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On the Spinor Rank of Fermi Fields

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Abstract. We show that any Wightman field satisfying equal-time anti-commutation relations involving space derivatives of degree at most r must have spinor rank $r+1$.

Let ψ be a Wightman Fermi field, transforming according to the representation $\mathcal{D}_{j,k}$ of SL(2, $\mathbb{C}$):

$$U(A)\psi_{(\mu)}(x)U(A)^{-1}=(\underbrace{A^{-1}\otimes\ldots\otimes A^{-1}}_{2j}\otimes\underbrace{A^{*-1}\otimes\ldots\otimes A^{*-1}}_{2k})_{(\mu)(\nu)}\psi_{(\nu)}(\Lambda(A)x)$$

where $A\to\Lambda(A)$ is the usual homomorphism from SL(2, $\mathbb{C}$) to the Lorentz group. Suppose also that ψ satisfies canonical anti-commutation relations at time zero in the form

$$\{\psi_{(\mu)}(0,\boldsymbol{x}),\psi^*_{(\nu)}(0,\boldsymbol{y})\}=P_{(\mu)(\nu)}(\nabla)\,\delta^3(\boldsymbol{x}-\boldsymbol{y})\,. \tag{1}$$

Here, P is a polynomial of degree r, and (μ), (ν) denote spinor indices, $2j$ of which are undotted and $2k$ of which are dotted[1]. We note that free fields of spin 1/2, 3/2, ... obey such relations, with $r=0, 2, \ldots$. From positivity, r must be even. If the spinor rank of ψ were $\leqq r-1$, then the left hand side of (1) would transform as a spinor of rank at most $2r-2$, i.e. the right hand side would be a polynomial in ∇ of degree at most $r-1$, a contradiction. Hence it is enough to show that the spinor rank $s=2(j+k)$, cannot exceed $r+1$, as it is odd by the spin-statistics theorem. We use the methods of [1].

Let $A=A(\lambda)\in$ SL(2, $\mathbb{C}$) be of the special form

$$A=A^*=\begin{pmatrix}\sqrt{\lambda} & 0\\ 0 & 1/\sqrt{\lambda}\end{pmatrix},\qquad \lambda>0$$

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[1] We denote dotted indices by dashed symbols.

corresponding to an acceleration, $x \to \Lambda(\lambda)x$ in the z-direction. Let $\psi(x)$ denote $\psi_{1\cdots 11'\cdots 1'}(x)$. Then ψ transforms according to

$$U(\Lambda)^{-1}\psi(x)U(\Lambda) = \sqrt{\lambda^s}\,\psi(\Lambda^{-1}(\lambda)x).$$

It follows that the two-point Wightman function

$$W(x-y) = \langle \Psi_0, \psi(x)\psi^*(y)\Psi_0\rangle$$

obeys the identity

$$W(x-y) = \lambda^s W(\Lambda^{-1}(x-y)).$$

Let $\xi = x-y$; there exists a space-like vector $\hat{\xi}$ such that $|\mathrm{Re}\,W(\hat{\xi})| = 2\delta > 0$; for, if not, $W=0$ and so $\psi(x)\Psi_0 = 0$. Since Ψ_0 is separating for the field, $\psi_{11\cdots 11'\cdots 1'}(x) = 0$, and this would imply that all components of ψ vanish.

Since $\hat{\xi}$ is a point of analyticity, there exists a space-like neighbourhood

$$R = \{x_1, x_2;\ |x_i^0 - \hat{x}_i^0| \leqq a,\ |x_i^k - \hat{x}_i^k| \leqq a;\ i=1,2,\ k=1,2,3\}$$

such that

$$|\mathrm{Re}\,W(\xi)| = \lambda^s |\mathrm{Re}\,W(\Lambda^{-1}\xi)| \geqq \delta\lambda^s \text{ provided } \Lambda^{-1}\xi \in R. \tag{2}$$

We now use this bound on the pointwise value of W to get a related bound on the smeared W-function.

Let $\Delta = [-a/2, a/2]$ and choose $h \in \mathscr{D}(\Delta)$ such that $h(t) \geqq 0$, $\int h(t)\,dt \equiv \|h\|_1 = 1$. Let $f \in \mathscr{D}(\Delta \times \Delta \times \Delta)$ be such that $f \geqq 0$, $\|f\|_1 = 1$ and set $f_\lambda(\boldsymbol{x}) = f(x_1, x_2, \lambda x_3)$. Then one finds $\|f_\lambda\|_2 = \lambda^{-1/2}\|f\|_2$, $\|f_\lambda\|_1 = \lambda^{-1}$ and

$$\|d^r f_\lambda / dx_3^r\|_2 = \lambda^{r-1/2}\|d^r f/dx_3^r\|_2 .$$

Let $H_\lambda(x_1, x_2) = \lambda^2 f_\lambda(\boldsymbol{x}_1) h(\lambda x_1^0) f_\lambda(\boldsymbol{x}_2) h(\lambda x_2^0)$. Then

$$\|H_\lambda\|_1 = \lambda^{-2}. \tag{3}$$

We now claim that if $\lambda \geqq 1$,

$$\mathrm{Re}\int W(\xi) H_\lambda(x_1 - \Lambda\hat{x}_1, x_2 - \Lambda\hat{x}_2)\, d^4x_1\, d^4x_2 \geqq \delta\lambda^{s-2} \tag{4}$$

or equivalently

$$\mathrm{Re}\int W(\xi + \Lambda\hat{\xi}) H_\lambda(x_1, x_2)\, d^4x_1\, d^4x_2 \geqq \delta\lambda^{s-2}. \tag{5}$$

