On the Time-Delay of Simple Scattering Systems

Ph. A. Martin

Laboratoire de Physique Théorique, Ecole Polytechnique Fédérale, CH-1006 Lausanne, Switzerland

Abstract. A new rigorous and simple study of the time-delay formula is presented.

I. Introduction

The time-delay of a scattering process may intuitively be considered to be the difference between the time spent by the colliding particles within the region of mutual interaction and the time that they would have spent in the same region had they moved freely. Consider a simple scattering system (H, H_0) on a Hilbert space $\mathcal{H}$ with free evolution $U_t = \exp(-iH_0 t)$ and total evolution $V_t = \exp(-iHt)$ [1]. The wave operators $\Omega_\pm = \text{s-lim}_{t \to \pm \infty} \exp(iHt)\exp(-iH_0 t)$ are assumed to be complete and the scattering operator $S = \Omega_+^* \Omega_-$ is unitary. Let P_r be the projection operator on a region Σ_r in configuration space in which the distance between particles does not exceed r. If $\psi_t = V_t \Omega_- \varphi$ is a scattering state which behaves as the freely evolving state $\varphi_t = U_t \varphi$ as $t \to -\infty$, the mean times spent in Σ_r by the interacting and the free particles are respectively:

$$\int_{-\infty}^{\infty} (\psi_t, P_r \psi_t) dt \quad \text{and} \quad \int_{-\infty}^{\infty} (\varphi_t, P_r \varphi_t) dt .$$

The time-delay for the scattering state $\Omega_- \varphi$ and the region Σ_r is then defined by [2, 3]:

$$T_r(\varphi) = \int_{-\infty}^{\infty} [(\psi_t, P_r \psi_t) - (\varphi_t, P_r \varphi_t)] dt . \tag{1}$$

This definition raises three mathematical questions which are in logical order:
  (i) Is the expression (1) meaningful for finite r and suitable states φ?
  (ii) Does there exist a time-delay $T(\varphi) = \lim_{r \to \infty} T_r(\varphi)$ for infinite space region?
  (iii) Is $T(\varphi)$ related to the classical formula of Eisenbud and Wigner [4] asserting that the time-delay is expressed by the derivative of the phase shift with respect to energy?

Question (i) can clearly be affirmatively answered on the grounds that scattering states propagate away from any bounded space region so that time integrals occuring in (1) are finite.

The point (ii), which is likely to be true for sufficiently short ranged interactions, is mathematically more delicate and it has received two different treatments using time dependent methods. The first one [3] shows the existence of the limit with the help of an abstract analysis based on the condition that the difference of the resolvants $(H-z)^{-1}-(H_0-z)^{-1}$ belongs to the trace-class of operators. The second one [5] makes use of the concept of H-smoothness and of commutator techniques developed in potential scattering. Finally in [3] the connection (iii) between time-delay and phase shift (which is not studied in [5]) is obtained by the rather elaborated tools of the theory of the spectral displacement function due to Birman and Krein (see ref. in [3]).

The purpose of this note is to present a rigorous and elementary derivation of the Eisenbud-Wigner formula, which is independent of the previously quoted works. Our study of time-delay relies essentially on simple asymptotic properties in time of scattering systems. While being of slightly less general applicability than that of ref. [3], it uses much more direct and transparent arguments and thus it has the merit of simplicity and conciseness.

II. The Time-Delay Formula

The main element of our demonstration consists in the following proposition which we now state and prove.

Proposition 1. *Let U_t be a strongly continuous unitary group on a Hilbert space $\mathscr{H}$ whose generator $H_0 = \int \lambda dE_\lambda$ has an absolutely continuous spectral family E_λ. Let P_r be a sequence of projections on $\mathscr{H}$ converging strongly to I as $r \to \infty$, and $A = \int A(\lambda) dE_\lambda$ be an operator function of H_0 (in the sense of functional calculus). Let φ and χ be in $\mathscr{H}$. We assume:*

(i) the functions $t \to \|P_r U_t \varphi\|$ and $t \to \|P_r U_t \chi\|$ belong to $\mathscr{L}^1(R, dt)$ for each r.

