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Scattering in the CAR Algebra

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Abstract. We obtain some results on time evolution in quasi local algebras, which are used to derive the scattering of quasi free evolution groups in the CAR algebra by certain inner perturbations.

Introduction

Consider E an infinite dimensional real vector space, and $C(E)$, the algebra of the canonical anticommutation relations over E. We want to consider the perturbation of a quasi-free evolution e^{tZ} of $C(E)$ by a possibly unbounded inner derivation of the algebra, to give a strongly continuous one-parameter group σ_t of *-automorphisms of $C(E)$. In the first section we employ some of the notation and technique of time development in quasi local algebras [3] to construct σ_t. In the second section we then show that under suitable conditions

$$\lim_{t \to \pm\infty} \sigma_t e^{-Zt} A$$

exists for all A in $C(E)$.

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§ 1. Time Evolution in Quasi-Local Algebras

Consider a C^*-algebra U, with e^{Zt} a strongly continuous one parameter group of *-automorphisms of U. Also consider the finite-dimensional lattice $\mathbb{Z}^\nu$, and denote by $\mathscr{F}$ the family of finite subsets of $\mathbb{Z}^\nu$. Suppose that for each $\Lambda \in \mathscr{F}$, there is a subalgebra $U(\Lambda)$ of U such that

(1.1) If $\Lambda, \Lambda_1 \in \mathscr{F}$, $\Lambda \subseteqq \Lambda_1$ then $U(\Lambda) \subseteqq U(\Lambda_1)$.

(1.2) $\cup\{U(\Lambda): \Lambda \in \mathscr{F}\}$ is dense in U.

(1.3) $U(\Lambda)$ is invariant under e^{tZ} for each $\Lambda \in \mathscr{F}$, $t \in R$.

We assign to each $\Lambda \in \mathscr{F}$ a potential $\Phi(\Lambda)$ in $U(\Lambda)$. We can then define, for each $\Lambda \in \mathscr{F}$ a "Hamiltonian" $H(\Lambda)$ in $U(\Lambda)$ by

(1.4) $H(\Lambda) = \sum_{X \subset \Lambda} \Phi(X)$.

The local hamiltonian $H(\Lambda)$ gives a perturbation of Z:

(1.5) $A \mapsto e^{t(Z + i \operatorname{ad} H(\Lambda))}(A)$; $\quad A \in U$, $\quad t \in R$.

We will demonstrate under suitable conditions that the limit as $\Lambda \to \infty$ of (1.5) exists and defines a strongly continuous one parameter group of *-automorphisms of U. The infinitesimal generator of this group will be interpreted as $Z + i\operatorname{ad}H$, where H is taken to denote the formal sum

$$\sum_{X \in \mathscr{F}} \Phi(X).$$

Definition 1.6. We let $\mathscr{B}$ denote the potentials Φ such that

(1.7) For each $X \in \mathscr{F}$

$U \ni x \mapsto i \operatorname{ad} \Phi(X) x$ is a *derivation on U.

(1.8) $\|\Phi\| = \sup_{x \in \mathbb{Z}^\nu} \left\{ \sum_{X \ni x} \|\Phi(X)\| \right\} < \infty$.

(1.9) Φ involves only a finite number of coordinates, i.e. $\Phi(X) = 0$ for all $X \in \mathscr{F}$ such that $|X| > N_\Phi$, where $N_\Phi < \infty$. (Here $|X|$ denotes the number of points in X.)

(1.10) Φ has finite range i.e. for each x in $\mathbb{Z}^\nu$, there exists $N_x < \infty$ such that $\{X \in \mathscr{F} : \Phi(X) \neq 0, X \ni x\}$ has cardinality $\leqq N_x$.

(1.11) If $X, \Lambda \in \mathscr{F}$ are disjoint then

$[\Phi(X), u] = 0 \quad \forall u \in U(\Lambda)$.

The following result extends [3] where $Z = 0$.