Because of (3), it is sufficient to prove that

$$\mathrm{Re}\,W(\xi + \Lambda\hat{\xi}) \geqq \delta\lambda^s \quad \text{on} \quad \mathrm{supp}\,H_\lambda .$$

According to (2), this holds if $\Lambda^{-1}\xi + \hat{\xi} \in R$ whenever $(x_1, x_2) \in \mathrm{supp}\,H_\lambda$. If $\lambda \geqq 1$ we have

$|\pm\lambda \pm \lambda^{-1}| \leqq 2\lambda$ and so from the Lorentz transformation

$$(\Lambda^{-1}x)^0 = \tfrac{1}{2}(\lambda + 1/\lambda)x^0 + \tfrac{1}{2}(\lambda - 1/\lambda)x^3$$

$$(\Lambda^{-1}x)^1 = x^1$$

$$(\Lambda^{-1}x)^2 = x^2$$

$$(\Lambda^{-1}x)^3 = \tfrac{1}{2}(\lambda - 1/\lambda)x^0 + \tfrac{1}{2}(\lambda + 1/\lambda)x^3$$

we conclude that

$$|(\Lambda^{-1}x_i)^0| < \tfrac{1}{2}|\lambda+\lambda^{-1}|\,|x_i^0| + \tfrac{1}{2}|\lambda-\lambda^{-1}|\,|x_i^3|$$
$$\leqq \lambda(|x_i^0|+|x_i^3|) \leqq a \quad \text{on} \quad \operatorname{supp} H_\lambda, \qquad i=1,2\,.$$

Also

$$(\Lambda^{-1}x_i)^1 = x_i^1 \in \Delta,\ (\Lambda^{-1}x_i)^2 = x_i^2 \in \Delta \text{ on } \operatorname{supp} H_\lambda, \text{ and}$$
$$|(\Lambda^{-1}x_i)^3| \leqq \tfrac{1}{2}(|\lambda-\lambda^{-1}|\,|x_i^0| + |\lambda^{-1}+\lambda|\,|x_i^3|) \leqq \lambda(|x_i^0|+|x_i^3|)$$
$$\leqq a \quad \text{on} \quad \operatorname{supp} H_\lambda.$$

Hence $\Lambda^{-1}\xi+\hat{\xi}\in R$ if $\xi\in\operatorname{supp} H_\lambda$ and our claims (4) and (5) are proved.

The CAR, Equation (1), leads to inequalities in the other direction. Let $g\in\mathscr{D}(\mathbb{R}^3)$ and Φ any unit vector; then if $\psi(0,g)$ denotes $\int \psi(0,\boldsymbol{x})g(\boldsymbol{x})d\boldsymbol{x}$,

$$\langle \Phi, \{\psi(0,g), \psi^*(0,g)\}\Phi\rangle \leqq \left|\int d\boldsymbol{x}_1\, g(\boldsymbol{x}_1) P(\nabla_2)\delta^3(\boldsymbol{x}_1-\boldsymbol{x}_2) g(\boldsymbol{x}_2)\, d\boldsymbol{x}_2\right|$$
$$\leqq \|g\|_2\, \|P(\nabla)g\|_2$$

from which it follows that

$$\|\psi(0,g)\| \leqq \|g\|_2^{\frac{1}{2}}\, \|P(\nabla)g\|_2^{\frac{1}{2}}\,.$$

Hence, if $\psi(h\otimes g)$ denotes $\int \psi(t,\boldsymbol{x})g(\boldsymbol{x})h(t)\,d\overset{3}{x}\,dt$,

$$\|\psi(h\otimes g)\| \leqq \|g\|_2^{\frac{1}{2}}\, \|P(\nabla)g\|_2^{\frac{1}{2}}\, \|h\|_1\,.$$

Hence, denoting $H(x_1-\Lambda\hat{x}_1, x_2-\Lambda\hat{x}_2)$ by $\hat{H}(x_1,x_2)$ etc.,

$$\int |W(x_1,x_2)\hat{H}_\lambda(x_1,x_2)|\,d\overset{4}{x}_1\,d\overset{4}{x}_2 = \lambda^2 |W(\hat{h}_{1\lambda}\otimes\hat{f}_{1\lambda}\otimes\hat{h}_{2\lambda}\otimes\hat{f}_{2\lambda})|$$
$$\leqq \lambda^2\, \|f_\lambda\|_2^{\frac{1}{2}}\, \|P(\nabla)f_\lambda\|_2^{\frac{1}{2}}\, \|h_\lambda\|_1\, \|f_\lambda\|_2^{\frac{1}{2}}\, \|P(\nabla)f_\lambda\|_2^{\frac{1}{2}}\, \|h_\lambda\|_1$$
$$= O(\lambda^2\cdot(\lambda^{-\frac{1}{4}}\lambda^{(r-\frac{1}{2})\frac{1}{2}}\lambda^{-1})^2) = O(\lambda^{r-1}), \qquad \lambda\to\infty\,. \tag{6}$$

Comparing (6) with (4), we obtain

$$r-1 \geqq s-2 \quad \text{i.e.} \quad s\leqq r+1, \qquad \text{as we claimed.}$$

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