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Distributional Limits of Renormalized Feynman Integrals with Zero-Mass Denominators

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Abstract. It is shown that the $\varepsilon \to 0$ limits of renormalized Feynman integrals exist and define Lorentz invariant tempered distributions in the external momenta. The proof applies to the case where some or all particle masses vanish.

Within the Bogoliubov-Parsiuk-Hepp-Zimmermann (BPHZ) framework of renormalized perturbation theory [1–3], the connected Green functions of elementary and composite fields are expressed as sums of contributions from Feynman diagrams, each of which corresponds to a subtracted momentum-space integral of the form

$$J_\varepsilon(p) = \int_{\mathbb{R}^{4M}} dk R_\varepsilon(p, k) \tag{1}$$

$p = (p_1, p_2, \ldots, p_N) =$ independent external momenta ($p_i \in \mathbb{R}^4$), $k = (k_1, k_2, \ldots, k_N) =$ independent internal (loop) momenta ($k_j \in \mathbb{R}^4$).

With Zimmermann's subtraction prescription [3], if all mass parameters are positive, the integral (1) converges absolutely for all $\varepsilon > 0$; moreover, as ε tends to zero, $J_\varepsilon(p)$ approaches, in the sense of tempered distributions, a Lorentz invariant limit [4]. Zimmermann's proof of the distributional limit was based on an earlier theorem of Hepp [2]. There, also, the non-vanishing of all masses was a crucial hypothesis.

In Ref. [5], one of us (J.H.L.) introduces a modified subtraction scheme such that the integral

$$T_\varepsilon(\phi) = \int_{\mathbb{R}^{4N}} \times \int_{\mathbb{R}^{4M}} dp dk \phi(p) R_\varepsilon(p, k) \tag{2}$$

converges absolutely for arbitrary $\varepsilon > 0$ and $\phi \in \mathscr{S}(\mathbb{R}^{4N})$, provided that a certain infrared power-counting criterion is fulfilled. There is no requirement that any of the masses of the unsubtracted integrand be positive (there is, however, at least one non-zero normalization mass appearing in subtractions terms). In the present article, we use the absolute convergence of (2) to show that T_ε approaches, when ε tends to zero, a Lorentz invariant limit as a tempered distribution. Again, some

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of the masses may vanish. Our proof shares some of the features of Hepp's proof of the purely massive case in [2, 6], and our technique of integration by parts is similar to that of [7].

Our main theorem (Theorem 1 below), combined with the subtraction prescriptions of [5, 8] and the absolute convergence theorems of [5, 9, 10], allow one to construct, within the BPHZ framework, the perturbative Green functions of a wide class of models involving zero-mass particles. This can now be done with the same level of mathematical precision attained by Blanchard and Seneor [11] using the method of Epstein and Glaser [12]. Normal product techniques based on [5, 8] have already proven extremely useful in deriving Ward identities and other structural relations in a number of theories with zero-mass particles [8, 12–15].

We begin with a (presumably subtracted) momentum-space Feynman integrand

$$R_\varepsilon(p, k) = P(p, k, \varepsilon) \Big/ \prod_{i=1}^{n} (\overline{l_i^2} - \overline{\mu_i^2}) .$$

P is a polynomial,

$$l_i = \sum_{j=1}^{N} \gamma_{ij} p_j + \sum_{j=1}^{M} \beta_{ij} k_j \tag{3}$$

a linear form in the momenta, and the ε dependence in the denominators is included via Zimmermann's prescription [3]:

$$\overline{l_i^2} = (l_i^0)^2 - \mathbf{l}_i^2(1 - i\varepsilon) ,$$
$$\overline{\mu_i^2} = \mu_i^2(1 - i\varepsilon) ,$$

with $\mu_i^2 \geqq 0$. We assume that $\operatorname{rank} \beta = M$.