Theorem 1.12. *Suppose $\Phi \in \mathscr{B}$. Then for all $t \in R$, $A \in U$,*

$$\sigma_t A = \lim_{\Lambda \to \infty} e^{(Z + i \operatorname{ad} H(\Lambda))t} A$$

*exists, and defines a strongly continuous one parameter group of *-automorphisms of U, where the limit is taken in the sense that Λ eventually contains every finite subset of $\mathbb{Z}^\nu$.*

Proof. Fix $\Lambda_0 \in \mathscr{F}$, and take $A \in U(\Lambda_0)$. Consider $\Lambda \in \mathscr{F}$, $\Lambda \supseteqq \Lambda_0$. Then for all t in R

$$\begin{aligned} e^{(Z + i \operatorname{ad} H(\Lambda))t} A &= e^{Zt} A + i \int_0^t [e^{Z(t-s_1)} H(\Lambda), e^{Zt} A] \, ds_1 \\ &\quad + (i)^2 \int_0^t \int_0^{s_2} [e^{Z(t-s_2)} H(\Lambda), [e^{Z(t-s_1)} H(\Lambda), e^{Zt} A]] \, ds_1 \, ds_2 \\ &\quad + \ldots = \sum_{n=0}^{\infty} a_n^\Lambda(t). \end{aligned} \tag{1.13}$$

where (in terms of potentials):

$$a_n^\Lambda(t) = \sum_{X_1, X_2, \ldots, X_n} (i)^n \int_0^t \int_0^{s_n} \ldots \int_0^{s_2} [e^{Z(t-s_n)}\Phi(X_n), [\ldots[e^{Z(t-s_1)}\Phi(X_1),$$
$$\cdot e^{Zt}A]\ldots] ds_1 \ldots ds_n$$

Since Φ has finite range (1.10), $a_n^\Lambda(t)$ becomes independent of Λ for $|\Lambda|$ sufficiently large. Hence we have

$$\lim a_n^\Lambda(t) = \sum_{X_1, X_2, \ldots, X_n} (i)^n \int_0^t \int_0^{s_n} \ldots \int_0^{s_2} [e^{Z(t-s_n)}\Phi(X_n),$$
$$\cdot [\ldots [e^{Z(t-s_1)}\Phi(X_1), e^{Zt}A] \ldots] ds_1 \ldots ds_n .$$

This means that the series (1.13) converges term by term. The following lemma is an adaptation of [3, Lemma 1].

Lemma 1.14. *If* $A \in U(\Lambda_0)$, $q_1 \ldots q_n \in R$, $\Lambda \in \mathscr{F}$

$$\|[e^{Zq_n}H(\Lambda), [e^{Zq_{n-1}}H(\Lambda), [\ldots [e^{Zq_1}H(\Lambda), A] \ldots]]\|$$
$$\leqq 2^n \|\Phi\|^n \|A\| \prod_{m=1}^{n} \{(m-1)(N_\Phi - 1) + |\Lambda_0|\} .$$

It follows from the Lemma that

$$\|a_n^\Lambda(t)\| \leqq \frac{|t|^n}{n!} 2^n \|\Phi\|^n \prod_{m=1}^{n} \{(m-1)(N_\Phi - 1) + |\Lambda_0|\} .$$

Hence for $2|t| \, \|\Phi\| \, (N_\Phi - 1) < 1$, the strong limit of

$$e^{(Z + i \operatorname{ad} H(\Lambda))t} A$$

exists as $\Lambda \to \infty$, $\Lambda \supseteqq \Lambda_0$. (This convergence is uniform in $2|t| \, \|\Phi\| \, (N_\Phi - 1) \leqq 1 - \varepsilon$ for any $\varepsilon > 0$.) We define this limit to be $\sigma_t(A)$. But $e^{(Z + i \operatorname{ad} H(\Lambda))t}$ is a *-automorphism for all $\Lambda \in \mathscr{F}$, t in R, and hence is isometric. It follows that

$$\|\sigma_t A\| = \|A\| \quad \forall A \in \bigcup_{\Lambda \in \mathscr{F}} U(\Lambda) .$$

Hence we can extend σ_t to the whole of U by continuity. Take any

$$0 < t_1 < \frac{1}{2\|\Phi\|(N_\Phi - 1)} .$$

Then the group property of $e^{(Z + i \operatorname{ad} H(\Lambda))t}$ for $|t| \leqq t_1$, implies that σ_t has the same property in $|t| \leqq t_1$. Hence we can extend σ_t to the whole of R, such that

$$\sigma_t A = \lim_\Lambda e^{(Z + i \operatorname{ad} H(\Lambda))t} A$$

for all A in U, t in R. This is a familiar argument, which can be found in detail in [2]. The strong continuity of σ_t follows from the series expansion for small t:

$$\sigma_t A = \sum_{n=0}^{\infty} a_n(t) \quad \text{for} \quad A \in U(\Lambda_0), \quad \text{where} \quad a_n(t) = \lim_\Lambda a_n^\Lambda(t) .$$

Moreover σ_t is a *-automorphism for each t in R.

