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On Lorentz Invariant Distributions

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Abstract. n-point Lorentz invariant tempered distributions with the supports for one-point only in $\bar{V}_+^\mu$ are described.

1. Introduction

Lorentz invariant one-point distributions were extensively investigated by P.-D. Methée [1–2]. n-point Lorentz invariant tempered distributions with supports for one-point only in $\bar{V}_+^\mu$ were studied by K. Hepp [3]. In this case the problem of the description of Lorentz invariant distributions is equivalent to the description of the rotation invariant tempered distributions of n three-vectors. For $n=1,2$ this problem was solved [3]. Rotation invariant distributions and the Lorentz invariant distributions were represented as distributions on the space of the SO(3)-invariants and conformably on the space of the $L_+^\uparrow$-invariants. In trying to generalize Hepp's results to $n>2$ one encounters the difficulty that the space of the $L_+^\uparrow$-invariants (and the SO(3)-invariants) is an algebraic variety with singularities, on which no reasonable spaces of testing functions have yet been defined [3].

In present paper SO(3)-harmonic analysis on the space $S'(R^3)$ is studied. Taking advantage of this analysis it is possible to describe the rotation invariant tempered distributions. As stated above the Lorentz invariant tempered distributions with supports in $\bar{V}_+^\mu \times R^{4n}$ were connected with the rotation invariant distributions. Hence we obtain the description of the Lorentz invariant distributions belonging to the space $S'(\bar{V}_+^\mu \times R^{4n})$.

The plan of this paper is as follows: Section 2 contains SO(3)-harmonic analysis on $S'(R^3)$; in Section 3 rotation invariant tempered distributions were studied. The Lorentz invariant distributions belonging to $S'(\bar{V}_+^\mu \times R^{4n})$ are under consideration in Section 4.

2. Spherical Harmonics

We shall consider first the spherical harmonics $Y_{lm}(\Theta, \varphi)$, i.e. the eigenvectors of the spherical part of the three-dimensional Lapalace operator. The spherical harmonics Y_{lm} and the associated Legendre functions P_l^m are related by ([4], p. 24).

$$Y_{lm}(\Theta, \varphi) = (-1)^m \left[\frac{(2l+1)(l-m)!}{4\pi(l+m)!}\right]^{1/2} P_l^m(\cos\theta) \exp im\varphi . \tag{2.1}$$

We define the harmonic polynomial $Y_{lm}(x)$, $x \in R^3$ as

$$Y_{lm}(x) = r^l Y_{lm}(\Theta, \varphi) ,$$

where r, Θ, φ are the spherical coordinates of x.

With the distribution $f(x) \in S'(R^3)$ and the harmonic polynomial $Y_{lm}(x)$ we relate the linear functional $f_{lm}(t)$ on the space $S(\bar{R}_+)$

$$(f_{lm}(t), \varphi(t)) = (f(x), Y_{lm}(x)\varphi(|x|^2)) . \tag{2.2}$$

The function $Y_{lm}(x)\varphi(|x|^2) \in S(R^3)$, and the relation (2.2) is well defined. We call $f_{lm}(t)$ the spherical harmonic of the distribution $f(x)$. It is evident that $f_{lm}(t) \in S'(\bar{R}_+)$. For further purposes we need to know how the continuity of $f_{lm}(t)$ depends on l. Let us estimate the seminorm $\| Y_{lm}(x)\varphi(|x|^2)\|_{n,k}$. $\| \; \|_{n,k}$ is a usual seminorm on the space $S(R^3)$

$$\|\varphi(x)\|_{n,k} = \sup_{R^3} (1+|x|^2)^n |\mathscr{D}^k \varphi(x)| ,$$

where $\mathscr{D}^k = \partial^{|k|}/\partial x_1^{k_1} \partial x_2^{k_2} \partial x_3^{k_3}$.