(ii) $A(\lambda)$ is differentiable and $A(\lambda)$ together with its derivative are Fourier transforms of $\mathscr{L}^1$ functions. Then

$$\lim_{r \to \infty} \int_0^\infty (U_t\varphi, [P_r, A]U_t\chi)dt = -i\left(\varphi, \frac{dA}{dH_0}\chi\right). \tag{2}$$

From the assumption (ii), A and $\dfrac{dA}{dH_0} = \int \dfrac{dA(\lambda)}{d\lambda} dE_\lambda$ are bounded operators on $\mathscr{H}$. Notice that in view of (i) the time integral in (2) converges absolutely.

Proof. Set $\varphi_t = U_t\varphi$ and $\chi_t = U_t\chi$. By the very definition of A and U_t one has:

$$(\varphi, A\chi) = \int A(\lambda)d(\varphi, E_\lambda\chi) = \int d(\varphi, E_\lambda\chi) \int \tilde{A}(\alpha)e^{i\alpha\lambda}d\alpha$$
$$= \int d\alpha\tilde{A}(\alpha) \int e^{i\lambda\alpha}d(\varphi, E_\lambda\chi) = \int d\alpha\tilde{A}(\alpha)(\varphi_\alpha, \chi). \tag{3}$$

Since the Fourier transform $\tilde{A}(\alpha)$ of $A(\lambda)$ belongs to $\mathscr{L}^1$ the exchange of integrals is allowed by Fubini's theorem.

Inserting (3) in (2) we get:

$$\int_0^\infty (\varphi_t, [P_r, A]\chi_t)dt = \int_0^\infty dt \int d\alpha \tilde{A}(\alpha)[(\varphi_t, P_r\chi_{t-\alpha}) - (\varphi_{t+\alpha}, P_r\chi_t)]$$

$$= \int d\alpha \tilde{A}(\alpha) \int_0^\infty dt [(\varphi_t, P_r\chi_{t-\alpha}) - (\varphi_{t+\alpha}, P_r\chi_t)]$$

$$= \int d\alpha \tilde{A}(\alpha) \int_0^\alpha (\varphi_t, P_r\chi_{t-\alpha})dt \qquad (4)$$

Owing to the fact that the integrand is majorized by the $\mathscr{L}^1$ function $|\tilde{A}(\alpha)|(\|P_r\varphi_t\| \ \|\chi\| + \|P_r\chi_t\| \ \|\varphi\|)$ in α and t, the exchange of the α and t integrals is also possible.

(4) results of the change of variable $t + \alpha \to t$ in the second time integral. The integrand of (4) is majorized uniformly in r by an integrable function of α

$$\left| \tilde{A}(\alpha) \int_0^\alpha (\varphi_t, P_r\chi_{t-\alpha})dt \right| \leqq \left| \tilde{A}(\alpha) \int_0^\alpha \|\varphi\| \ \|\chi\| dt \right| = \|\varphi\| \ \|\chi\| \ |\alpha\tilde{A}(\alpha)| \,.$$

Therefore the dominated convergence theorem applies in (4) yielding

$$\lim_{r\to\infty} \int_0^\infty (\varphi_t, [P_r, A]\chi_t)dt = \int d\alpha \tilde{A}(\alpha) \int_0^\alpha \lim_{r\to\infty} (\varphi_t, P_r\chi_{t-\alpha})dt = \int d\alpha \tilde{A}(\alpha)\alpha(\varphi_\alpha, \chi) \,.$$

The same argument which gave (3) now shows that this last expression precisely equals $-i\left(\varphi, \dfrac{dA}{dH_0} \chi \right)$ and this concludes the proof. $\square$

Consider a scattering system (H, H_0) whose free Hamiltonian H_0 is assumed to have an absolutely continuous simple spectrum on $\mathscr{H}$. Then the scattering operator, which commutes with H_0, is necessarily given by a function $S(\lambda)$ on the spectrum of H_0. If such a scattering system has suitable asymptotics in time (i.e. the asymptotic condition holds with a sufficiently fast rate of convergence), Proposition 1 can be used for a straightforward derivation of the Eisenbud-Wigner formula. This is the content of Proposition 2.

Proposition 2. *Let $\mathscr{D}$ be the set of Schwartz functions with compact support on the spectrum of H_0 and let $\varphi \in \mathscr{D}$.*

a) Let P_r be a sequence of projections on $\mathscr{H}$ converging strongly to I as $r \to \infty$, and such that the functions $t \to \|P_r U_t\varphi\|$ and $t \to \|P_r U_t S\varphi\|$ belong to $\mathscr{L}^1(R, dt)$ for each r.

b) Assume that the functions $t \to \|(V_t\Omega_- - U_t)\varphi\|$ and $t \to \|(V_t\Omega_- - U_tS)\varphi\|$ are integrable at $t \to -\infty$ and $t \to +\infty$ respectively.

c) Suppose moreover that $S(\lambda)$ is twice continuously differentiable.

