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A Definition of Gibbs State for a Compact Set with Z^ν Action

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Abstract. The definition of Gibbs states used in the equilibrium statistical mechanics of lattice spin systems is extended to apply to a compact metrizable space, where Z^ν acts by an expansive group of homeomorphisms.

Introduction

In this paper we give an extension of the definition of Gibbs states used in the equilibrium statistical mechanics of classical lattice spin systems [1]. The assumptions, definitions and results are stated in Section 1, the proofs are given in Section 3, and examples are discussed in Section 2. The discussion of the example of the toral diffeomorphism could be generalized to apply to the basic sets for diffeomorphisms satisfying Smale's Axiom A [2].

1. Results

1.1. Assumptions. Let Ω be a compact metrizable space, and $\{T^k, k \in Z^\nu\}$ a group of homeomorphisms of Ω, *expansive* with expansive constant γ. That is $\gamma > 0$ is such that if $x, y \in \Omega$, and $d(T^k x, T^k y) \leqq \gamma \, \forall k \in Z^\nu$, then $x = y$.

1.2. Definitions. Two points $x, y \in \Omega$ are *conjugate* if $d(T^k x, T^k y) \xrightarrow[|k| \to \infty]{} 0$.

Let $O \subset \Omega$ be open: a mapping $\varphi : O \to \Omega$ is *conjugating* if $d(T^k x, T^k \varphi(x)) \xrightarrow[|k| \to \infty]{} 0$ uniformely with respect to $x \in O$.

1.3. Theorem. *Suppose that for every pair of conjugate points $x, y \in \Omega$ there is an open set $O \subset \Omega$, $O \ni x$ and a mapping $\varphi : O \to \Omega$ conjugating, continous at x and such that $\varphi(x) = y$.*

Then for every such mapping one can find an open set $\tilde{O} \ni x$, $\tilde{O} \subset O$ such that φ is a homeomorphism of $\tilde{O}$ to $\varphi(\tilde{O})$. If φ' is a mapping with the same properties as φ, φ and φ' coincide on some neighbourhood of x.

1.4. Assumption. From now on we assume that the condition of the Theorem hold.

1.5. Definitions. A *family of multipliers* is a family $\mathscr{I}=(f_{(O,\varphi)})$ such that:

i) The index set consists of all pairs (O,φ) where O is an open subset of Ω and φ is a conjugating homeomorphism defined on O; $f_{(O,\varphi)}$ is a real positive function continous on O.

ii) If $O'\subset O$ and $\varphi'=\varphi|O'$ then $f_{(O',\varphi')}=f_{(O,\varphi)}|O'$.

iii) If $O\subset\varphi'^{-1}(\varphi'O'\cap O'')$ then $f_{(O,\varphi''\circ\varphi')}=(f_{(O',\varphi')}|O)(f_{(O'',\varphi'')}\circ\varphi'|O)$. The family $\mathscr{I}$ is an *invariant* family if $f_{(T^kO,T^k\varphi T^{-k})}=f_{(O,\varphi)}\circ T^k$ for every $k\in Z^\nu$, (O,φ).

A probability measure σ on Ω is a *Gibbs state* for the family of multipliers $\mathscr{I}$ if

$$(*)\quad \varphi_*(f_{(O,\varphi)}\sigma|O)=\sigma|\varphi(O)\quad\text{for every}\quad(O,\varphi).$$

1.6. Proposition. *The set of Gibbs states on Ω for a given $\mathscr{I}$ (if it is not empty) is convex, compact, and a Choquet simplex.*

2. Examples

2.1. Classical Lattice Systems. Let F be a finite set, $\mathscr{D}$ a translation invariant family of subsets of Z^ν, which have bounded diameter, $\bar{\Omega}_\Delta$ a subset of F^Δ (product of a copy of F for each element of Δ) for each $\Delta\in\mathscr{D}$. We assume that $\bar{\Omega}_{\Delta+k}$ is the image of $\bar{\Omega}_\Delta$ by the canonical map $F^\Delta\to F^{\Delta+k}$, for each $k\in Z^\nu$. We assume that the configuration space

$$\Omega=\{x\in F^{Z^\nu}:x|\Delta\in\bar{\Omega}_\Delta\quad\text{for all}\quad\Delta\in\mathscr{D}\}$$

is not empty.

With the usual topology, Ω is compact, metrizable, and the group of translations $\{T^k,k\in Z^\nu\}$, $(T^kx)_{k'}=x_{k+k'}$ is expansive.

$x,y\in\Omega$ are conjugate if and only if there is a finite subset Λ of Z^ν such that $x|Z^\nu\backslash\Lambda=y|Z^\nu\backslash\Lambda$.

Let then $x,y\in\Omega$, $x|Z^\nu\backslash\Lambda=y|Z^\nu\backslash\Lambda$, and choose a finite $M\subset Z^\nu$, $M\supset\Lambda$ such that if $\Delta\in\mathscr{D}$ and $\Delta\cap\Lambda\neq\phi$ then $\Delta\subset M$. Let $O(x;M)=\{x'\in\Omega:x'|M=x|M\}$. The mapping $\varphi_{xy}:O(x;M)\to O(y;M)$; $\varphi_{xy}(x')|M=y|M$ $\varphi_{xy}(x')|Z^\nu\backslash M=x'|Z^\nu\backslash M$ is a conjugating homeomorphism.

