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Statistical Mechanics of a One-dimensional Lattice Gas with Exponential-polynomial Interactions

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Abstract. Some properties of the transfer-matrix for a one-dimensional classical lattice-gas with exponential-polynomial pair interactions are studied using Hilbert space techniques.

I. Introduction and Statement of Results

We are concerned here with the statistical mechanics of a classical, one-dimensional lattice-gas, or equivalently of a spin system with exponentially decreasing pair interactions of the type

$$\varphi_1(n)=\lambda^n \sum_{i=0}^{p} c_i n^i \qquad (0<\lambda<1) \tag{1.1}$$

as well as potentials which are a finite sum of decreasing exponentials,

$$\varphi_2(n)=\sum_{i=1}^{k} c_i \lambda_i^n \qquad (0<\lambda<1) \tag{1.2}$$

potential (1.1) will be termed exponential-polynomial type. Ruelle [1][1] has established the absence of phase transitions in one-dimensional systems with translationally invariant two-body interactions that satisfy the condition

$$\sum_{i\in\mathbb{N}} i|\varphi(0,i)|<\infty \tag{1.3}$$

where $\mathbb{N}$ is the set of all integers >0.

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[1] Ruelle's results actually extend to many-body translationally invariant interactions which satisfy the following criterion

$$\sum_{l>0} \sum_{0<i_1<i_2<\ldots<i_l} i_l|\varphi^{(l+1)}(0,i_1,i_2,\ldots,i_l)|<\infty$$

where $\varphi^{(l+1)}$ is the $(l+1)$ body potential.

Furthermore Ruelle [1] has shown that the study of the statistical mechanics of one-dimensional lattice systems that satisfy (1.3) is greatly simplified by introducing the following operator $\mathscr{L}$ on the space $C(K_+)$ of functions continuous on $K_+=[0,1]^{\mathbb{N}}$. If $f\in C(K_+)$, $x\in K_+$; i.e., $x=\{x_i\}_{i\in\mathbb{N}}$:

$$\mathscr{L}f(x)=f(0,x)+\gamma\exp\left(-\sum_{i\in\mathbb{N}}x_i\varphi(i)\right)f(1,x)\,. \tag{1.4}$$

The operator $\mathscr{L}$ defined above is continuous but not compact. In order to introduce a compact operator one proceeds as follows [2]. One first notes that in [1], one utilizes the Banach space character of $C(K_+)$ which contains the functions $\mathscr{L}^n\mathbb{1}$, $n\in\mathbb{N}$. However, $\mathscr{L}^n\mathbb{1}$ is an entire function of $\sum_{i\in\mathbb{N}}x_i\varphi(i)$. This suggests that we consider changing to a variable $z(x)$ defined by

$$z(x)=\sum_{k=1}^{\infty}x_k\lambda^k;\qquad x=\{x_k\}_{k\in\mathbb{N}}\in K_+\,. \tag{1.5}$$

Let D be a closed disk with center at the origin and of radius $R>\lambda/(1-\lambda)$. Let, further,

$$A_\lambda(D)=(f:f\in C(K_+),\ f(z)=\varphi(z(x))$$

where $\varphi(z)$ is an analytic function in a circle of radius $|z|<R$ such that, if $\varphi(z)=\sum_{n=0}^{\infty}C_nz^n$, then $\sum_{n=1}^{\infty}R^{2n}|C_n|^2<\infty$. Then the restriction $\mathscr{L}_D$ of the operator $\mathscr{L}$ acting on $A_\lambda(D)$ can be seen to be defined by

$$\mathscr{L}_D\varphi(z)=\varphi(\lambda z)+\gamma\exp(-cz)\varphi(\lambda+\lambda z)\,. \tag{1.6}$$

Proposition 1. $\mathscr{L}_DA_\lambda(D)\subset A_\lambda(D)$.

The proof follows immediately from the above definition.