Our main result is

Theorem 1. *Suppose that, for $\varepsilon > 0$, R_ε defines an element in $\mathscr{S}'(\mathbb{R}^{4N})$ via an integral* (2) *which is absolutely convergent for every $\phi \in \mathscr{S}(\mathbb{R}^{4N})$. Then*

$$T_0 = \lim_{\varepsilon \to 0+} T_\varepsilon$$

exists in $\mathscr{S}'(\mathbb{R}^{4N})$; T_0 is a distribution of order at most $n - 2M - [r/2]$, where r is the degree of P in k. If $P(p, k, 0)$ is a Lorentz covariant polynomial, then T_0 is a distribution of the same Lorentz covariance.

The first step in proving Theorem 1 is to introduce Feynman parameters, so that (2) becomes

$$T_\varepsilon(\phi) = (n-1)! \int dp dk \phi(p) P(p, k, \varepsilon) \int_{\mathscr{D}} d\alpha \left[\sum_{i=1}^{n} \alpha_i (\overline{l_i^2} - \overline{\mu_i^2}) \right]^{-n} , \tag{4}$$

where $\mathscr{D} = \{\alpha | \alpha_i \geqq 0, \sum \alpha_i = 1\}$. Zimmermann [3] has shown that (4) is absolutely convergent; his proof applies to the case considered here in which some μ_i may vanish. By Fubini's theorem, we may then interchange the α and k integrations. The k integrations are evaluated (for almost every α) by completing the square and diagonalizing the resulting quadratic form in the denominator (it is here we need rank $\beta = M$). We find

$$T_\varepsilon(\phi) = (1-i\varepsilon)^{-(3M/2+[r'/2])} I_\varepsilon^1(\phi),$$

where r' is the degree of P in $\boldsymbol{k}$, and

$$I_\varepsilon^1 = \int_{\mathbb{R}^{4N}} dp \int_{\mathscr{D}} d\alpha N(\alpha, p, \varepsilon)\phi(p) F_\varepsilon^1(\alpha, p)^{-t}. \tag{5}$$

Here $N(\alpha, p, \varepsilon)$ is a polynomial in p and ε, rational in α;

$$\begin{aligned} t &= n - 2M - [r/2]; \\ F_\varepsilon^1(\alpha, p) &= \sum_{i,j=1}^{N} p_i^0 A_{ij} p_j^0 - \left(\sum_{i,j=1}^{n} \sum_{\mu=1}^{3} p_i^\mu A_{ij} p_j^\mu + M^2 \right)(1-i\varepsilon) \\ &= p^0 A p^0 - (\boldsymbol{p} A \boldsymbol{p} + M^2)(1-i\varepsilon); \\ M^2 &= \sum_{i=1}^{n} \alpha_i \mu_i^2; \end{aligned}$$

and A is an $N \times N$ quadratic form, rational in α, continuous in $\mathscr{D}$, and positive definite when all α_i are positive. We note that, if $P(p, k, 0)$ is Lorentz covariant, so is $N(\alpha, p, 0)$. Thus Theorem 1 will follow from

Theorem 2. *If the integral* (5) *is absolutely convergent for all* $\varepsilon > 0$ *and all* $\phi \in \mathscr{S}(\mathbb{R}^{4N})$, *then*

$$I_0 = \lim_{\varepsilon \to 0+} I_\varepsilon^1$$

exists in $\mathscr{S}'(\mathbb{R}^{4N})$; I_0 *is a distribution of order at most* t. *If* $N(p, k, 0)$ *is Lorentz covariant, so is* I_0.

Theorem 2 can be expected to have independent application to α-space renormalization methods.