Recursion relations for the associated Legendre functions $P_l^m(x)$ ([4], pp. 23–24) are combined for finding the relations for derivatives of $Y_{lm}(x)\varphi(|x|^2)$

$$\begin{aligned}
&(\partial/\partial x_1 + i\partial/\partial x_2) Y_{lm}(x)\varphi(|x|^2) \\
&= -2\alpha_{lm} Y_{l+1,m+1}(x)\varphi'(|x|^2) + \alpha_{l-1,-m-1} Y_{l-1,m+1}(x)\Phi_l(x) \\
&\partial/\partial x_3 (Y_{lm}(x)\varphi(|x|^2)) \\
&= 2\beta_{l+1,m} Y_{l+1,m}(x)\varphi'(|x|^2) + \beta_{lm} Y_{l-1,m}(x)\Phi_l(x) \\
&(\partial/\partial x_1 - i\partial/\partial x_2) Y_{lm}(x)\varphi(|x|^2) \\
&= 2\alpha_{l,-m} Y_{l+1,m-1}(x)\varphi'(|x|^2) - \alpha_{l-1,m-1} Y_{l-1,m-1}(x)\Phi_l(x)
\end{aligned} \tag{2.3}$$

where the coefficients

$$\begin{aligned}
\alpha_{lm} &= [(l+m+1)(l+m+2)/(2l+1)(2l+3)]^{1/2} \\
\beta_{lm} &= [(l-m)(l+m)/(2l-1)(2l+1)]^{1/2} \\
\Phi_l(x) &= (2l+1)\varphi(|x|^2) + 2|x|^2 \varphi'(|x|^2) .
\end{aligned}$$

Combining the equations (2.3) and the inequality

$$|Y_{lm}(x)| < (2l+1)^{1/2} |x|^l$$

we obtain the estimate

$$\| Y_{lm}(x)\varphi(|x|^2)\|_{n,k} \leqq C(2l+1)^{|k|+1} \max_{q\leqq|k|} \|\varphi\|^{(l)}_{n+|k|,q,|k|} \tag{2.4}$$

where constant C depends on n and k, and the seminorm

$$\|\varphi(t)\|^{(l)}_{n,q,k}=\sup_{\bar{R}_+} t^{\frac{1}{2}(l-k)_+}(1+t)^n|\varphi^{(q)}(t)|$$

(here $(l-k)_+=\max(l-k,0)$) is a usual seminorm on the space $S(\bar{R}_+)$.

The application of the inequality (2.4) to the distribution $f(x)$ gives the continuity of the spherical harmonics $f_{lm}(t)$.

Let $S(\bar{R}_+\times \hat{\mathrm{SO}}(3))$ be the space of the sequences $\{\varphi_{lm}(t)\}$ ($l=0,1,\ldots; m=-l,\ldots,l$) of the infinitely differentiable functions $\varphi_{lm}(t)$ in $\bar{R}_+$ such that

$$\max_{l,m}(2l+1)^p\|\varphi_{lm}(t)\|^{(l)}_{n,q,k}<\infty \tag{2.5}$$

for any p, n, q, k. $S(\bar{R}_+\times \hat{\mathrm{SO}}(3))$ is locally convex topological vector space with the topology defined by the seminorms that are finite according to (2.5). The inequality (2.4) gives us $\{f_{lm}(t)\}\in S'(\bar{R}_+\times \hat{\mathrm{SO}}(3))$.

Let the spherical harmonics f_{lm} be given. We shall consider the problem of the distribution $f(x)$ reconstruction.

Let the function $g(x)\in S(R^3)$. We rewrite it in spherical coordinates: $g(r,\theta,\varphi)=g(r\sin\theta\cos\varphi, r\sin\Theta\sin\varphi, r\cos\Theta)$. We define the function $\tilde{g}_{lm}(r)$ as follows:

$$\tilde{g}_{lm}(r)=\int_{S^2} d\Omega\, \bar{Y}_{lm}(\Omega)g(r,\Omega)\,. \tag{2.6}$$

It is easy to see that $\tilde{g}_{lm}(r)\in S(R^1)$ for any l, m. (We allow a negative r.) We shall study the properties of $\tilde{g}_{lm}(r)$.