We note that there is a natural action of $\mathbb{Z}^\nu$ on $\mathscr{F}$. If we strengthen the translation invariance property (1.8) we can remove the hypothesis of finite range of our potentials.

Definition 1.15. We let $\mathscr{B}_0$ denote the potentials Φ such that

(1.16) For each $X \in \mathscr{F}$,
$U \ni x \mapsto i \operatorname{ad} \Phi(X)x$ is a *-derivation on U.

(1.17) $\|\Phi(X+a)\| = \|\Phi(X)\| \quad \forall X \in \mathscr{F}, \quad a \in \mathbb{Z}^\nu$.

(1.18) $\|\Phi\| \equiv \sum_{X \ni 0} \|\Phi(X)\| < \infty$

(1.19) Φ involves only a finite number of coordinates i.e. $\Phi(X)=0$ for all $X \in \mathscr{F}$ such that $|X| > N_\Phi$.

(1.20) If $X, \Lambda \in \mathscr{F}$ are disjoint
$[\Phi(X), u] = 0 \quad \forall u \in U(\Lambda)$.

For this class of potentials we can again take the limit as $\Lambda \to \infty$ in (1.5). Note that there is no need to assume that there is an action of $\mathbb{Z}^\nu$ on $\cup U(\Lambda)$, such that each a in $\mathbb{Z}^\nu$ takes $U(\Lambda)$ onto $U(\Lambda+a)$ for all Λ in $\mathscr{F}$.

Theorem 1.21. *Suppose $\Phi \in \mathscr{B}_0$. Then for all t in R, A in U*

(1.22) $\sigma_t A = \lim_{\Lambda \to \infty} e^{(Z + i \operatorname{ad} H(\Lambda))t} A$

*exists and defines a strongly continuous one parameter group of *automorphisms on U.*

Proof. We note that under the norm $\|\cdot\|$ (1.18), $\mathscr{B}_0$ is a real normed space. Moreover $\mathscr{B} \cap \mathscr{B}_0$ is dense in $\mathscr{B}_0$; if $\Phi \in \mathscr{B}_0$ define $\Phi_n \in \mathscr{B} \cap \mathscr{B}_0$ by

$$\Phi_n(X) = \begin{cases} \Phi(X) & \text{if diam} \quad X \leqq n \\ 0 & \text{otherwise}. \end{cases}$$

Then $\quad \|\Phi_n - \Phi\| \to 0 \quad$ and $\quad \|\Phi_n\| \leqq \|\Phi\|$.

The following is an elementary extension of [3, Lemma 2].

Lemma 1.22. *If $A \in U(\Lambda_0)$, $\Phi_1, \Phi_2 \in \mathscr{B}_0$, and N_Φ is such that $\Phi_1(X) = 0 = \Phi_2(X)$ if $|X| > N_\Phi$ then for all $q_1, \ldots, q_n \in R$, $\Lambda \in \mathscr{F}$*

$$\begin{aligned} &\|[e^{Zq_n} H_{\Phi_1}(\Lambda), [e^{Zq_{n-1}} H_{\Phi_1}(\Lambda), \ldots [e^{Zq_1} H_{\Phi_1}(\Lambda), A] \ldots] \\ &- [e^{Zq_n} H_{\Phi_2}(\Lambda), [e^{Zq_{n-1}} H_{\Phi_2}(\Lambda), [\ldots [e^{Zq_1} H_{\Phi_2}(\Lambda), A] \ldots]\| \\ &\leqq n 2^n \|\Phi_1 - \Phi_2\| \, \|\Phi\|^{n-1} \|A\| \prod_{m=1}^{n} \{(m-1)(N_\Phi - 1) + |\Lambda_0|\}. \end{aligned}$$