Let $\Lambda\subset Z^\nu$, and $\Omega_\Lambda=\{x\in F^\Lambda:x|\Delta\in\bar{\Omega}_\Delta$ for all $\Delta\in\mathscr{D}$, $\Delta\subset\Lambda\}$. An interaction Φ is a real function on $\bigcup_{\Lambda\text{ finite}\subset Z^\nu}\Omega_\Lambda$, such that $\Phi|\Omega\phi=0$ and, for each $k\in Z^\nu$, $\|\Phi\|_k=\sum_{\Lambda\ni k}\sup_{x\in\Omega_\Lambda}|\Phi(x)|<+\infty$. If $z'\in\Omega_\Lambda$, $z''\in\Omega_{Z^\nu\backslash\Lambda}$, denote by $z'\vee z''$ the element $z\in F^{Z^\nu}$ such that $z|\Lambda=z'$, $z|Z^\nu\backslash\Lambda=z''$, and, for finite $\Lambda\subset Z^\nu$, let f_Λ be the real, continous function on $\Omega_\Lambda\times\Omega_{Z^\nu\backslash\Lambda}$ defined by $f_\Lambda(z',z'')=\exp\Big(-\sum_{M\subset Z^\nu,M\cap\Lambda\neq\phi}\Phi(z'\vee z''|M)\Big)$ if $z'\vee z''\in\Omega$ $f_\Lambda(z',z'')=0$ otherwise.

In the equilibrium statistical mechanics [1, 3] a Gibbs state for the interaction Φ is a probability measure on Ω such that, for any finite $\Lambda\subset Z^\nu$, the conditional probability that $z|\Lambda=z'$, knowing that $z|Z^\nu\backslash\Lambda=z''$ is given by

$$f_\Lambda(z',z'')/\sum_{w\in\Omega_\Lambda}f_\Lambda(w,z'').$$

Every such measure is then a Gibbs state corresponding to the family of multipliers:

$$f_{(O(x;M),\varphi_{xy})}(x')=f_\Lambda(y|\Lambda,x'|Z^\nu\backslash\Lambda)/f_\Lambda(x|\Lambda,x'|Z^\nu\backslash\Lambda).$$

2.2. *Toral Automorphism.* Let Ω be the two dimensional torus, $\Omega = \{(x_1, x_2) \bmod 1\}$ and T the automorphism induced on Ω by the matrix $\tilde{T} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ [4]. The group $\{T^n, n \in Z\}$ is then expansive.

Let $A \subset \Omega$ be the set of points $a = (a_1, a_2)$ such that for some integers k, l, m, n $(a_1 + k, a_2 + l)$ and $(a_1 + m, a_2 + n)$ are eigenvectors of $\tilde{T}$ corresponding to the eigenvalues lower and greater than one, respectively. A is a dense subgroup of the topological group Ω, invariant under T. Two points $x, y \in \Omega$ are conjugate if and only if $x - y \in A$. The set of conjugating homeomorphisms is the set of transformations $\{\varphi_a : O \to O + a, a \in A\}$ where $\varphi_a(x) = x + a$.

Each conjugating homeomorphism extends thus uniquely to the whole torus.

A family $\mathscr{I}$ of multipliers is then defined by a family $\{f_a, a \in A\}$ of continous positive functions on Ω, such that $f_{a_1 + a_2}(x) = f_{a_1}(x) f_{a_2}(x + a_1) \forall x \in \Omega$, $a_1, a_2 \in A$. The Haar measure is a Gibbs state: the corresponding invariant family $\mathscr{I}$ consists of the functions that are identically one.

3. Proofs

3.1. *For any $\varepsilon > 0$, there is a finite set $\Lambda_\varepsilon \subset Z^\nu$ such that $d(T^k x, T^k y) \leqq \gamma \forall k \in \Lambda_\varepsilon \Rightarrow d(x, y) < \varepsilon$.*

Proof. By compactness.

3.2. *For any finite set $\Lambda \subset Z^\nu$, there is $\varepsilon_\Lambda > 0$ such that $d(x, y) \leqq \varepsilon_\Lambda \Rightarrow d(T^k x, T^k y) \leqq \gamma \forall k \in \Lambda$.*

Proof. By continuity of the T^k.

3.3. *Two points $x, y \in \Omega$ are conjugate if and only if there is a finite set $\Lambda \subset Z^\nu$ such that $d(T^k x, T^k y) \leqq \gamma \forall k \notin \Lambda$.*

Proof. For any given $\varepsilon > 0$, $\Lambda \subset Z^\nu$ finite, let Λ_ε be given as in 3.1, and $M(\Lambda, \varepsilon) = \{k \in Z^\nu : k = l + m, l \in \Lambda, -m \notin \Lambda_\varepsilon\}$. Then $d(T^k x, T^k y) \leqq \gamma \forall k \notin \Lambda$ implies $d(T^k x, T^k y) < \varepsilon$ $\forall k \notin M(\Lambda, \varepsilon)$. The necessity of the condition follows from 1.2 and 3.2.