If follows from (2.6) that $\tilde{g}_{lm}(0)=0$ for $l>0$. Let us introduce the differential operators

$$d_1=(8\pi/3)^{1/2}(\partial/\partial x_1-i\partial/\partial x_2)$$
$$d_0=(4\pi/3)^{1/2}\,\partial/\partial x_3$$
$$d_{-1}=-(8\pi/3)^{1/2}(\partial/\partial x_1+i\partial/\partial x_2)\,.$$

We may use the spherical harmonics Y_{1m} for computing the derivative of $g(r,\Omega)$

$$(\partial g/\partial r)(r,\Omega)=\sum_{m=-1}^{1} Y_{1m}(\Omega)d_m g(r,\Omega)\,.$$

In virtue of (2.6) this implies

$$(d^k\tilde{g}_{lm}/dr^k)(0)=\int_{S^2} d\Omega\,\bar{Y}_{lm}(\Omega)\left(\sum_{n=-1}^{1} Y_{1n}(\Omega)d_n\right)^k g(0)\,.$$

Thus the problem of computing $\tilde{g}^{(k)}_{lm}(0)$ is reduced to that of computing the integrals over the product of the spherical harmonics $Y_{lm}(\Omega)$. These integrals equal zero, if the resultant angular momentum of the addition of the angular momentum l and k angular momenta 1 isn't zero [4].

Hence

$$(d^k\tilde{g}_{lm}/dr^k)(0)=0 \qquad k=0, \ldots, l-1\,. \tag{2.7}$$

Now we show that function $\tilde{g}_{lm}(r)$ has the definite parity. On taking into account the relation (2.6) and

$$g(-r, \Theta, \varphi)=g(r, \pi-\Theta, \varphi+\pi)$$

$$Y_{lm}(\pi-\Theta, \varphi-\pi)=(-1)^l Y_{lm}(\Theta, \varphi)$$

we find

$$\tilde{g}_{lm}(-r)=(-1)^l\tilde{g}_{lm}(r)\,. \tag{2.8}$$

In view of (2.7) and (2.8) the even function $r^{-l}\tilde{g}_{lm}(r)\in S(R^1)$. Thus there exists the function $g_{lm}(t)\in S(\bar{R}_+)$ such that

$$g_{lm}(r^2)=r^{-l}\tilde{g}_{lm}(r)$$

(see, for example, [5]). We call $g_{lm}(t)$ the spherical harmonic of the function $g(x)$. We shall prove that for $g(x)\in S(R^3)$ the sequence $\{g_{lm}(t)\}\in S(\bar{R}_+\times \hat{\mathrm{SO}}(3))$. We must show that any seminorm (2.5) is finite on $\{g_{lm}(t)\}$. First we consider the factor $(2l+1)^p$ in (2.5). The spherical harmonic Y_{lm} is the eigen function of the spherical part Δ_Ω of the Laplace operator Δ with the eigenvalue $-l(l+1)$ ([4], p. 21). Hence

$$(2l+1)^{2p}g_{lm}(t)=t^{-l/2}\int_{S^2} d\Omega\,\bar{Y}_{lm}(\Omega)(1-4\Delta_\Omega)^p g(t^{1/2}, \Omega)\,. \tag{2.9}$$

However the function $(1-4\Delta_\Omega)^p g(r, \Omega)$ is the function

$$g_{(p)}(x)=2^{2p}\left[\left(\frac{1}{2}+\sum_{i=1}^{3} x_i\partial/\partial x_i\right)^2-|x|^2\Delta\right]^p g(x)$$

in spherical coordinates. It is clear that $g_{(p)}(x)\in S(R^3)$. Thus we have

$$(2l+1)^{2p}g_{lm}(t)=g_{(p)lm}(t)\,. \tag{2.10}$$

Let us estimate now the seminorm $\|g_{lm}\|^{(l)}_{n,q,k}$. First note that

$$t^n(d^k f/dt^k)(t)=(td/dt-(n-1))t^{n-1}(d^{k-1}f/dt^{k-1})(t)\,.$$

This implies for $l\geqq 2q+k$

$$\|g_{lm}(t)\|^{(l)}_{n,q,k}$$
$$\leqq C(2l+1)^q \max_{s\leqq q}\|r^{l-k-2q}g_{lm}(r^2)\|_{n+q,s}\,. \tag{2.11}$$