Our proof of the Lorentz covariance is similar to that of Zimmermann [3]: we show that I_0 is also the limit of distributions which are manifestly covariant. Thus define

$$\begin{aligned} F_\varepsilon^2 &= pAp - M^2 + i\varepsilon \\ I_\varepsilon^2(\phi) &= \int dp \int_{\mathscr{D}} d\alpha N(\alpha, p, \varepsilon)\phi(p) F_\varepsilon^2(\alpha, p)^{-t}; \end{aligned} \tag{6}$$

F_ε^2 is Lorentz invariant. We will write $F_\varepsilon^i = pAp + i\varepsilon Q^i$ $(i = 1, 2)$, where

$$Q^i = \begin{cases} \boldsymbol{p} A \boldsymbol{p} + M^2 & i = 1 \\ 1 & i = 2 \end{cases}.$$

Lemma 1. *The integral* (6) *is absolutely convergent for all* $\varepsilon > 0$ *and all* $\phi \in \mathscr{S}(\mathbb{R}^{4N})$.

Proof. Since A is continuous on the compact set $\mathscr{D}$,

$$\boldsymbol{p} A \boldsymbol{p} + M^2 \leqq K(1 + \sum |\boldsymbol{p}_i|^2)$$

for some constant K; we may take $K \geqq 1$. Then

$$|F_\varepsilon^1(\alpha, p)|^2 = |pAp - M^2|^2 + \varepsilon^2 |\boldsymbol{p}A\boldsymbol{p} + M^2|$$
$$\leqq (|pAp - M^2| + \varepsilon^2) K(1 + \sum |\boldsymbol{p}_i|^2) = |F_\varepsilon^2(\alpha, p)|^2 K(1 + \sum |\boldsymbol{p}_i|^2).$$

Since

$$\psi(p) = \phi(p)[K(1 + \sum |\boldsymbol{p}_i|^2)]^{t/2}$$

is in $\mathscr{S}(\mathbb{R}^{4N})$, the integrand for $I_\varepsilon^2(\phi)$ is dominated by the absolute value of the integrand for $I_\varepsilon^1(\psi)$.

Lemma 2. *Let* $N(\alpha, p, \varepsilon) = \sum_{r,a \geqq 0} N_{ra}(\alpha, p)\varepsilon^r$, *where* N_{ra} *is homogeneous of degree* a *in* $p^0 = (p_1^0, \ldots, p_N^0)$, *and let*

$$J_{ra\varepsilon}^i = \int dp \int_{\mathscr{D}} d\alpha N_{ra}(\alpha, p)\phi(p)F_\varepsilon^i(\alpha, p)^{-t}, \tag{7}$$

$i = 1, 2$. *Then* (7) *is absolutely convergent.*

Proof. For fixed $\lambda > 0$, we make the variable charge $(p_j^0, \boldsymbol{p}_j) = (\lambda q_i^0, \boldsymbol{q}_i)$ in (5), to obtain

$$I_\varepsilon^i = \int d\alpha dq \sum_{a,r} \lambda^{N+a}\varepsilon^r N_{ra}(\alpha, q)\tilde{\phi}(q)F_\varepsilon^i(\alpha, p(q))^{-t}, \tag{8}$$

where $\tilde{\phi}(q) = \phi(p)$ defines $\tilde{\phi} \in \mathscr{S}'(\mathbb{R}^{4N})$. Note that for λ, ε, and δ in a fixed compact subset of $R^+ = \{x | x > 0\}$, there is a constant K with

$$|(\lambda x - 1 + i\varepsilon)/(x - 1 + i\delta)| < K$$

for all $x \in \mathbb{R}$. Thus

$$|F_\varepsilon^1(\alpha, p(q))/F_\delta^1(\alpha, q)| = |(\lambda D - 1 + i\varepsilon)/(D - 1 + i\delta)| < K \tag{9}$$

for almost every α, p, where

$$D(\alpha, p) = p^0 A p^0/(\boldsymbol{p}A\boldsymbol{p} + M^2). \tag{10}$$

Since (8) is absolutely convergent, (9) implies that

$$\int d\alpha dq \sum_{a,r} \lambda^{N+a}\varepsilon^r N_{ra}(\alpha, q)\tilde{\phi}(q)F_\delta^1(\alpha, q)^{-t} \tag{11}$$

is absolutely convergent for fixed $\delta > 0$ and λ, ε in an open set; the integral $J_{ra\delta}^1$ defining the coefficient of $\lambda^{N+a}\varepsilon^r$ in (11) is therefore also absolutely convergent. The convergence of $J_{ra\varepsilon}^2$ now follows from Lemma 1.