Definition. Define on $A_\lambda(D)$ a scalar product

$$\langle f|g\rangle=\sum_{n=0}^{\infty}R^{2n}\bar{C}_n\gamma_n$$

where $f(z)=\sum C_nZ^n$ and $g(z)=\sum\gamma_nz^n$. Then $A_\lambda(D)$ becomes a Hilbert space $\mathscr{H}(D)$. Ferrero [2] has shown that, provided $0<\lambda<1/2$, $\mathscr{L}$ in (1.6) is compact. It is not necessary in what follows to restrict $\lambda<1/2$. We shall require only that $0<\lambda<1$. In this article we establish some further properties of $\mathscr{L}$ and elucidate its connection with the transfer matrix. In particular we establish the following theorems.

Theorem 1. *The operator $\mathscr{L}_D^N$ in (1.6) is a trace-class operator $\forall N\geqq 1$, and its largest eigenvalue coincides with the largest eigenvalue of $\mathscr{L}^N$ on $C(K_+)$ (which is unique and positive).*

Corollary 1. *The principal eigenvector of $\mathscr{L}$ is of the form $h(x)=\varphi(z(x))$, where $\varphi(z)$ is an entire function of z.*

Corollary 2. *The largest eigenvalue of $\mathscr{L}$ on $C(K_+)$ depends analytically on γ in the neighborhood of γ real.*

In what follows we drop the suffix D on the operator $\mathscr{L}_D$.

Theorem 2. $\mathrm{Tr}(\mathscr{L}^N)$ *is, up to a multiplicative constant* $(1-\lambda^N)$, *the partition function* Q_N *for a one-dimensional lattice-gas containing* N*-sites interacting through a pair potential*

$$\varphi(n)=c\lambda^n \tag{1.7}$$

with periodic boundary conditions.

By this we mean: A given site $i(0\leqq i\leqq N)$ interacts with all the sites of $\mathbb{Z}$ to its right. ($\mathbb{Z}=$ the set of integers $\geqq 0$.) The occupation x_i for $i\geqq N$ is determined by

$$x_{i+N}=x_i \tag{1.8}$$

where

$$x_i=\begin{cases}0 & \text{if site } i \text{ is empty}\\ 1 & \text{if site } i \text{ is occupied}.\end{cases} \tag{1.9}$$

Theorem 3. *We form the function*

$$\Xi(z)=\exp\left(\sum_{N=1}^{\infty}(z^N/_N)Q_N\right) \tag{1.10}$$

where

$$Q_N=(1-\lambda^N)\,\mathrm{Tr}\,\mathscr{L}^N. \tag{1.11}$$

Then $\Xi(z)$, *which is analytic in the neighborhood of* $z=0$, *extends by analytic continuation to a meromorphic function in the entire* z*-plane.*

In Section III, we extend our results to systems with exponential-polynomial interactions of the form (1.1). The operator $\mathscr{L}$ now acts on a Hilbert-space $\mathscr{H}(D)$ of functions of $(p+1)$ complex variables, holomorphic on open polydisc $D_{(p+1)}$.

$$\mathscr{L}f(z)=f(\lambda Az)+\gamma\exp(-\tilde{c}\cdot z)f(\lambda(Az+I)) \tag{1.12}$$

where

$$z=\begin{bmatrix} z_0\\ z_1\\ z_2\\ \vdots\\ z_p\end{bmatrix} \tag{1.13a}$$

$$\left.\begin{aligned}(A)_{ij}&=\binom{i}{j} \quad \text{if} \quad j\leqq i\\ &=0 \quad \text{otherwise}\end{aligned}\right\}0\leqq i,j\leqq p, \tag{1.13b}$$

i.e. A is a $(p+1)\times(p+1)$ triangular matrix.

$$I=\begin{bmatrix}1\\1\\\vdots\\1\end{bmatrix} \quad \text{and} \quad \tilde{c}\circ z=\sum_{i=0}^{p} c_i z_i\,. \tag{1.13c}$$

Finally we indicate how some obvious generalizations can be made to systems with pair-interactions of the form 1.2.