We wrote the result in terms of the coordinate r, $r^2=t$. The seminorm $\|\ \|_{n,q}$ is usual seminorm on the space $S(R^1)$

$$\|\varphi(r)\|_{n,q}=\sup_{R^1}(1+r^2)^n|\varphi^{(q)}(r)|\,.$$

Similarly for $l<2q+k$

$$\|g_{lm}(t)\|^{(l)}_{n,q,k}$$
$$\leqq C(2l+1)^q \max_{s\leqq \frac{1}{2}(l-k)_+}\|(r^{-1}d/dr)^{q-(l-k)_+/2}g_{lm}(r^2)\|_{n+q,s}\,. \tag{2.12}$$

Using (2.7), (2.11) and the formula for the Taylor remainder terms we have for $l \geqq 2q+k$

$$\|g_{lm}(t)\|_{n,q,k}^{(l)} \leqq C(2l+1)^q \max_{|s| \leqq 3q+k} \|g(x)\|_{n+q,s} \tag{2.13}$$

where constant C depends on q and k only. It is a simple matter to extend this estimate to $l<2q+k$ by using (2.8) and (2.12). In order to prove $\{g_{lm}(t)\} \in S(\bar{R}_+ \times \hat{SO}(3))$ it is sufficient now to use (2.10) and the estimate (2.13) for the function $g_{(p+q)}(x)$. Whence

$$\begin{aligned}&\max_{l,m} (2l+1)^p \|g_{lm}(t)\|_{n,q,k}^{(l)}\\ &\leqq C \max_{|s| \leqq 3q+k} \|g_{(p+q)}(x)\|_{n+q,s}\end{aligned} \tag{2.14}$$

and consequently $\{g_{lm}(t)\} \in S(\bar{R}_+ \times \hat{SO}(3))$. In particular this implies that for the sequence $\{f_{lm}(t)\}$ of the spherical harmonics of the distribution $f(x) \in S'(R^3)$ the expansion

$$\sum_{l,m} (f_{lm}, g_{lm})$$

is convergent. We prove it converge to $(f(x), g(x))$. Note that the series

$$\sum_{l,m} Y_{lm}(x) g_{lm}(|x|^2) \tag{2.15}$$

absolutely converge to $g(x)$ at every point [6]. In view of (2.4) and (2.14) it converges to $g(x)$ in the topology of $S(R^3)$. By definition (2.2) of the spherical harmonic $f_{lm}(t)$ of the distribution $f(x)$ we have

$$(f(x), g(x)) = \sum_{l,m} (f_{lm}(t), g_{lm}(t)) \,. \tag{2.16}$$

Summing up:

Theorem 1. *The relation*

$$(f(x), g(x)) = \sum_{l,m} (f_{lm}(t), t^{-l/2} \int_{S^2} d\Omega \bar{Y}_{lm}(\Omega) g(t^{1/2}, \Omega)) \tag{2.17}$$

implies the isomorphism between two topological spaces: $S'(R^3)$ and $S'(\bar{R}_+ \times \hat{SO}(3))$.

In Section 3 we consider the rotation invariant distributions from the space $S'(R^{3n})$.

3. Rotation Invariant Distributions

Let the distribution $f(x_1, \ldots, x_n) \in S'(R^{3n})$. Its spherical harmonics $f_{l_1 m_1 \ldots l_n m_n}(t_1, \ldots, t_n)$ are defined in the similar way to the spherical harmonics of the distribution $f(x) \in S'(R^3)$. The sequence $\{f_{l_1 m_1 \ldots l_n m_n}\}$ belongs to the space $S'((\bar{R}_+ \times \hat{SO}(3))^n)$. The proof of this is exactly analogous and can be omitted. The distribution f and its spherical harmonics are related by

$$(f, \varphi) = \sum_{l,m} (f_{lm}, \varphi_{lm}) \tag{3.1}$$

where $\varphi_{l_1m_1\dots l_nm_n}(t_1, \dots, t_n)$ is n-dimensional spherical harmonics of the function $\varphi(x_1, \dots, x_n) \in S(R^{3n})$

$$\varphi_{lm} = t_1^{-l_1/2} \dots t_n^{-l_n/2} \int\limits_{S^{2\times n}} d^n\Omega \bar{Y}_{l_1m_1}(\Omega_1) \dots \bar{Y}_{l_nm_n}(\Omega_n)\varphi\,. \tag{3.2}$$