Lemma 3. *If the integral* (7) *defining* $J_{ra\varepsilon}^i$ *is absolutely convergent for all* $\varepsilon > 0$, *then*

$$J_{ra0}^i(\phi) = \lim_{\varepsilon \to 0+} J_{ra\varepsilon}^i(\phi)$$

exists and defines a tempered distribution of order at most t; *moreover*, $J_{ra0}^1 = J_{ra0}^2$.

We will prove this lemma shortly, but first note that it implies our main result.

Proof of Theorem 2.

$$\begin{aligned}I_0 &\equiv \lim_{\varepsilon\to 0+} I_\varepsilon^1\\ &= \lim_{\varepsilon\to 0+} \int d\alpha dp \sum_{r,a} N_{ra}\varepsilon^r \phi(p) F_\varepsilon^1(\alpha,p)^{-t}\\ &= \sum_r \lim_{\varepsilon\to 0+} \varepsilon^r \int d\alpha dp \sum_a N_{ra}\phi(p) F_\varepsilon^1(\alpha,p)^{-t}\\ &= \lim_{\varepsilon\to 0+} \int d\alpha dp N(\alpha,p,0)\phi(p) F_\varepsilon^1(\alpha,p)^{-t}\\ &= \lim_{\varepsilon\to 0+} \int d\alpha dp N(\alpha,p,0)\phi(p) F_\varepsilon^2(\alpha,p)^{-t}.\end{aligned}$$

These limits exist as distributions of order at most t by Lemma 3. Finally, if $N(\alpha,p,0)$ is Lorentz covariant, the last equality represents I_0 as a limit of distributions with the same covariance.

Proof of Lemma 3. Let $\chi(x)$ be a C^∞ function of $x\in\mathbb{R}$ with $0\leqq\chi(x)\leqq 1$ and

$$\chi(x)=\begin{cases}0, & \text{if } x<1/3,\\ 1, & \text{if } x>2/3.\end{cases}$$

We will write $J^i_{ra\varepsilon}=K^i_\varepsilon+L^i_\varepsilon$, where

$$K_\varepsilon=\int d\alpha dp N_{ra}(\alpha,p)\phi(p)F_\varepsilon(\alpha,p)^{-t}(1-\chi(D)), \tag{12}$$

$$L_\varepsilon=\int d\alpha dp N_{ra}(\alpha,p)\phi(p)F_\varepsilon(\alpha,p)^{-t}\chi(D); \tag{13}$$

here $D=D(\alpha,p)$ is given by (10). Now for $(1-\chi(D))\neq 0$, $|D-1|>1/3$, and

$$|D-1+i\varepsilon|/|D-1|\leqq 1+3\varepsilon.$$

Hence

$$\begin{aligned}|pAp-M^2| &= |\boldsymbol{p}A\boldsymbol{p}+M^2|\,|D-1|\\ &\geqq |\boldsymbol{p}A\boldsymbol{p}+M^2|\,|D-1+i\varepsilon|(1+3\varepsilon)^{-1}=|F_\varepsilon^1(\alpha,p)|(1+3\varepsilon)^{-1}.\end{aligned}$$