II. Proof of Theorems 1–3

Lemma 1. *The operator $\mathscr{L}$ defined in* (1.6) *admits the following representation.*

$$\mathscr{L}=\sum_{x=0,1}\sum_{n=0}^{\infty}\lambda^n|\varphi_n^{(x)}\rangle\langle\psi_n^{(x)}|\cdot\rangle_{\mathscr{H}^*} \tag{2.1}$$

where

$$|\varphi_n^{(x)}\rangle=\gamma^x e^{-xcz}z^n \tag{2.2}$$

and

$$\langle\psi_n^{(x)}|f\rangle_{\mathscr{H}^*}=1/2\pi i\oint_{\partial_0 D} f(z)(z-\lambda x)^{-(n-1)}dz$$
$$=1/n!\,f^n(\lambda x)\equiv T_x^n f\,. \tag{2.3}$$

Note that $\{\varphi_n^{(0)}\}_{n\geqq 0}$ is a complete orthonormal basis in $\mathscr{H}(D)$. Furthermore $\langle\psi_n^{(x)}|f\rangle_{\mathscr{H}^*}$ is a linear functional in the dual $\mathscr{H}^*$ of $\mathscr{H}(D)$ and hence it follows from Riesz' theorem that there exists a unique $\xi_n\in\mathscr{H}(D)$ such that

$$\langle\psi_n^{(x)}|f\rangle_{\mathscr{H}^*}=\langle\xi_n^x|f\rangle_{\mathscr{H}(D)} \qquad \forall f\in\mathscr{H}(D)$$

and that

$$\|\xi_n^{(x)}\|_{\mathscr{H}(D)}=\|T_x^n\|_{\mathscr{H}^*}\,. \tag{2.4}$$

Proof of Lemma 1. Let $f(z)\in\mathscr{H}(D)$. Then

$$\mathscr{L}f(z)=\sum\lambda^n|\varphi_n^{(x)}\rangle\langle\psi_n^{(x)}|f\rangle_{\mathscr{H}^*}$$
$$=\sum_{n\geqq 0}\lambda^n z^n f^n(0)/n!+\gamma\sum_{n\geqq 0}e^{-cz}(\lambda z)^n f^n(\lambda)/n!$$
$$=f(\lambda z)+\gamma e^{-cz}f(\lambda+\lambda z)\,. \tag{2.5}$$

Proof of Theorem 1. Recall [3, 4] that an operator A is of trace class if and only if

$$\mathrm{Tr}[A]<\infty \tag{2.6}$$

where $[A]=+\sqrt{A^*A}$.

Now by Riesz' theorem, $\mathscr{L}$ can be written as

$$\mathscr{L}=\sum \lambda^n|\varphi_n^{(x)}\rangle\langle\xi_n^{(x)}|\cdot\rangle_{\mathscr{H}(D)}\,. \tag{2.7}$$

Denote $\operatorname{Tr}[\mathscr{L}]$ by $\tau(\mathscr{L})$, then

$$\begin{aligned}\tau(\mathscr{L}) &= \operatorname{Tr}[\sum \lambda^n|\varphi_n^{(x)}\rangle\langle\xi_n^{(x)}|\cdot\rangle]\\ &\leqq \sum_{n,x}\lambda^n\tau(|\varphi_n^{(x)}\rangle\langle\xi_n^{(x)}|\cdot\rangle)\\ &= \sum_{n,x}\lambda^n\|\varphi_n^{(x)}\|_{\mathscr{H}(D)}\|\xi_n^{(x)}\|_{\mathscr{H}(D)}\,.\end{aligned} \tag{2.8}$$

The following estimates are easily verified.

$$\left.\begin{aligned}&\|\varphi_n^{(0)}\|=R^n\\ &\|\varphi_n^{(1)}\|\leqq\gamma R^n\exp(c^2R^2/2)\\ &\|\psi_n^{(0)}\|\leqq 1/R^n\\ &\|\psi_n^{(1)}\|\leqq R^{-n}(1-\lambda/R)^{-(n+1)}\end{aligned}\right\} \tag{2.9}$$

substituting these estimates in (2.8) one finds that

$$\tau(\mathscr{L})\leqq 1/(1-\lambda)\{1+R\gamma\exp(C^2R^2/2)/(R-\lambda/(1-\lambda))\}$$

provided $R>\lambda/(1-\lambda)$. This is precisely the restriction we had imposed on the radius R of the disk at the beginning. Compactness of $\mathscr{L}$ follows at once as a corollary of Theorem 1.