Let us study how the spherical harmonics of the distribution f vary under the rotation $u \in \mathrm{SO}(3)$. Applying (3.1) for the function $\varphi_u = \varphi(ux_1, \dots, ux_n)$ we get

$$(f, \varphi_u) = \sum_{l,m,k} D^{(l_1)}_{k_1m_1}(u) \dots D^{(l_n)}_{k_nm_n}(u)(f_{lm}, \varphi_{lk}) \tag{3.3}$$

where the matrice $D^{(l)}_{km}(u)$ represents the rotation u in the $(2l+1)$-dimensional irreducible representation of the group SO(3). Unitary matrice $D^{(l)}_{km}(u)$ continuously depends on u. This implies that the series (3.3) is integrable with respect to du, where du is the invariant normalized Haar measure of SO(3). In virtue of (3.3) we get for the rotation invariant distribution $f(x_1, \dots, x_n)$

$$(f, \varphi) = \sum_{l,m,k} (f_{lm}, \varphi_{lk}) \int\limits_{\mathrm{SO}(3)} du D^{(l_1)}_{k_1m_1}(u) \dots D^{(l_n)}_{k_nm_n}(u)\,. \tag{3.4}$$

Thus the problem of the description any rotation invariant tempered distribution is reduced to that of computing the integral over the product of n D's. In order to compute this integral we note that the product of two D's may be expressed in terms of one D function by using the Clebsh-Gordan coefficients $(l_1m_1l_2m_2|l_1l_2jm)$ ([4], (4.3.1)) and the integral over the product of three D's equals the product of two $3-j$ symbols of Wigner ([4], (4.6.2))

$$\begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix}.$$

Then

$$\begin{aligned} &\int\limits_{\mathrm{SO}(3)} du D^{(l_1)}_{k_1m_1}(u) \dots D^{(l_n)}_{k_nm_n}(u) \\ &= \sum_j \begin{pmatrix} l_1 \dots l_n \\ m_1 \dots m_n \end{pmatrix}_{j_1 \dots j_{n-3}} \begin{pmatrix} l_1 \dots l_n \\ k_1 \dots k_n \end{pmatrix}_{j_1 \dots j_{n-3}} \end{aligned} \tag{3.5}$$

where the generalized Wigner's symbol[1]

$$\begin{aligned} &\begin{pmatrix} l_1 \dots l_n \\ m_1 \dots m_n \end{pmatrix}_{j_1 \dots j_{n-3}} \\ &= \sum_p (l_1m_1l_2m_2|l_1l_2j_1p_1) \dots \begin{pmatrix} j_{n-3} & l_{n-1} & l_n \\ p_{n-3} & m_{n-1} & m_n \end{pmatrix}. \end{aligned} \tag{3.6}$$

Let us consider the properties of the generalized Wigner's symbols. The properties of the Clebsh-Gordan coefficients may be used to obtain the invariance property and the ortogonal property of the generalized Wigner's symbols stating from the definition (3.6).

[1] These generalized Wigner's symbols differ in factors only from those introduced in [7].

We have

$$\sum_k D^{(l_1)}_{m_1k_1}(u)\dots D^{(l_n)}_{m_nk_n}(u)\begin{pmatrix} l_1 \dots l_n \\ k_1 \dots k_n\end{pmatrix}_j=\begin{pmatrix} l_1 \dots l_n \\ m_1 \dots m_n\end{pmatrix}_j \tag{3.7}$$