This inequality and the convergence of the integral (7) defining $J^1_{ra\varepsilon}$ imply that the integral

$$K_0=\int\int d\alpha dp N_{ra}(\alpha,p)\phi(p)|pAp-M^2|^{-t}[1-\chi(D)]$$

is absolutely convergent; the Lebesgue dominated convergence theorem and the inequality

$$|F_\varepsilon^i(\alpha,p)|\geqq|pAp-M^2|$$

imply that

$$\lim_{\varepsilon\to 0} K_\varepsilon^i=K_0, \qquad i=1,2.$$

To study L_ε^i we write

$$F_\varepsilon^i(\alpha,p)^{-t}=(-1)^{t-1}2^{-t}(t-1)!^{-1}\left[(p^0Ap^0)^{-1}\sum_{i=1}^N p_i^0\,\partial/\partial p_i^0\right]^t \log F_\varepsilon^i(\alpha,p). \tag{14}$$

If we consider (13) as an iterated integral we may insert (14) and integrate by parts with respect to p^0, to find

$$L_\varepsilon^i = -2^{-t}(t-1)!^{-1} \int d\alpha d\boldsymbol{p} \int dp^0 \log F_\varepsilon^i(\alpha, p) \cdot \left\{ \left[\sum_{i=1}^{N} \partial/\partial p_i^0 \, p_i^0/(p^0 A p^0) \right]^t \phi(p) N_{ra}(p, \alpha) \chi(D) \right\}. \tag{15}$$

The integration by parts is justified, and hence (15) converges as an iterated integral, since for almost every α and $\boldsymbol{p}$, $F_\varepsilon^i(\alpha, p)$ and $\chi(D)$ are C^∞ functions of p^0, and the fast decrease of ϕ enables us to discard boundary terms.

We wish to show that (15) is in fact absolutely convergent. The quantity in brackets has the form

$$N_{ra}(p^0 A p^0)^{-t} \sum_b \tilde{\phi}_b(p) \tilde{\chi}_b(\alpha, p),$$

where $\tilde{\phi}_b \in \mathscr{S}(\mathbb{R}^{4N})$ and $\tilde{\chi}_b$ is a bounded function with

$$\operatorname{supp} \tilde{\chi}_b \subset H = \{(\alpha, p) | D(\alpha, p) \geqq 1/3\}$$

(it is here that we use the homogeneity of N_{ra} in p^0). Since

$$\left| \frac{F_\varepsilon^1(\alpha, p)}{p^0 A p^0} \right| \leqq 1 + (1+\varepsilon)/D$$

is bounded on H, the absolute convergence of (13) for any $\phi \in \mathscr{S}'(\mathbb{R}^{4N})$ implies that

$$\int d\alpha dp N_{ra}(\alpha, p)(p^0 A p^0)^{-t} \tilde{\phi}_b(p) \tilde{\chi}_b(p) \tag{16}$$

is absolutely convergent. We will show in the Appendix (Corollary 1) that this implies that

$$\int d\alpha dp N_{ra}(p^0 A p^0)^{-t} \tilde{\phi}_b \tilde{\chi}_b \log|pAp - M^2| \tag{17}$$

and

$$\int d\alpha dp N_{ra}(p^0 A p^0)^{-t} \tilde{\varphi}_b \tilde{\chi}_b \log|\boldsymbol{p}A\boldsymbol{p} + M^2| \tag{18}$$

are absolutely convergent. Now we use the inequality, valid for $c > 0$ and $d_0 > d > 0$,

$$|\log(c+d)| \leqq |\log c| + |\log d_0| + 1,$$

which implies

$$\begin{aligned} |\log F_\varepsilon^i| &\leqq \tfrac{1}{2} \log |F_\varepsilon^i|^2 + \pi \\ &\leqq |\log|\boldsymbol{p}A\boldsymbol{p} + M^2|| + |\log \varepsilon_0 Q^i| + \pi + \tfrac{1}{2}, \end{aligned}$$

whenever $\varepsilon \leqq \varepsilon_0$. Thus (15) is dominated by a fixed linear combination of (16)–(18); it is absolutely convergent, and the Lebesque dominated convergence theorem implies that

$$\lim_{\varepsilon \to 0} L_\varepsilon^i = \sum_b \int d\alpha dp N_{ra}(p^0 A p^0)^{-t} \tilde{\varphi}_b \tilde{\chi}_b \log|pAp - M^2|$$

for $i = 1, 2$. Finally, note that $\tilde{\varphi}_b$ is obtained from φ by at most t differentiations, so the limiting distribution is at order at most t.