Proposition 2.

$$\operatorname{Tr}\mathscr{L}=1/(1-\lambda)\left\{1+\gamma\exp\left(-c\sum_{n=1}^{\infty}\lambda^n\right)\right\}. \tag{2.10}$$

Remark. $(1-\lambda)\operatorname{Tr}\mathscr{L}$ can be interpreted as the partition function for a system with one site (site 1) interacting with all other sites $n\geqq 1$ to the right with the pair potential (1.7) and $x_{1+n}=x_n$. For all $n\geqq 1$.

Proof. Choose an orthonormal basis $\{x_n\}$ in $\mathscr{H}(D)$. Then

$$\operatorname{Tr}\mathscr{L}=\sum\langle x_n|\mathscr{L}|x_n\rangle \tag{2.11a}$$

$$=\sum_{m,x}\lambda^m\langle\psi_m^{(x)}|\varphi_n^{(x)}\rangle_{\mathscr{H}^*} \tag{2.11b}$$

$$=\sum_{m,x}\gamma^x\lambda^m\oint_{\partial^0 D}e^{-xcz}\cdot z^m(z-\lambda x)^{-(m+1)}dz/2\pi i \tag{2.11c}$$

$$=\sum_x\gamma^x/(1-\lambda)\oint_{\partial_0 D}e^{-xcz}(z-\lambda x/(1-\lambda))^{-1}dz/2\pi i \tag{2.11d}$$

$$=1/(1-\lambda)\left\{1+\gamma\exp-\left(c\sum_{n=1}^{\infty}\lambda^n\right)\right\}. \tag{2.11e}$$

(2.11d) follows from the fact that $\sum(\lambda z/(z-\lambda))^m$ is uniformly convergent for $|z|>\lambda/(1-\lambda)$ $(0<\lambda<1)$.

Proof of Theorem 2.

$$\operatorname{Tr}\mathscr{L}^N=\sum_{\{x_i\}}\operatorname{Tr}(\mathscr{L}_{x_1}\mathscr{L}_{x_2}\dots\mathscr{L}_{x_N}) \tag{2.12a}$$

$$=\sum_{\{x_i\}}\sum_{\{n_i\}}\prod_i(\lambda^{n_i})\langle\psi_{n_N}^{(x_N)}|\varphi_{n_1}^{(x_1)}\rangle_{\mathscr{H}^*}\langle\psi_{n_1}^{(x_1)}|\varphi_{n_2}^{(x_2)}\rangle_{\mathscr{H}^*}$$
$$\times\dots\langle\psi_{n_{N-1}}^{(x_{N-1})}|\varphi_{n_N}^{(x_N)}\rangle_{\mathscr{H}^*} \tag{2.12b}$$

$$=\sum_{\{x_i\}}\left(\gamma^{\sum_k x_k}\right)\sum_{\{n_i\}}\prod_i(\lambda^{n_i})\oint_{\partial_0 D}\prod_{k=1}^N(dz_k)$$
$$\cdot(2\pi i)^{-N}\exp\left(-c\sum_{k=1}^N x_{k+1}z_k\right)\prod_{k=1}^N(z_k)^{n_{k+1}}\Big/\prod_{k=1}^N(z_k-\lambda x_k)^{n_k+1}\,. \tag{2.12c}$$

In above $\mathscr{L}_0$ is that part of $\mathscr{L}$ that corresponds to $x=0$ in (2.1) and $\mathscr{L}_1$ is the part with $x=1$.

$$x_{k+N}=x_k \quad\text{and}\quad n_{k+N}=n_k\,. \tag{2.13}$$

In (2.12c) $\partial_0 D$ is the distinguished boundary of D^N; i.e.

$$\partial_0 D=\partial_0 D_1\times\partial_0 D_2\times\dots\times\partial_0 D_N\,.$$

