Commun. math. Phys. 47, 167—170 (1976)

Communications in
Mathematical
Physics

On Uniqueness of KMS States of One-dimensional Quantum Lattice Systems

A. Kishimoto

Department of Physics, Kyoto University, Kyoto, Japan

Abstract. We present a proof of the theorem on the uniqueness of KMS states of one-dimensional quantum lattice systems, which is based on some equicontinuity.

1. Introduction

Araki [1] has proved, in full generality, the uniqueness of KMS states of one-dimensional quantum lattice systems under the condition that for some increasing family of finite volumes the corresponding surface energies are bounded. (See also [8, 3, 5, 9].) We present another proof of this fact in the same setting as in [1, 9]. The reader is referred to [1] for the connection with one-dimensional lattice systems.

2. Theorem

Let A be a UHF algebra and δ a normal *-derivation on A, i.e., the domain $D(\delta)$ of δ is the union of an increasing family $\{A_n\}$ of finite type I factors (which is dense in A). There exists $h_n = h_n^* \in A$ for each n satisfying $\delta(a) = \delta_{ih_n}(a) \equiv [ih_n, a]$ for all $a \in A_n$. Let τ be the unique tracial state on A and P_n the canonical conditional expectation of A onto A_n, i.e., $k_n \equiv P_n h_n \in A_n$ satisfies $\tau(h_n a) = \tau(k_n a)$ for all $a \in A_n$. If $\{\|h_n - k_n\|\}$ is bounded, the closure of δ generates a one parameter automorphism group ϱ_t satisfying

$$\varrho_t(X) = \lim e^{ik_n t} x e^{-ik_n t}, \qquad x \in A .$$

(For the proof, see [6].) Since ϱ_t is approximately inner, there exists at least one KMS state for any temperature [7]. On the uniqueness of KMS states we have

Theorem. *If $\{\|h_n - k_n\|\}$ is uniformly bounded, then A has only one ϱ_t-KMS state for each inverse temperature β.*

3. Proof

Let ψ be an extremal KMS state at β and $(\mathfrak{H}, \pi, \Psi)$ the GNS representation of A associated with ψ. Then Ψ is a cyclic and separating vector relative to $\mathfrak{M} \equiv \pi(A)''$.

Let Δ be the modular operator (for Ψ relative to $\mathfrak{M}$). Now we define the following function of z in the strip region $I_{\beta/2} \equiv \{z; \operatorname{Im} z \in [0, \beta/2]\}$ for each $x \in A$:

$$F_n(z; x) = (e^{i\bar{z}(-H+W_n)}\Psi | \pi(z) e^{-iz(-H+W_n)}\Psi)$$

where $W_n = \pi(h_n - k_n)$ and $H = -\beta^{-1}\log\Delta$. Then $F_n(z) = F_n(z; x)$ is a bounded continuous function of z in $I_{\beta/2}$ and holomorphic in the interior of $I_{\beta/2}$ [2]. For real t,

$$F_n(t) = (\varphi_t(iW_n)\Psi | \pi(z)\varphi_{-t}(iW_n)\Psi)\,.$$

Here

$$\varphi_t(iW_n) = e^{it(-H+W_n)}e^{itH}$$

$$= \sum_{m=0}^{\infty} \int_0^t dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{m-1}} dt_m \sigma_{t_m}(iW_n) \dots \sigma_{t_1}(iW_n)\,,$$

$$\sigma_t(Q) = e^{-itH} Q e^{itH}\,, \qquad Q \in \mathfrak{M}\,.$$

On the other boundary,

$$F_n\left(t + \frac{i\beta}{2}\right) = (\varphi_t(iW_n)\Psi(\sigma_t(\beta W_n)) | \pi(z)\varphi_{-t}(iW_n)\Psi(\sigma_{-t}(\beta W_n)))$$

where

$$\Psi(\sigma_t(\beta W_n)) = \exp[\tfrac{1}{2}(\log\Delta + \sigma_t(\beta W_n))]\Psi\,.$$

It is shown as follows:

$$e^{it(-H+W_n)} e^{\frac{\beta}{2}(-H+W_n)} \Psi$$

$$= e^{it(-H+W_n)} e^{itH} e^{-itH} e^{\frac{\beta}{2}(-H+W_n)} e^{itH} \Psi$$

$$= \varphi_t(iW_n) e^{\frac{\beta}{2}(-H+\sigma_t(W_n))} \Psi\,.$$

Now we can prove:

Lemma. *If* $\{\|W_n\|\}$ *is bounded, the families of functions* $\{F_n(t)\}$ *and* $\left\{F_n\left(t + \frac{i\beta}{2}\right)\right\}$ *are uniformly bounded and equicontinuous.*

Proof.

$$\|\varphi_t(iW_n)\| = 1\,,$$

$$\left\|\frac{d}{dt}\varphi_t(iW_n)\right\| = \|\varphi_t(iW_n)\sigma_t(iW_n)\|$$

$$= \|W_n\|\,,$$

$$\|\Psi(\sigma_t(\beta W_n))\| \leqq \exp\frac{\beta}{2}\|W_n\|\,,$$

$$\left\|\frac{d}{dt} e^{(it+\frac{\beta}{2})(-H+W_n)}\Psi\right\| = \|(-H+W_n)\Psi(\beta W_n)\|$$

$$= \|j(W_n)\Psi(\beta W_n)\| \leqq \|W_n\| \exp\frac{\beta}{2}\|W_n\|$$

where we have used the fact that $\log\Delta + \beta W_n - j(\beta W_n)$ is the modular operator for $\Psi(\beta W_n)(j(\beta W_n) = J\beta W_n J$, J is the modular conjugation operator, [cf. 2]). Q.E.D.