Then $\displaystyle\lim_{r\to\infty} T_r(\varphi) = -i\left(\varphi, S^* \dfrac{dS}{dH_0} \varphi \right)$.

Proof. Set $\psi_t = V_t\Omega_-\varphi$ and $\varphi_t = U_t\varphi$, $\|\varphi\| = 1$. We evaluate separately in (1) the integral over negative and positive times. For negative times the integrand is

majorized uniformly in r by an integrable function at $t \to -\infty$:

$$|(\psi_t, P_r\psi_t) - (\varphi_t, P_r\varphi_t)| \leqq |((\psi_t - \varphi_t), P_r\varphi_t)| + |(\psi_t, P_r(\psi_t - \varphi_t))| \leqq 2\|\psi_t - \varphi_t\| .$$

We have similarly for positive times

$$|(\psi_t, P_r\psi_t) - (S\varphi_t, P_rS\varphi_t)| \leqq 2\|\psi_t - S\varphi_t\|$$

which is integrable at $t \to +\infty$.

Hence by dominated convergence the integral over the negative times does not contribute to the time-delay in the limit $r \to \infty$ and we are left with

$$\lim_{r \to \infty} T_r(\varphi) = \lim_{r \to \infty} \int_0^\infty [(S\varphi_t, P_rS\varphi_t) - (\varphi_t, P_r\varphi_t)]dt$$

$$= \lim_{r \to \infty} \int_0^\infty (S\varphi_t, [P_r, S]\varphi_t)dt . \tag{5}$$

In obtaining (5), use has been made of the unitarity of the S operator. We now apply Proposition 1 to (5) in the following way. Let $g(\lambda)$ be a function on the spectrum of H_0 which belongs to $\mathscr{D}$ and equals one on the support of φ. Then $S_g\varphi = S\varphi$ with $S_g(\lambda) = g(\lambda)S(\lambda)$, and S_g can be substituted to S in (5). With the help of two integrations by part one verifies immediately from c) that $S_g(\lambda)$ as well as its first derivative are Fourier transforms of $\mathscr{L}^1$ functions. Therefore Proposition 1 applies to (5) with S_g in place of S, and we get

$$\lim_{r \to \infty} T_r(\varphi) = -i\left(S_g\varphi, \frac{dS_g}{dH_0}\varphi\right) = -i\left(\varphi, S^*\frac{dS}{dH_0}\varphi\right). \quad \square$$

Proposition 2 gives in a simple manner positive answers to the questions (ii) and (iii) stated in the introduction. It remains to exhibit a class of interactions for which conditions b) and c) hold.

III. Potential Scattering

Let $H_0 = -\varDelta$ be the Laplacian acting on $\mathscr{L}^2(R^3)$ and $H = -\varDelta + \mathscr{V}(x)$, $x = |\pmb{x}|$. $\mathscr{V}(x)$ is a spherically symmetric potential for which the asymptotic condition holds with complete wave operators. (See [6] X § 3.)

Lemmas 1 and 2 supplement the asymptotic condition with an information on the rate of convergence as t goes to infinity.

Lemma 1. *Assume that* $(1 + |\pmb{x}|^{1+\varepsilon})\mathscr{V}(x) \in \mathscr{L}^2(R^3)$ *for some* $\varepsilon > 0$. *Let* φ *be a vector whose momentum representation* $\tilde{\varphi}(\pmb{k})$ *is a Schwartz* $\mathscr{D}$ *function with no support at the origin* $\pmb{k} = 0$. *Then* $t \to \|(V_t\Omega_- - U_t)\varphi\|$ *is integrable at* $t \to -\infty$.