The N-fold summation is uniformly convergent if $\lambda|z_k|<|z_{k+1}-\lambda|(k=1,\dots,N)$ and in particular if the radii R_k of $\partial_0 D_k$ are all equal and such that $R_k=R_0>\lambda/(1-\lambda)$; $(\forall k)$. Thus

$$\operatorname{Tr}\mathscr{L}^N=\sum_{\{x_i\}}\gamma^{\sum_k x_k}\oint_{\partial_0 D}\prod_k dz_k(2\pi i)^N$$
$$\cdot\exp\left(-c\sum_k x_{k+1}z_k\right)\Big/\prod_{k=1}^N(z_{k+1}-\lambda x_{k+1}-\lambda z_k)\,. \tag{2.14}$$

Define a new variable

$$z_{k+1}-z_k=w_{k+1}\quad\text{with}\quad w_{k+N}=w_k\qquad(k=1,\dots,N)\,. \tag{2.15}$$

In matrix notation (2.15) reads

$$AZ=W \tag{2.16}$$

where

$$A=\begin{array}{cc} N\text{ Columns} & \\ \begin{bmatrix} 1 & 0 & \cdot & \cdot & \cdot & & -\lambda\\ -\lambda & 1 & 0 & \cdot & & & 0\\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\ & & & & & & \\ 0 & \cdot & \cdot & \cdot & & -\lambda & 1\end{bmatrix} & N\text{-Rows}\end{array}\quad Z(\text{or } W)=\begin{bmatrix} z_1(w_1)\\ z_2(w_2)\\ \cdot\\ \cdot\ \cdot\\ z_N(w_N)\end{bmatrix}. \tag{2.17}$$

Inverting (2.16) we get

$$z_i=1/(1-\lambda^N)\sum_{k=1}^N w_{k+i}\lambda^{N-k}\,, \tag{2.18}$$

$$\prod_i(dz_i)=1/(1-\lambda^N)\prod_i dw_i\,. \tag{2.19}$$

Inserting (18) and (19) into (2.14), one gets

$$\mathrm{Tr}\mathscr{L}^N = 1/(1-\lambda^N)\sum_{\{x_i\}}\oint_{\partial_0 D'}\prod_k dw_k\left(\gamma^{\sum_k x_k}\right)$$

$$\cdot\exp\left(-c/(1-\lambda^N)\sum_k x_{k+1}\sum_i w_{k+1}\lambda^{N-i}\right)(2\pi i)^{-N}\Big/\prod_{k=1}^N (w_k-\lambda x_k)\,.$$

Clearly, if $|z_k| > \frac{\lambda}{(1-\lambda)}$ $(\forall k)$, on applying Cauchy's theorem we pick up the contribution from the poles at $w_k=\lambda x_k \forall k$, and we obtain the following.

$$\mathrm{Tr}\,\mathscr{L}^N = 1/(1-\lambda^N)\sum_{\{x_i\}}\left(\prod_k \lambda^{x_k}\right)\exp\left(-c\lambda/(1-\lambda^N)\sum_{k=1}^N x_{k+1}\sum_{i=1}^N \lambda^{N-i}x_{k+i}\right) \quad (2.21\,\mathrm{a})$$

$$= 1/(1-\lambda^N)\sum_{\{x_i\}}\left(\prod_k \gamma^{x_k}\right)\exp\left(-c\sum_{k=1}^N x_{k+1}\sum_{l=1}^N x_{k+i}\sum_{l=0}^\infty \lambda^{(Nl+N-i+1)}\right). \quad (2.21\,\mathrm{b})$$

Let $k+i \pmod N = s$, $(0\leqq k, i\leqq N)$ and $N+1-i=t$. Imposing the condition $x_{i+N}=x_i$ $(0\leqq i\leqq N)$ we can write (2.21 b) as

$$\mathrm{Tr}\mathscr{L}^N = 1/(1-\lambda^N)\sum_{\{x_i\}}\exp\left(-c\sum_{s=1}^N x_s\sum_{t=1}^\infty x_{s+t}\lambda^t\right)\left(\gamma^{\sum_k x_k}\right). \quad (2.22)$$

Thus $\mathrm{Tr}\,\mathscr{L}^N$ is $(1-\lambda^N)$ times the partition function for a lattice gas of N sites subject to the boundary conditions imposed earlier.