$$\begin{aligned}&\sum_m \begin{pmatrix} l_1 \dots l_n \\ m_1 \dots m_n\end{pmatrix}_j \begin{pmatrix} l_1 \dots l_n \\ m_1 \dots m_n\end{pmatrix}_{j'}\\ &=\delta_{j_1j_1'}\dots\delta_{j_{n-3},j_{n-3}'}\delta(l_1\dots l_n, j_1\dots j_{n-3})\end{aligned} \tag{3.8}$$

where $\delta(l_1\dots l_n, j_1\dots j_{n-3})=1$ if the natural numbers $l_1, \dots, l_n; j_1, \dots, j_{n-3}$ satisfy the polygonal condition, and is zero otherwise. The polygonal condition for $l_1, \dots, l_n$; $j_1, \dots, j_{n-3}$ is as follows: one can construct the polygon such that $l_1, \dots, l_n$ correspond to the lengths of sides and $j_1, \dots, j_{n-3}$ correspond to the lengths of the diagonals which get going at the vertex where sides l_1 and l_n intersect. By P_n we denote the set of $l_1, \dots, l_n; j_1, \dots, j_{n-3}$ satisfying the polygonal condition.

The invariance property (3.7) and the ortogonal property (3.8) of the generalized Wigner's symbols are analogous to those of the $3-j$ symbols of Wigner ([4], (4.3.3), (3.7.8)). In view of (3.8) the generalized Wigner's symbol

$$\begin{pmatrix} l_1 \dots l_n \\ m_1 \dots m_n\end{pmatrix}_{j_1\dots j_{n-3}}$$

equals zero if $(l_1, \dots, l_n; j_1, \dots, j_{n-3}) \notin P_n$.

Let us return now to the rotation invariant distributions. Let $p=(l_1, \dots, l_n$; $j_1, \dots, j_{n-3})\in P_n$. We define the invariant harmonic of the distribution $f(x_1, \dots, x_n)\in S'(R^{3n})$ by

$$f_p=\sum_m \begin{pmatrix} l_1 \dots l_n \\ m_1 \dots m_n\end{pmatrix}_{j_1\dots j_{n-3}} f_{lm}(t_1, \dots, t_n) \tag{3.9}$$

where $f_{l_1m_1\dots l_nm_n}$ are the spherical harmonics of f.

Similarly for $\varphi\in S(R^{3n})$

$$\varphi_p=\sum_m \begin{pmatrix} l_1 \dots l_n \\ m_1 \dots m_n\end{pmatrix}_{j_1\dots j_{n-3}} \varphi_{lm}(t_1, \dots, t_n)\,. \tag{3.10}$$

In virtue of (3.5) the relation (3.4) implies

$$(f(x), \varphi(x))=\sum_{p\in P_n}(f_p, \varphi_p)\,. \tag{3.11}$$

It follows from the invariance property (3.7) that any term in the sum (3.11) is a rotation invariant distribution from $S'(R^{3n})$.

Let us express the invariant harmonics f_p and φ_p in terms of f and φ. We relate $p=(l_1, \dots, l_n; j_1, \dots, j_{n-3})\in P_n$ to the invariant spherical harmonic

$$Y_p=\sum_m \begin{pmatrix} l_1 \dots l_n \\ m_1 \dots m_n\end{pmatrix}_j Y_{l_1m_1}(\Omega_1)\dots Y_{l_nm_n}(\Omega_n)\,.$$

It corresponds to the invariant polynomial

$$Y_p(x_1, \dots, x_n)=r_1^{l_1}\dots r_n^{l_n} Y_p(\Omega_1, \dots, \Omega_n)\,.$$

By definition of the spherical harmonics of f we get

$$(f_p(t), \Psi(t)) = (f(x_1, \ldots, x_n), Y_p(x_1, \ldots, x_n)\Psi(|x_1|^2, \ldots, |x_n|^2)) \tag{3.12}$$

for every $\Psi \in S(\bar{R}^n_+)$.