Now take $A \in U(\Lambda_0)$, and $\Lambda \supset \Lambda_0 \in \mathscr{F}$. Now suppose $\Phi, \Phi_1 \in \mathscr{B}_0$ and $\Phi_1(X) = 0 = \Phi(X)$ if $|X| > N_\Phi$ and also $\|\Phi_1\| \leqq \|\Phi\|$. Then for all t in R.

$$\begin{aligned} &\|e^{t(Z + i \operatorname{ad} H_\Phi(\Lambda))} A - e^{t(Z + i \operatorname{ad} H_{\Phi_1}(\Lambda))} A\| \\ &\leqq 2\|A\| \, \|\Phi - \Phi_1\| \sum_{n=1}^{\infty} \frac{2^n |t|^n \|\Phi\|^{n-1}}{(n-1)!} \prod_{m=1}^{n} \{(m-1)(N_\Phi - 1) + |\Lambda_0|\} \end{aligned}$$

by Lemma 1.22.

This estimate is uniform in Λ, and the right hand side converges for

$$2|t|\,\|\Phi\|\,(N_\Phi-1)<1\,.$$

Given $\Phi\in\mathscr{B}_0$, we take the sequence Φ_n in $\mathscr{B}\cap\mathscr{B}_0$, defined above, and express the limit in (1.22) as a double limit. The above estimate justifies the interchange of limits and the existence of the interchanged limit under the condition

$$2|t|\,\|\Phi\|\,(N_\Phi-1)<1\,.$$

(Here we have used Theorem 1.12 for the potential $\Phi_n\in\mathscr{B}$.)

The remainder of the proof is similar to that in Theorem 1.12.

Thus if $\Phi\in\mathscr{B}\cup\mathscr{B}_0$, $A\in U$

$$\lim_{\Lambda\to\infty} e^{(Z+i\,\mathrm{ad}\,H(\Lambda))t}e^{-Zt}A=\sigma_t e^{-Zt}A$$

where σ_t is a strongly continuous one parameter group of *-automorphisms of U, and the limit existing in the norm topology.

Now suppose that $\mathscr{D}$ is the subset of A in U such that the following limits

$$\lim_{t\to\pm\infty} e^{(Z+i\,\mathrm{ad}\,H(\Lambda))t}e^{-Zt}A$$

exist in the norm topology for all Λ in $\mathscr{F}$, with uniform convergence over Λ (for fixed A). Then both repeated limits exist on $\mathscr{D}$ (and are equal);

i.e. $\lim\limits_{t\to\pm\infty}\lim\limits_{\Lambda\to\infty} e^{(Z+i\,\mathrm{ad}\,H(\Lambda))t}e^{-Zt}A$ exists for all A in $\mathscr{D}$,

i.e. $\lim\limits_{t\to\pm\infty}\sigma_t e^{-Zt}A$ exists for all A in $\mathscr{D}$.

Hence if the subalgebra $\mathscr{D}$ is dense in U, the wave operators exist on U. We shall see that this approach is applicable in the CAR algebra.

§ 2. Scattering in the CAR Algebra

We are now going to study in some detail scattering in a particular C^*-algebra, namely the algebra of the canonical anticommutation relations. We briefly recall its construction, for more details see [1, 5].

Let E be an infinite dimensional real vector space with $(\ ,\ )$, a positive definite, non degenerate symmetric form. Let $C_0(E)$ be the complex Clifford algebra over E. The unique tracial state on $C_0(E)$ gives a pre-hilbertian structure for $C_0(E)$. Thus left multiplication on $C_0(E)$ induces a C^*-norm $\|\cdot\|$ on $C_0(E)$, and the CAR algebra $C(E)$ is the completion of $C_0(E)$ with respect to this norm. Let $f\to[f]$ be the canonical embedding of E in $C_0(E)$.

If e^{tZ} is a strongly continuous one parameter group of orthogonal maps on E, e^{tZ} can be uniquely extended to a strongly continuous one parameter group of *-automorphisms of $C(E)$.

In our first result, we consider a perturbation of the free evolution e^{tZ} by an inner derivation of the algebra $C(E)$. A similar result was obtained by D. W. Robinson [4, Application 2]. We remark that it is only necessary to show that the wave operators exist on a subset $\hat{U}$ of $C(E)$, such that the algebra generated by $\hat{U}$ is dense in $C(E)$.

