such that if $(g, \pi) \in V$ then there is an $N \in \mathscr{M}^p_{s+1,\delta}$ such that $p=0$ at $\lambda=1$. (By a suitable choice of scale, we may assume $\lambda=1$ will be reached by the dynamics.)

Proof. Let $F = \mathscr{C}_0 \times \mathscr{M}^p_{s+1,\delta}(\mathbb{R}^3, \mathbb{R}) \to M^p_{s-1,\delta+2}(\mathbb{R}^3, \mathbb{R})$ be defined (on a suitable open set) by

$$F((g, \pi), N) = \{\text{the function } p \text{ at } \lambda = 1 \text{ determined by Equations (1), (2), (5)}\}\,.$$

Then using smoothness properties of the evolution equations (see [8]), F is a smooth mapping. The derivative with respect to N at $((g_0, \pi_0), 0)$ in the direction δN is

$$D_N F((g_0, \pi_0), 0)\cdot\delta N = \left(\int_0^1 (k_0(\lambda)\cdot k_0(\lambda) - \nabla^2_\lambda)d\lambda\right)\delta N \tag{6}$$

where $k_0(\lambda)$ is the evolution of k_0 for the given (g_0, π_0) and ∇^2_λ is the Laplacian for $g_0(\lambda)$.

Since we are only considering functions that are independent of λ, it follows easily from Lemma 1 that the operator (6) is an isomorphism (see also [9]). Thus by the implicit function theorem we can uniquely solve $F((g, \pi), N) = 0$ for $N(g, \pi)$ near 0 and (g, π) near (g_0, π_0). This proves step 2. □

Step 3 (Globalization). Let (g_0, π_0) be joined to (g, π) be a continuous curve $(g(\alpha), \pi(\alpha))$ in $\mathscr{C}_0$, $\alpha \in [0, 1]$. Let J be the set of α for which the resulting space time has a maximal slice. Then $0 \in J$ and step 2 shows that J is open. We can always work in a neighborhood of the curve $(g(\alpha), \pi(\alpha))$ so that the evolution times used in step 2, can be chosen to be uniform along the curve.

To show J is closed, let $\alpha_m \in J$ and $\alpha_m \to \alpha$. Let N_m be the unique lapse functions given by step 2. In order to demonstrate that ∇_m^2 remains uniformly elliptic and the slices "uniformly spacelike", we may take a sequence of coordinate transformations f_m on the slices S_m chosen so as to keep the eigenvalues of g_m (relative to the flat background metric) bounded away from zero. Since $k_0(m)$ remains uniformly bounded and ∇_m^2 remains uniformly elliptic for α_m, $m \to \infty$, the N_m will converge to a function N. This N is the required zero of F.

Thus $J = [0, 1]$ and our proof is complete. □

Note Added in Proof. The hypotheses of Lemma 1 should include that a_0 is non-positive. In our application, $a_0 = -\int_0^1 k_0(\lambda) \cdot k_0(\lambda) d\lambda \leqq 0$.

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