Similarly

$$\varphi_p = t_1^{-l_1/2} \ldots t_n^{-l_n/2} \int_{S^{2\times n}} d^n\Omega \bar{Y}_p(\Omega)\varphi(t_1^{1/2}, \Omega_1, \ldots, t_n^{1/2}, \Omega_n) . \tag{3.13}$$

Let $S(\bar{R}^n_+ P_n)$ be the space of the sequences $\{\Psi_p\}$ $(p=(l_1, \ldots, l_n; j_1, \ldots, j_{n-3}) \in P_n)$ of the infinitely differentiable functions $\Psi_p(t_1, \ldots, t_n)$ in $\bar{R}^n_+$ such that for any k, m, q, s.

$$\max_{p \in P_n} \left(2 \sum_{i=1}^{n} l_i + 1\right)^k \|\Psi_p(t)\|_{m,q,s}^{(p)} < \infty \tag{3.14}$$

where the seminorm $\|\Psi(t)\|_{m,q,s}^{(p)}$ equals

$$\sup_{\bar{R}^n_+} t_1^{(l_1-s)_+/2} t_n^{(l_n-s)_+/2} \left(1 + \sum_{i=1}^{n} t_i\right)^m |\mathscr{D}^q \Psi(t)| .$$

$S(\bar{R}^n_+ P_n)$ is locally convex topological vector space with the topology defined by the seminorms that are finite according to (3.14).

It is easy to see from the ortogonal property (3.8) that a modulus of any generalized Wigner's symbol is less than one. This implies $\{\varphi_p\} \in S(\bar{R}^n_+ P_n)$ and $\{f_p\} \in S'(\bar{R}^n_+ P_n)$ in virtue of (3.10) and (3.9). Inversely any sequence $\{f_p\} \in S'(\bar{R}^n_+ P_n)$ defines a rotation invariant distribution from $S'(R^{3n})$ by (3.11). More precisely we have

Theorem 2. *The relation*

$$(f, \varphi) = \sum_{p \in P_n} (f_p, t_1^{-l_1/2} \ldots t_n^{-l_n/2} \int_{S^{2\times n}} d^n\Omega \bar{Y}_p(\Omega)\varphi) \tag{3.11}$$

implies the topological isomorphism between the space of SO(3)*-invariant tempered distributions from $S'(R^{3n})$ and the space $S'(\bar{R}^n_+ P_n)$.*

For $n=1$ the set P_n is one point $l=0$ (the only connected polygon). Thus the Theorem 2 coincides in this case with the well-known theorem on rotation invariant distributions from $S'(R^3)$ [5].

Note that a rotation invariant polynomial $Y_p(x_1, \ldots, x_n)$ may be represented as a polynomial of the scalar products (x_i, x_j) [8]. Thus the sequence of the invariant harmonics $\{f_p\}$ defined by (3.12) is probably a way to define a distribution on the variety of the SO(3)-invariants.

4. Lorentz Invariant Distributions

In this section we shall consider the Lorentz invariant tempered distributions with the supports in $\bar{V}^\mu_+ \times R^{4n}$, where $\bar{V}^\mu_+ = \{p | p_0 \geqq \sqrt{|p|^2 + \mu^2}\}$.

Let $\bar{R}_+^{(\mu^2)}$ be the interval $\mu^2 \leqq t < \infty$. K. Hepp proved that the subspace of $L_+^\uparrow$-invariant distributions from $S'(\bar{V}_+^\mu \times R^{4n})$ is topologically isomorphic to the subspace of the distributions from $S'(\bar{R}_+^{(\mu^2)} \times R^n \times R^{3n})$ which are SO(3)-invariant in the last variables [3]. We shall describe this isomorphism.

For any $y \in V_+$ let $L(y)$ be the Lorentz transformation corresponding to the $A(y) \in \mathrm{SL}(2, C)$ (σ_i: Pauli matrices)

$$A(y) = [2(y, y)^{1/2}((y, y)^{1/2} + y_0)]^{-1/2}\{((y, y)^{1/2} + y_0)\sigma_0 + \boldsymbol{y\sigma}\}. \tag{4.1}$$