Proof. A standard estimate yields for $\varphi \in \mathscr{D}$

$$\|(V_t\Omega_- - U_t)\varphi\| \leqq \int_{-\infty}^t \|\mathscr{V}U_\tau\varphi\|d\tau .$$

The norm $\|\mathscr{V} U_t \varphi\|$ is evaluated with the help of the explicit form of the kernel of the free propagator in configuration space, noting that $\left(\dfrac{1}{2\pi}\right)^{3/2} \int \varphi(y) d^3 y = \tilde{\varphi}(0) = 0$,

$$\|\mathscr{V} U_t \varphi\|^2 = \left(\frac{1}{4\pi|t|}\right)^3 \int d^3 x |\mathscr{V}(x)|^2 \left| \int d^3 y \left(\exp\left(\frac{i|x-y|^2}{4t}\right) - 1\right) \varphi(y) \right|^2$$

$$= \left(\frac{1}{4\pi|t|}\right)^3 \left(\frac{1}{4|t|}\right)^{2\mu} \int d^3 x |\mathscr{V}(x)|^2 \left| \int d^3 y \left[\frac{\exp\left(\frac{i|x-y|^2}{4t}\right) - 1}{\left(\frac{|x-y|^2}{4t}\right)^\mu}\right] |x-y|^{2\mu} \varphi(y) \right|^2 .$$

Taking into account that $\left| \dfrac{e^{ix} - 1}{x^\mu} \right| \leqq 2$, $|x - y|^{2\mu} \leqq 2(|x|^{2\mu} + |y|^{2\mu})$ for $0 \leqq \mu \leqq 1$, this integral is estimated as follows:

$$\|\mathscr{V} U_t \varphi\|^2 \leqq \frac{C_1}{|t|^{3+2\mu}} \int d^3 x |\mathscr{V}(x)|^2 (\int d^3 y (|x|^{2\mu} + |y|^{2\mu}) |\varphi(y)|)^2$$

$\int |\varphi(y)| d^3 y$ and $\int |y|^{2\mu} |\varphi(y)| d^3 y$ are clearly finite with our choice of φ and hence

$$\|\mathscr{V} U_t \varphi\| \leqq \frac{C_2}{|t|^{3/2+\mu}} \|(1 + |x|^{2\mu}) \mathscr{V}\|_2 . \tag{6}$$

Choosing $\frac{1}{2} < \mu \leqq \frac{1}{2}(1 + \varepsilon)$ and integrating the inequality (6) with respect to t yields the desired result. $\square$

Lemma 2. *Let $\mathscr{V}(x)$ be as in Lemma 1. Consider the decomposition $\oplus \displaystyle\sum_{\substack{l=0 \\ |m| \leqq l}}^{\infty} \mathscr{H}_{lm}$ of $\mathscr{L}^2(R^3)$ into the direct sum of invariant subspaces under the rotation group. Let $\varphi \in \mathscr{H}_{lm}$ be a vector whose representative $\varphi_{lm}(k)$, $k = |k|$, is a $\mathscr{D}$ function with no support at the origin. If $S_l(k)$ is three times continuously differentiable, then $t \to \|(V_t \Omega_- - U_t S)\varphi\|$ is integrable at $t \to +\infty$.*

Proof. From the fact that the wave operators are complete we get

$$\|(V_t \Omega_- - U_t S)\varphi\| = \|(V_t \Omega_+ - U_t)S\varphi\| \leqq \int_t^\infty \|\mathscr{V} U_\tau S\varphi\| d\tau .$$

Thus the argument leading to (6) can be reproduced without modification and the conclusion of the Lemma will hold, provided that $\int d^3 y |(S\varphi)(y)|$ and $\int d^3 y |y|^{2\mu} |(S\varphi)(y)|$ are finite for some $\mu > \frac{1}{2}$. In view of the asymptotic properties of the spherical Bessel function j_l we conclude from three successive integrations by part that $(S\varphi)(y) = 4\pi Y_{lm}(\vartheta, \gamma) \int k^2 dk j_l(ky) S_l(k) \varphi_{lm}(k)$, $y = (y, \vartheta, \gamma)$, can be written in the form $Y_{lm}(\vartheta, \gamma) \dfrac{f(y)}{1 + y^4}$ with $f(y) \in \mathscr{L}^2(R^+, dy)$.

Hence by Schwartz inequality we have

$$\int d^3 y |y|^{2\mu} |(S\varphi)(y)| \leqq C \|f\|_2 \left(\int_0^\infty \left(\frac{y^{2(1+\mu)}}{1 + y^4} \right)^2 dy \right)^{1/2}$$

which is finite for $\mu < \frac{3}{4}$. $\square$

Finally Lemma 3 links the differentiability of the S matrix to the decrease of the potential at infinity. We merely sketch its proof which uses standard methods [7].