Proof of Theorem 3. From (1.10) and (1.11) it follows that

$$\Xi(z) = \exp\left(\sum_{N=1}^\infty (z^N/_N)(1-\lambda^N)\,\mathrm{Tr}\mathscr{L}^N\right)$$

$$= \exp\left(\sum_{N=1}^\infty (z^N/_N)(1-\lambda^N)\sum_{\{k\}}\lambda_k^N\right) \quad (2.23)$$

where $\{\lambda_k\}$ are the eigenvalues of $\mathscr{L}$ repeated according to their multiplicity. For convenience we assume that the set $\{\lambda_k\}$ is ordered, i.e.; $\lambda_0\geqq\lambda_1\geqq\lambda_2\geqq\ldots\geqq\lambda_k\geqq\ldots$. From the compactness of $\mathscr{L}$ it follows that $\lambda_k\to 0$ as $k\to\infty$. Performing the sum over N, we obtain, provided that $|z|<1/|\lambda_0|$, the following

$$\Xi(z) = \exp\left(\sum_{\{k\}}\{\ln(1-\lambda_k\lambda z)-\ln(1-\lambda_k z)\}\right) \quad (2.24\,\mathrm{a})$$

$$= \prod_{\{k\}}(1-\lambda\lambda_k z)\Big/\prod_{\{k\}}(1-\lambda_k z) \quad (2.24\,\mathrm{b})$$

$$= f(\lambda z)/f(z)\,. \quad (2.24\,\mathrm{c})$$

The infinite products in (2.24 b) are convergent on any compact domain of the complex plane and thus define in (2.24 c) a ratio of two entire functions of z. Thus $\Xi(z)$ which is analytic in a small neighborhood of z, extends by analytic continuation into a meromorphic function in the entire complex plane.

So far we have used the lattice-gas language. To translate it into the spin language we only need to redefine $\mathscr{L}$ as follows

$$\begin{aligned} &\mathscr{L}=\mathscr{L}_{+}+\mathscr{L}_{-} \\ &\mathscr{L}_{+} f(z)=\exp(cz) f(-\lambda+\lambda z) \\ &\mathscr{L}_{-} f(z)=\exp(-cz) f(\lambda+\lambda z)\,. \end{aligned} \tag{2.25}$$

Then all the results in this section are valid and in particular

$$\operatorname{Tr}\mathscr{L}^{N}=1/(1-\lambda^{N}) \sum_{\{\sigma_{i}=\pm 1\}} \exp\left(-c \sum_{i=0}^{N} \sigma_{i} \sum_{l=1}^{\infty} \sigma_{i+l}\lambda^{l}\right). \tag{2.26}$$

III. Exponential-Polynomial Interactions

We briefly indicate how the above analysis can be carried over to accomodate potentials of the form (1.1);

$$\varphi(n)=\lambda^{n} \sum_{i=0}^{p} c_{i} n^{i}\,. \tag{1.1}$$

Ferrero [2] has shown that the operator $\mathscr{L}$ defined in (1.12) is again compact.

Proposition 3. *The operator* $\mathscr{L}$ *defined in* (1.12), *acting on a Hilbert space* $\mathscr{H}(D)$ *of functions of* $(p+1)$ *variables, holomorphic on an open polydisk D, admits the following representation*

$$\mathscr{L}=\sum_{\{n_{i};\,i=0,\ldots p\}} \prod_{i} (\lambda^{n_{i}}) \sum_{x=0,1} |\varphi_{\{n_{i}\}}^{(x)}\rangle \langle \psi_{\{n_{i}\}}^{(x)}|\ldots\rangle_{\mathscr{H}^{*}} \tag{3.1}$$

where

$$|\varphi_{\{n_{i}\}}^{(x)}\rangle=\exp(-x\tilde{c}_{0} z) \prod_{i=0}^{p} (Az)_{i}^{n_{i}} \tag{3.2a}$$

and

$$\langle \psi_{\{n_{i}\}}^{(x)} | f(z)\rangle_{\mathscr{H}^{*}}=\oint_{\partial_{0} D} \prod_{i=0}^{p} (dz_{i})(2\pi i)^{-(p+1)} f(\{z_{i}\}) \Big/ \prod_{i=0}^{p} (z_{i}-\lambda x)^{(n_{i}+1)}\,. \tag{3.2b}$$

(See Section 1 for notations.) We have dropped the factor γ in (3.2a) as this can easily be incorporated in the end if one wishes.