This argument also shows that:

3.4. *A mapping $\varphi : O \to \Omega$ is conjugating if and only if there is a finite set $\Lambda \subset Z^\nu$ such that $d(T^k x, T^k \varphi(x)) \leqq \gamma \forall k \notin \Lambda$, $x \in O$.*

3.5. *Let $O \subset \Omega$ be an open set, $O \ni x$, and $\varphi : O \to \Omega$ a conjugating mapping, continous at x. Then there is a neighbourhood $\tilde{O}$ of x such that $\varphi | \tilde{O}$ is injective, and continous.*

Proof. Let Λ be such that if $x' \in O$, $k \notin \Lambda$ $d(T^k x', T^k \varphi(x')) \leqq \gamma/4$. Choose a neighbourhood $\tilde{O}$ of x such that $\tilde{O} \subset O$ and $d(T^k x, T^k x') \leqq \gamma/2$ if $x' \in \tilde{O}$, $k \in \Lambda$. Then if $x', x'' \in \tilde{O}$ $d(T^k x', T^k x'') \leqq \gamma$ for any $k \in \Lambda$, and if $k \notin \Lambda$, $d(T^k x', T^k x'') \leqq d(T^k x', T^k \varphi(x')) + d(T^k x'', T^k \varphi(x'')) + d(T^k \varphi(x'), T^k \varphi(x'')) \leqq \gamma + d(T^k \varphi(x'), T^k \varphi(x''))$.

Therefore $\varphi(x') = \varphi(x'')$ implies $x' = x''$, and $\varphi | \tilde{O}$ is injective. Since φ is continous at x, we may also choose $\tilde{O}$ such that if $x', x'' \in \tilde{O}$ $d(T^k \varphi(x'), T^k \varphi(x'')) \leqq \gamma$ for any $k \in \Lambda$. By the choice of Λ if $x', x'' \in O$, $k \notin \Lambda$ $d(T^k \varphi(x'), T^k \varphi(x'')) \leqq d(T^k x', T^k \varphi(x')) + d(T^k (x''), T^k \varphi(x'')) + d(T^k x', T^k x'') \leqq \gamma/2 + d(T^k x', T^k x'')$.

Therefore if $x', x'' \in \tilde{O}$ and $d(T^k x', T^k x'') \leqq \gamma/2$ for any $k \in \Delta$ for some finite Δ, $d(T^k \varphi(x'), T^k \varphi(x'')) \leqq \gamma$ for any $k \in \Delta$ that is $\varphi | \tilde{O}$ is continous (3.1, 3.2).

3.6. *If two mappings φ and φ', defined in a neighbourhood O of x, are conjugating, continous at x, and such that $\varphi(x)=\varphi'(x)$, then there is a neighbourhood $\tilde{O}$ of x such that $\varphi|\tilde{O}=\varphi'|\tilde{O}$.*

Proof. Choose Λ such that for any $x' \in O$, $k \notin \Lambda$ $d(T^k x', T^k \varphi(x')) \leqq \gamma/2$ $d(T^k x', T^k \varphi'(x')) \leqq \gamma/2$, and choose a neighbourhood $\tilde{O} \subset O$ of x such that if $x' \in \tilde{O}$, $d(T^k \varphi(x'), T^k \varphi(x)) \leqq \gamma/2$, $d(T^k \varphi'(x'), T^k \varphi'(x)) \leqq \gamma/2$ for any $k \in \Lambda$. Then if $x' \in \tilde{O}$ $d(T^k \varphi(x'), T^k \varphi'(x')) \leqq \gamma$ for any $k \in Z^\nu$, and $\varphi|\tilde{O}=\varphi'|\tilde{O}$.

Proof of the Theorem. Let $x, y \in \Omega$ be conjugate. It follows from 3.5 that under the hypothesis of the theorem one can find two conjugating mappings φ and ψ, defined in neighbourhoods of x, y, injective, continous and such that $\varphi(x)=y$, $\psi(y)=x$. Then the mappings $\psi \circ \varphi$ and $\varphi \circ \psi$ are continous, conjugating, such that $\psi \circ \varphi(x)=x$, $\varphi \circ \psi(y)=y$ and by uniqueness (3.6) are equal to the identity map on suitable neighbourhoods of x, y. Therefore φ and ψ are inverse homeomorphisms. This concludes the proof of the first statement. The second was proven in 3.6.

3.7. It is clear that the set of Gibbs states for a family $\mathscr{I}$, if it is not empty, is convex and compact. If μ is any real measure on Ω satisfying the conditions $(*)$ also the absolute value $|\mu|$ satisfyies these conditions, therefore the set of real measures on Ω, satisfying $(*)$ is a lattice, respect to the usual order relation on measures. Then the Gibbs states form a Choquet simplex [3].

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