We are now ready to start the proof of Theorem. Let ω_n be the state such that $\omega_n(x) = \tau(e^{-\beta k_n}x)/\tau(e^{-\beta k_n})$. Then ω_n is a KMS state at β relative to $e^{t\delta i k_n}(x) = e^{itk_n}xe^{-itk_n}$ $(x \in A)$. First of all we choose a subsequence $\{n_k\}$ such that $\omega_{n_k} \to \omega$ in the vague topology. Then ω is a KMS state at β relative to $\varrho_t(x) = \lim e^{t\delta i k_n}(x)$ $(x \in A)$ [7].

We notice

$$F_n\left(\frac{i\beta}{2}; x\right) = \psi^{(\beta W_n)}(x) \equiv (\Psi(\beta W_n)|\pi(x)\Psi(\beta W_n))$$

where $\psi^{(\beta W_n)}/\psi^{(\beta W_n)}(1)$ is a KMS state at β relative to

$$\varrho_t^{(W_n)}(x) = \pi^{-1}(e^{it(H-W_n)}\pi(z)e^{-it(H-W_n)}).$$

Since $\varrho_t^{(W_n)}/A_n = e^{t\delta i k_n}/A_n$, we have

$$\psi^{(\beta W_n)}/A_n = \psi^{(\beta W_n)}(1)\omega_n/A_n.$$

(These facts are all due to the equivalence of the KMS condition and the Gibbs condition [4, 2, 1]). By choosing a suitable subsequence $\{m\}$ of $\{n_k\}$ we have convergences

$$F_m(z; y) \to F_\infty(z; y)$$

$$F_m(z; 1) \to F_\infty(z; 1)$$

for arbitrarily chosen $y \in \cup A_n$, where the convergence is uniform in z on every compact set in $I_{\beta/2}$ (by Lemma and the theory of normal families). Since $\|\delta_{iW_n}\| \leqq 2\|W_n\|$ and

$$\lim \delta_{iW_n}(\pi(a)) = \lim\{\pi(\delta_{ik_n}(a)) - \pi(\delta_{ik_n}(a))\}$$
$$= \lim \pi \circ (1-P_n)\delta(a) = 0 (a \in D(\delta) \equiv \cup A_n),$$

we obtain $\lim \delta_{iW_n}(\pi(z)) = 0$ for all $x \in A$. We can conclude

$$\lim \|[\varphi_t(iW_n)^*, \pi(x)]\| = 0.$$

This implies that $\varphi_t(iW_m)^*\varphi_{-t}(iW_m)$ converges weakly to $F_\infty(t, 1)1$, because $\mathfrak{M} \cap \mathfrak{M}'$ is trivial by the extremality of ψ. Hence

$$F_\infty(t; y) = \psi(y)F_\infty(t; 1)$$

which implies

$$F_\infty\left(\frac{i\beta}{2}; y\right) = \psi(y)F_\infty\left(\frac{i\beta}{2}; 1\right)$$

by the analytic continuation. On the other hand

$$F_\infty\left(\frac{i\beta}{2}; y\right) = \lim \psi^{(\beta W_m)}(y)$$
$$= \lim \psi^{(\beta W_m)}(1)\omega_m(y)$$
$$= \omega(y)F_\infty\left(\frac{i\beta}{2}; 1\right)$$

Since $\psi^{(\beta W_m)}(1) = \|\Psi(\beta W_m)\|^2 \geqq \exp \psi(\beta W_m)$ [2], $F_\infty\left(\frac{i\beta}{2}; 1\right) \neq 0$. Therefore $\psi(y) = \omega(y)$, i.e., $\psi = \omega$. Since an arbitrary extremal KMS state is equal to the fixed KMS state ω, the set of KMS states consists of only one state.

Acknowledgement. The author is deeply indebted to Professor H. Araki for his useful comments.

References

1. Araki, H.: On uniqueness of KMS states of one-dimensional quantum lattice systems (0000)
2. Araki, H.: Positive cone, Radon-Nikodym theorems, relative Hamiltonian and the Gibbs condition in statistical mechanics. An application of the Tomita-Takesaki theory (0000)
3. Araki, H.: Commun. math. Phys. **14**, 120—157 (1968)
4. Araki, H., Ion, P. D. F.: Commun. math. Phys. **35**, 1—12 (1974)
5. Dobrushin, R. L.: Funktsional'nyi Analiz i Ego Prilozheniya **2** (4), 44—57 (1968)
6. Kishimoto, A.: Dissipations and derivations
7. Powers, R. T., Sakai, S.: Commun. math. Phys. **39**, 273—288 (1975)
8. Ruelle, D.: Commun. math. Phys. **9**, 267—278 (1968)
9. Sakai, S.: On commutative normal *-derivations II, III

Communicated by H. Araki

Received September 5, 1975