Appendix

We want to prove the statement made earlier that the addition of a logarithm to a convergent integral does not destroy convergence. We use the method of resolution of singularities, and follow the notation and ideas of [16].

Lemma 4. *Let X be a real analytic n-manifold with $f, g, h_1, \dots h_k$ real analytic functions on X and ω an analytic n-form on X. Let*

$$G=\{x\in X \mid h_i(x)\geqq 0 \text{ for all } i\}$$

and suppose that G is compact. Then if

$$\int_G f^{-1}\omega \tag{19}$$

is absolutely convergent, so is

$$\int_G f^{-1}\log|g|\omega .$$

Proof. Because G is compact it suffices to prove integrability in the neighborhood of every point $x_0\in G$. Let $t_1, \dots t_n$ be local coordinates at x_0, so that

$$\omega = w(t)dt_1 \wedge \dots \wedge dt_n$$

in a neighborhood at x, with w real analytic. We apply the Resolution Theorem [16] to resolve the set $A=\{t \mid f(t)g(t)w(t)\prod h_i(t)=0\}$; this produces a neighborhood U of x_0, a real analytic manifold $\tilde{U}$, and a proper analytic map $\varphi:\tilde{U}\to U$ such that $\varphi:(\tilde{U}-\tilde{A})\to U-A$ is a homeomorphism, with $\tilde{A}=\varphi^{-1}(A)$. Moreover, if $\tilde{f}=f\circ\varphi$, etc., then for any $\tilde{x}\in\tilde{U}$ and $y_1, \dots y_n$ local coordinates centered at $\tilde{x}$ in a suitable neighborhood $\tilde{V}$ of $\tilde{x}$, each of $\tilde{f}, \tilde{g}, \tilde{h}_i$, and $\tilde{w}$ have the form

$$\alpha(y)\prod_{i=1}^{n} y_i^{k_i}, \tag{20}$$

with $\alpha\neq 0$ in $\tilde{V}$. Finally,

$$\varphi^*(dt_1\wedge\dots\wedge dt_n)=r(y)dy_1\wedge\dots\wedge dy_n$$

in $\tilde{V}$, with $r(y)$ real analytic; $r(y)$ cannot vanish for $y\notin\tilde{A}\cap\tilde{V}$. Since φ extends to a map $\varphi_c: U_c\to U_c$ of the complexifications, $r(y)$ is complex analytic and vanishes only on $\bigcup_{i=1}^{n}\{y_i=0\}$ in a complex neighborhood of $\tilde{x}$; hence $r(y)$ also has the form (20).

Now

$$\int_{U\cap G} f^{-1}\omega = \int_{\tilde{U}\cap\varphi^{-1}(G)} \tilde{f}^{-1}\varphi^*\omega .$$

The set $\varphi^{-1}(G)\cap\tilde{V}$ is a union of certain octants $\tilde{G}_i$ of $\tilde{V}$, i.e., subsets in which each y_i has a fixed sign. Thus (19) becomes locally a sum of terms

$$\int_{\tilde{G}_i}\alpha(y)\prod_{i=1}^{n} y_i^{j_i}dy_1\wedge\dots\wedge dy_n , \tag{21}$$

with $\alpha(y)\neq 0$; the absolute convergence of (19) and hence (21) implies $j_i \geqq 0$, $i=1,\ldots n$. Since $\log|\tilde{g}| = \log|\alpha'(y)| + \sum_{i=1}^{n} k_i \log|y_i|$ from (20), the inclusion of a factor $\log|\tilde{g}|$ in (21) does not affect convergence.