Proof. The representation (3.1) is an obvious generalization of Lemma 1 to $p+1$ complex variables.

Tr $\mathscr{L}$ and Tr$\mathscr{L}^{N}$ can be calculated in a manner analogous to the calculations in Section 2. However there are some important differences in the convergence arguments. This is best illustrated by calculating Tr$\mathscr{L}$. From (3.1) and (3.2), it follows that

$$\operatorname{Tr}\mathscr{L}=\sum_{\{n_{i}\}} \sum_{\{x\}} \prod_{i=0}^{p} (\lambda^{n_{i}}) \langle \psi_{\{n_{i}\}}^{(x)} | \varphi_{\{n_{i}\}}^{(x)}\rangle_{\mathscr{H}^{*}} \tag{3.3a}$$

$$\begin{aligned} &=\sum_{\{n_{i}\}} \sum_{(x)} \prod_{i=0}^{p} (\lambda^{n_{i}}) \oint_{\partial_{0} D^{(x)}} \prod_{\alpha} (dz_{\alpha})(2\pi i)^{-(p+1)} \prod_{i} (\sum_{j} A_{ij} z_{j})^{n_{i}} \\ &\quad \cdot \exp\left(-x \sum_{0}^{p} c_{i} z_{i}\right) \Big/ \prod_{i} (z_{i}-\lambda x)^{(n_{i}+1)}\,. \end{aligned} \tag{3.3b}$$

Now the series

$$\sum_{n_i=0}^{\infty}\left(\lambda\sum_j A_{ij}z_j\right)^{n_i}/(z_i-\lambda x)^{n_i}\qquad (i=0,1,\ldots,p)$$

are uniformly convergent provided that

$$\left|\sum_j A_{ij}z_j/(z_i-\lambda x)\right|<1/\lambda\qquad (i=0,1,\ldots,p).$$

This is accomplished if one chooses the radii $R_0^{(x)}, R_1^{(x)},\ldots,R_p^{(x)}$ of the polydisk $\partial_0 D^{(x)}=\partial_0 D_0^{(x)}\times\ldots\times\partial_0 D_p^{(x)}$ such that $R_p^{(x)}>R_{p-1}^{(x)}>\ldots>R_0^{(x)}$ and

$$R_i^{(x)}>\lambda/(1-\lambda)\left[x+\sum_{j=0}^{i-1}\binom{i}{j}R_j^{(x)}\right].\tag{3.4}$$

Thus for $x=0$ one chooses the radii $R_i^{(0)}$ in (3.3 b) such that

$$R_i^{(0)}>\lambda/(1-\lambda)\left[\sum_{j=0}^{i-1}\binom{i}{j}R_j^{(0)}\right]\tag{3.5}$$

and for $x=1$

$$R_i^{(1)}>1/(1-\lambda)\left[1+\sum_{j=0}^{i-1}\binom{i}{j}R_j^{(1)}\right].\tag{3.6}$$

Now the summation and integration can be interchanged and we obtain for

$$\mathrm{Tr}\mathscr{L}=\sum_{(x)}\oint_{\partial_0 D^{(x)}}\prod_{\alpha=0}^{p}dz_\alpha\exp\left(-z\sum_i c_iz_i\right)(2\pi i)^{-(p+1)}\Big/\prod_i\left(z_i-\lambda x-\lambda\sum_j A_{ij}z_j\right).\tag{3.7}$$