It is convenient to define $S(\bar{V}_+^\mu \times R^{4n})$ as the quotient space of $S(V_+^\nu \times R^{4n})$ $(0 < \nu < \mu)$ by the subspace of those functions which are zero on $\bar{V}_+^\mu \times R^{4n}$. Any function $\varphi \in S(V_+^\nu \times R^{4n})$ are related to the function $M\varphi(t_0, \ldots, t_n, x_1, \ldots, x_n) \in S(R_+^{(\nu 2)} \times R^n \times R^{3n})$ by

$$M\varphi = \int dy \delta((y, y) - t_0)\varphi(y, L(y)(t_1, x_1), \ldots, L(y)(t_n, x_n)). \tag{4.2}$$

The mapping M implies the above mentioned isomorphism. More precisely for any Lorentz invariant $F \in S'(\bar{V}_+^\mu \times R^{4n})$ there exists a rotation invariant $f(t, x) \in S'(\bar{R}_+^{(\mu^2)} \times R^n \times R^{3n})$ such that

$$(F, \varphi) = (f(t, x), M\varphi(t, x)). \tag{4.3}$$

Let us use the SO(3)-Fourier transform of $f(t, x)$ in the variables $x_1, \ldots, x_n$.

The invariant harmonics $f_p(t_0, \ldots, t_n, t_{n+1}, \ldots, t_{2n}) \in S'(\bar{R}_+^{(\mu^2)} \times R^n \times \bar{R}_+^n)$ are defined in exactly the same way as for the distributions from $S'(R^{3n})$. Let the function $\Psi \in S(\bar{R}_+^{(\mu^2)} \times R^n \times \bar{R}_+^n)$. We have

$$(f_p, \Psi) = (f(t, x), Y_p(x)\Psi(t_0, \ldots, t_n, |x_1|^2, \ldots, |x_n|^2)). \tag{4.4}$$

The space $S(\bar{R}_+^{(\mu^2)} \times R^n \times \bar{R}_+^n P_n)$ may be defined in the similar way to the space $S(\bar{R}_+^n P_n)$; for a sequence $\{\varphi_p(t_0, \ldots, t_n, t_{n+1}, \ldots, t_{2n})\}$ an index $p \in P_n$ is related to the variables $t_{n+1}, \ldots, t_{2n}$ only.

It is clear that the sequence $\{f_p\} \in S'(\bar{R}_+^{(\mu^2)} \times R^n \times \bar{R}_+^n P_n)$. The distribution $f(t, x)$ and the sequence $\{f_p\}$ of its invariant harmonics are connected by the relation which is analogous to (3.11). Substituting in this relation the invariant harmonics

$$M_p\varphi = t_{n+1}^{-l_1/2} \ldots t_{2n}^{-l_n/2} \int_{S^{2 \times n}} d^n\Omega \bar{Y}_p(\Omega) M\varphi \tag{4.5}$$

of the function $M\varphi$ we obtain (F, φ) in virtue of (4.3).

In summary:

Proposition 1. *The retation*

$$(F, \varphi) = \sum_{p \in P_n} (f_p, M_p\varphi) \tag{4.6}$$

implies the topological isomorphism between the space of $L_+^\uparrow$-invariant tempered distributions from $S'(\bar{V}_+^\mu \times R^{4n})$ and the space $S'(\bar{R}_+^{(\mu^2)} \times R^n \times \bar{R}_+^n P_n)$.

It is anticipated that our isomorphism (4.6) can be extended to more general case.

References

1. Methée, P.-D.: Comm. math. helv. **28**, 225—269 (1954)
2. Gårding, L., Lions, J. L.: Nuovo Cimento, Suppl. **14**, 9—66 (1959)
3. Hepp, K.: Helv. Phys. Acta **37**, 639—658 (1964)
4. Edmonds, A. R.: Angular Momentum in Quantum Mechanics. Princeton N.J.: Princeton U.P. 1957
5. Schwartz, L.: Séminaire 1954/55, Exposé no. 7
6. Beers, B. L., Dragt, A. J.: J. Math. Phys. **11**, 2313—2328 (1970)
7. Jucys, A., Levinsonas, J., Vanagas, V.: Mathematical Apparatus of the Theory of Angular Momentum Vilinius 1960. Translation: New York: Gordon and Breach 1964
8. Weyl, H.: The Classical Groups. Princeton N.J.: Princeton U.P. 1939

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