Theorem 2.1. *Let e^{Zt} be a strongly continuous one parameter group of orthogonal maps on E. Suppose $f^i_{j,k} \in E$ for $1 \leqq k \leqq 2i$, $i, j = 1, 2, \ldots$ with $\|f^i_{j,k}\| = 1$ for all such (i, j, k); and $\alpha^i_j \in \mathbb{C}$ for $(i, j) \in \mathbb{N}^2$, such that*

$$\sum_{i,j} |\alpha^i_j| < \infty \,,$$

and

$$a = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \alpha^i_j \,[f^i_{j,1}] \ldots [f^i_{j,2i}]$$

*gives a *-derivation $x \mapsto i \,\mathrm{ad}\, a\,(x)$ on $C(E)$.*

Also suppose that

$$\mathscr{C} = \left\{ f \in E \colon \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \sum_{k=1}^{2i} |\alpha^i_j| \int |(e^{Zt} f^i_{j,k}, f)| dt < \infty \right\}$$

is such that $C_0(\mathscr{C})$ is dense in $C(E)$.

Then the wave operators

$$W_{\pm} = \operatorname*{st\,lim}_{t \to \pm\infty} e^{t(Z + i\,\mathrm{ad}\, a)} e^{-tZ}$$

exist.

Proof. If $(i, j) \in \mathbb{N}^2$, and $f \in E$

$$[f^i_{j,1}] \ldots [f^i_{j,2i}]\,[f] - [f]\,[f^i_{j,1}] \ldots [f^i_{j,2i}]$$

$$= 2 \sum_{k=1}^{2i} [f^i_{j,1}] \ldots [f^i_{j,k-1}]\,(f, f^i_{j,k})\,(-1)^k\,[f^i_{j,k+1}] \ldots [f^i_{j,2i}] \,.$$

Hence

$$a[f] - [f]a$$

$$= 2 \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \sum_{k=1}^{2i} \alpha^i_j \,[f^i_{j,1}] \ldots [f^i_{j,k}]\,(-1)^k (f, f^i_{j,k})\,[f^i_{j,k+1}] \ldots [f^i_{j,2i}] \,.$$

Take $t > s > 0$, $f \in \mathscr{C}$. Then

$$\|e^{t(Z + i\,\mathrm{ad}\, a)} e^{-tZ} f - e^{s(Z + i\,\mathrm{ad}\, a)} e^{-sZ} f\,\|$$

$$= \left\| \int_s^t e^{u(Z + i\,\mathrm{ad}\, a)} ad(a) e^{-uZ} f\, du \right\|$$

$$\leqq \int_s^t \|\mathrm{ad}\, a e^{-Zu} f\,\|\, du \leqq 2 \int_s^t \left\{ \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \sum_{k=1}^{2i} |\alpha^i_j (e^{-Zu} f, f^i_{j,k})| \right\} du \,.$$

Thus by the homomorphism property of $e^{t(Z + i\,\mathrm{ad}\, a)} e^{-tZ}$ for all t in R, and their uniform boundedness, we see that the wave operators exist on $C_0(\mathscr{C})$, and finally on $C(E)$.

Taking this a step further, we consider perturbations induced by formal sums, and exploit the computations of the previous section. Again let e^{Zt} be a strongly continuous one parameter group of orthogonal maps on E. Suppose $\{f_i : i \in \mathbb{Z}\}$, is a countable set in E with disjoint energy spectra, i.e.

(2.2) $(e^{Zt}f_i, f_j)=0$ if $i\neq j$, for all t in R.

Without much loss of generality we shall take

$$E=\text{linear span } \{e^{Zt}f_i : t\in R, i\in\mathbb{Z}\}.$$

In the notation of the previous section, consider $\mathscr{F}$, the family of finite subsets of $\mathbb{Z}$ (i.e. $\nu=1$). For $\Lambda\in\mathscr{F}$, we define $U(\Lambda)$ to be the subalgebra of $C_0(E)$ generated by $\{[e^{Zt}f_i] : t\in R, i\in\Lambda\}$, Clearly $\{U(\Lambda):\Lambda\in\mathscr{F}\}$ satisfy the isotony relation (1.1). Moreover $C(E)=\{\cup U(\Lambda):\Lambda\in\mathscr{F}\}^-$ and each $U(\Lambda)$ is invariant under e^{Zt}. We now define a potential Φ.