Let

$$w_i=z_i-\lambda\sum_j A_{ij}z_j\qquad (i=0,1,\ldots,p).\tag{3.8}$$

In matrix form

$$W=(1-\lambda A)z.\tag{3.9}$$

Since $(1-\lambda A)$ is non-singular, we can invert (3.9) to obtain

$$z_i=\sum_j B_{ij}w_j\tag{3.10}$$

where

$$B=(1-\lambda A)^{-1}.\tag{3.11}$$

Clearly $\det B=(1-\lambda)^{p+1}$ and B is again triangular matrix. The Jacobian of the transformation is easily seen to be $(1-\lambda)^{-(p+1)}$. Applying Cauchy's theorem to (3.7), the only contribution comes from the point $W=\lambda I$. Hence

$$\begin{aligned}\mathrm{Tr}\mathscr{L}&=(1-\lambda)^{-(p+1)}\sum_{(x)}\exp(-x\lambda\bar{C}(1-\lambda A)^{-1}I\\&=(1-\lambda)^{-(p+1)}\sum_{(x)}\exp\left(-\lambda x\sum_{n=0}^{\infty}\tilde{C}(\lambda A)^nI\right).\end{aligned}\tag{3.12}$$

Now

$$
\begin{aligned}
\tilde{C}A^n I &= \sum_{i=0}^{p} \sum_{j=0}^{p} c_i (A^n)_{ij} \\
&= \sum_{i=0} \sum_{j=0} \sum_{k_1} \dots \sum_{k_{n-1}} c_i A_{ik_1} A_{k_1 k_2} \dots A_{k_{n-1} j} \\
&= \sum_i \sum_j \sum_{\{k_i\}} c_i \binom{i}{k_1} \binom{k_1}{k_2} \dots \binom{k_{n-1}}{j} \\
&= \sum_{i=0}^{p} c_i (n+1)^i ,
\end{aligned} \tag{3.13}
$$

$$
\mathrm{Tr}\mathscr{L} = (1-\lambda)^{-(p+1)} \left\{ 1 + \exp\left(- \sum_{n=1}^{\infty} \lambda^n \sum_{\alpha=0}^{p} c_\alpha n^\alpha \right) \right\} . \tag{3.14}
$$

Similarly one can calculate $\mathrm{Tr}\mathscr{L}^N$ and one finds the following

$$
\mathrm{Tr}\mathscr{L}^N = (1-\lambda^N)^{-(p+1)} \sum_{\{x_i\}} \exp\left(- \sum_{s=1}^{N} x_s \sum_{n=1}^{\infty} x_{s+n} \lambda^n \sum_{\alpha=0}^{p} c_\alpha n^\alpha \right) . \tag{3.15}
$$

One constructs $\Xi(z)$ analogous to (1.10)

$$
\begin{aligned}
\Xi(z) &= \exp\left(\sum_{N=1}^{\infty} z^N / N (1-\lambda^N)^{p+1} \, \mathrm{Tr}\mathscr{L}^N \right) \\
&= \exp\left(\sum_{N=1}^{\infty} z^N / N \sum_{\alpha=0}^{p+1} \binom{p+1}{\alpha} (-\lambda^N)^\alpha \sum_k \lambda_k^N \right)
\end{aligned} \tag{3.16}
$$

where $\{\lambda_k\}$ are the eigenvalues of $\mathscr{L}$ repeated according to their multiplicity. Performing the sum over N, we obtain

$$
\Xi(z) = \exp\left(\sum_{\alpha=0}^{p+1} (-1)^{\alpha+1} \binom{p+1}{\alpha} \ln \prod_k (1 - \lambda_k \lambda^\alpha z) \right) . \tag{3.17}
$$

Defining $f(z)$ by

$$
f(z) = \prod_k (1 - \lambda_k z) \tag{3.18}
$$

which is an entire function z, one obtains for 3.17

$$
\begin{aligned}
\Xi(z) &= \exp\left(\sum_{\alpha=0}^{p+1} (-1)^{\alpha+1} \binom{p+1}{\alpha} \ln f(\lambda^\alpha z) \right) \\
&= \prod_{\alpha=0}^{p+1} [f(\lambda^\alpha z)]^{(-1)^{\alpha+1} \binom{p+1}{\alpha}} .
\end{aligned} \tag{3.19}
$$

Thus $\Xi(z)$ is a ratio of a finite product of entire functions. Hence it is meromorphic.

The approach to the transfer matrix outlined above can easily be generalized to pair potentials which are a finite sum of exponentials of the form

$$
\Phi(n) = \sum_{i=1}^{M} \lambda_i^n \sum_{j=1}^{p} (c_{ij} n^j) . \tag{3.20}
$$

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