Corollary 1. *If*

$$\int_{\mathscr{D}} d\alpha \int_{\mathbb{R}^{4N}} dp R(\alpha, p)\phi(p)\chi_H(\alpha, p) \tag{22}$$

converges absolutely for every $\varphi \in \mathscr{S}(\mathbb{R}^{4N})$, *where* R *is rational and* χ_H *is the characteristic function of* $H = \{(\alpha, p)|D(\alpha, p) > a\}$ *for* D *rational, then so does*

$$\int_{\mathscr{D}} d\alpha \int_{\mathbb{R}^{4N}} dp R(\alpha, p) \log|S(\alpha, p)| \phi(p)\chi_H(\alpha, p)\,, \tag{23}$$

where $S(\alpha, p)$ *is rational.*

Proof. Let

$$B(p) = 1 + \sum_{i,\mu} |p_i^\mu|^2\,.$$

It is an easy exercise to see that (22) will converge absolutely if $\phi(p)$ is replaced by $B(p)^{-r}$ for some sufficiently large r; moreover, it suffices to prove the convergence of (23) with this same replacement.

We define $X_1 = \{\alpha \in \mathbb{R}^n | \sum \alpha_i = 1\}$,

$$X_2 = S^{4N} = \left\{(y, \bar{y}) \in \mathbb{R}^{4N+1} \,\middle|\, \sum_{i=1}^{N} \sum_{\mu=0}^{3} (y_i^\mu)^2 + \bar{y}^2 = 1\right\}$$

and $X = X_1 \times X_2$. There is an injection $\lambda : \mathbb{R}^{4N} \to X_2$ given by

$$\lambda(p) = (2p/B(p), 1 - 2/B(p))\,,$$

and an analytic $(n+4N)$-form $\hat{\omega}$ on X with

$$(\mathbb{1} \times \lambda)^* \hat{\omega} = B(p)^{-s} d\alpha_1 \wedge \ldots d\alpha_{n-1} \wedge dp_1^0 \wedge \ldots \wedge dp_N^3\,,$$

for some s. Then (22) becomes

$$\int_G R(\alpha, p) B(p)^{s-r} \hat{\omega}\,, \tag{24}$$

where $G \subset X$, G the closure of $\{(\alpha, p)|\alpha_i \geqq 0,\ i=1,\ldots n,\ D(\alpha, p) > 1/3\}$.

We can rewrite (24) in the form (19) of Lemma 5 as follows. Let

$$\begin{aligned} B(p)^{s-r} R(\alpha, p) &= R_1(\alpha, p)/R_2(\alpha, p) \\ D(\alpha, p) &= D_1(\alpha, p)/D_2(\alpha, p) \\ S(\alpha, p) &= S_1(\alpha, p)/S_2(\alpha, p)\,, \end{aligned}$$

where R_i, D_i, and S_i are polynomials; let k denote the maximal degree in p of these polynomials. If we define

$$\omega = R_1(\alpha, p)B(p)^{-k}\hat{\omega},$$

$$f(\alpha, p) = R_2(\alpha, p)B(p)^{-k},$$

$$h_i(\alpha) = \alpha_i, \quad 1 \leqq i \leqq n,$$

$$h_{n+1}(\alpha, p) = [D_1(\alpha, p) - aD_2(\alpha, p)]B(p)^{-k},$$

then ω, f, and $h_1, \ldots h_{n+2}$ are analytic on X, (24) becomes

$$\int_G f^{-1}\omega,$$

and $G = \{(\alpha, p) | h_i(\alpha, p) \geqq 0,\ i = 1, 2, \ldots n+1\}$. Finally, we note

$$\log|S(\alpha, p)| = \log|S_1(\alpha, p)B(p)^{-k}| - \log|S_2(\alpha, p)B(p)^{-k}|$$

and apply Lemma 5, with $g = S_1 B^{-k}$ and $g = S_2 B^{-k}$ in turn, to deduce the absolute convergence of (23).

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