Lemma 3. *Assume that* $\int\limits_0 x^p |\mathscr{V}(x)| dx < \infty$. $p = 0, 1 \ldots s$, *s being a positive integer. Then* $S_l(k)$ *is continuously differentiable for* $k \neq 0$ *up to order s.*

Proof. From the wellknown relation between the Jost solution $f(k, x)$ of the radial Schrödinger equation and the scattering matrix [7, Chap. 5] it is sufficient to show that $f(k, x)$ together with $\dfrac{\partial}{\partial x} f(k, x)$ are s time continuously differentiable in k for some $x > 0$. $f(k, x) = \sum\limits_{n=0}^{\infty} f_n(k, x)$, where the iterated functions $f_n(k, x)$ obey the equation

$$f_n(k, x) = \int\limits_x^\infty B(k, x, y) \mathscr{V}(y) f_{n-1}(k, y) dy$$

with kernel $B(k, x, y) = \dfrac{i}{2k} (f_0(k, y) f_0(-k, x) - f_0(k, x) f_0(-k, y))$, $f_0(k, x) = (-i)^{l+1} kx h_l^{(2)}(kx)$, $h^{(2)}$ being the second spherical Hankel function. When k and x lie in the domains $k \geq k_0 > 0$, $x \geq 1$, we can find constants C_1 and C_2 independent of k and x such that

$$\left| \frac{\partial^p}{\partial k^p} f_0(k, x) \right| \leq C_1 x^p, \left| \frac{\partial^p}{\partial k^p} B(k, x, y) \right| \leq C_2 x^p y^p, \quad p = 0, 1, \ldots, s.$$

Then one proves easily by induction that $f_n(k, x)$ is continuously differentiable up to order s and that for $k \geq k_0$ and $x \geq 1$,

$$\left| \frac{\partial^p}{\partial k^p} f_n(k, x) \right| \leq C_1 (n+1)^p \frac{x^p}{n!} \left(C_2 \int\limits_x^\infty y^p |\mathscr{V}(y)| dy \right)^n \quad \begin{matrix} p = 0, 1, \ldots, s \\ n = 0, 1, 2, \ldots \end{matrix}$$

Thus the series of derivatives converges uniformly for $k \geq k_0$, from which the differentiability properties of $f(k, x)$ follow.   □

Let $\mathscr{V}(x)$ be a locally square integrable spherically symmetric potential which is $O\left(\dfrac{1}{x^\gamma}\right)$, $x \to \infty$, with $\gamma > 4$ and consider the restriction of the scattering system $(-\varDelta, -\varDelta + \mathscr{V}(x))$ to a subspace $\mathscr{H}_{lm}$. Then, on $\mathscr{H}_{lm}$, H_0 has a simple spectrum and one concludes from Lemmas 1, 2, and 3 that Proposition 2 applies for all $\varphi \in \mathscr{H}_{lm}$ which are $\mathscr{D}$ functions on the energy spectrum having no support at the origin $\lambda = |k|^2 = 0$. Notice that condition a) of Proposition 2 holds for projections P_r associated with any family of bounded space regions Σ_r approaching R^3 as $r \to \infty$. Therefore the time-delay for infinite space region is independent of its finite space-region approximating sequences.

Our result should be compared with that of [3] where it is shown that the energy shell components $T_r(\lambda)$ of the time-delay converge in the distribution sense to the derivative of the phase shift, when the difference of the resolvents

belongs to the trace-class. Since the latter condition is fulfilled in potential scattering if and only if $\mathcal{V}(x) = O\left(\dfrac{1}{x^{\gamma}}\right)$, $\gamma > 3$ [8], it allows a slightly slower decrease of the potential at infinity than that we have to require for the validity of the present analysis.

References

1. Jauch, J. M.: Helv. Phys. Acta **31**, 127 (1958)
2. Jauch, J. M., Marchand, J. P.: Helv. Phys. Acta **40**, 217 (1968)
3. Jauch, J. M., Sinha, K., Misra, B.: Helv. Phys. Acta **45**, 398 (1972)
4. Wigner, E. P.: Phys. Rev. **98**, 145 (1955)
5. Lavine, R.: In: Scattering in Mathematical Physics, p. 141. Dordrecht: D. Reidel Publishing Company 1974
6. Kato, T.: Perturbation Theory for Linear Operator. Berlin-Heidelberg-New York: Springer 1966
7. Alfaro, de, V., Regge, T.: Potential Scattering. Amsterdam: North-Holland 1965
8. Martin, Ph., Misra, B.: J. Math. Phys. **14**, 997 (1973)