We put $\Phi(X)=0$, unless X is even, and if $X=\{i_1, \ldots, i_{2n}\}$, put

$$\Phi(X)=\alpha_{i_1,\ldots,i_{2n}}[f_{i_1}]\ldots[f_{i_{2n}}], \quad \text{where} \quad \alpha_{i_1,\ldots,i_{2n}}\in\mathbb{C}.$$

For convenience [and consistency with the CAR's and (2.2)] we impose the condition that if π is any permutation of $1, \ldots, 2n$, that

$$\alpha_{i_{\pi(1)},\ldots,i_{\pi(2n)}}=\operatorname{sgn}(\pi)\alpha_{i_1,\ldots,i_{2n}}.$$

Also if $(i_1, \ldots, i_{2n})\in\mathbb{Z}^{2n}$, with not all i_r distinct, we put

$$\alpha_{i_1,\ldots,i_{2n}}=0.$$

If $i\neq j$, $[e^{Zt}f_i]$ and $[f_j]$ anticommute because of (2.2) and the CAR's. Hence if $X\in\mathscr{F}$, and $i\notin X$, $[\Phi(X), [e^{Zt}f_i]]=0$ $\forall t\in R$. Thus if $\Lambda, X\in\mathscr{F}$ are disjoint

$$[\Phi(X), u]=0, \qquad \forall u\in U(\Lambda).$$

Finally under suitable conditions of the values of $\{\alpha_{i_1,\ldots,i_{2n}}\}$, Φ is an element of $\mathscr{B}\cup\mathscr{B}_0$.

Theorem 2.3. *For all* $A\in C(E)$, $t\in\mathbb{R}$,

$$\sigma_t A=\lim_{\Lambda\to\infty} e^{(Z+i\,\mathrm{ad}\,H(\Lambda))t}A$$

*exists in the norm topology, and defines a strongly continuous one-parameter group of *-automorphisms of* $C(E)$.

If $\int_{-\infty}^{\infty}|(e^{Zs}f_i, f_i)|ds<\infty$ *for all* i *in* $\mathbb{Z}$, *the wave operators*

$$W_\pm=\operatorname*{st\,lim}_{t\to\pm\infty}\sigma_t e^{-Zt}$$

exist on the CAR *algebra* $C(E)$.

Proof. The first statement follows from Theorems 1.12 and 1.21.

Suppose $(e^{Zs}f_i, f_i)\in L^1(R)$ for each i in $\mathbb{Z}$. By the remarks at the end of §1, it is enough to show for fixed i, that

$$\lim_{t\to\pm\infty} e^{t(Z+i\,\mathrm{ad}\,H(\Lambda))t}e^{-tZ}[f_i]$$

exists uniformly over $\Lambda\in\mathscr{F}$, in the norm topology.

Now if $i \in \Lambda$,

$$\operatorname{ad} H(\Lambda)\,[e^{-Zu} f_i]$$

$$= 2 \sum_{n=1}^{\infty} \sum_{\{j_r \in \Lambda : r = 1, 2, \ldots 2n-1\}} \alpha_{j_1, \ldots, j_{2n-1}, i} \, [f_{j_1}] \ldots [f_{j_{2n-1}}] \, (e^{-uZ} f_i, f_i) \, .$$

Hence for all $t > s \in \mathbb{R}$, $\Lambda \ni i$

$$\| e^{t(Z + i \operatorname{ad} H(\Lambda))} e^{-tZ} f_i - e^{s(Z + i \operatorname{ad} H(\Lambda))} e^{-sZ} f_i \|$$

$$\leqq \int_s^t \| \operatorname{ad} H(\Lambda) e^{-Zu} f_i \| du$$

$$\leqq 2 \int_s^t \sum_{n=1}^{\infty} \sum_{j_r} |\alpha_{j_1, \ldots, j_{2n-1}, i}| \, \| [f_{j_1}] \ldots [f_{j_{2n-1}}] \| \, |(e^{-Zu} f_i, f_i)| du$$

$$\leqq 2 \frac{\| \Phi \|}{\| f_i \|} \int_s^t |(e^{-Zu} f_i, f_i)| du$$

which tends to zero as $t, s \to \pm \infty$ independently of Λ.

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