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Macrocausality Conditions from Analyticity Properties of Scattering Kernels in Field Theory

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Abstract. From the analyticity properties of the scattering kernels in axiomatic field theory we derive macrocausality conditions for the scattering amplitudes in the sense of Iagolnitzer-Stapp. By applying the generalized Laplace transform theorem of Bros-Iagolnitzer, we show the equivalence of such conditions with the exact size and shape of the analyticity domains.

1. Introduction

In the last ten years, several authors have looked for a direct space-time interpretation of the analyticity properties of scattering amplitudes of elementary particles. As a matter of fact one can adopt two different attitudes concerning investigations of this type:

i) One believes that analyticity should be derived from "physically-admissible" space-time properties of the collision amplitudes; the latter properties should in particular express the short-range character of interactions together with a certain form of causality. This point of view was supported for the first time by Omnès in [1], where a derivation of the analyticity of two-body amplitudes in t-ellipses, was proposed on the basis of a certain short-range hypothesis. An analogous derivation was given in a more rigorous form in [2]. The same point of view had also been adopted by the tenants of axiomatic S-matrix theory [3]. In this context, a set of general "macrocausality conditions" was defined by Iagolnitzer and Stapp in [4], which was proved to be equivalent to local analyticity properties of the n-body collision amplitudes. These macrocausality conditions express in an appropriate mathematical language the fact that in collision processes, all energy-momentum transfers which are not carried by stable elementary particles give rise to short range phenomena in space-time.

We note that in the above formalism (as well as in [1] and [2]), the short range properties are always assumed to hold in the sense of *exponential* decrease, the latter being essential to yield analyticity.

ii) Starting from the analyticity properties of the scattering kernels which have been proved in axiomatic field theory, one tries to derive equivalent space-time properties of transition amplitudes. This is the point of view which we adopt in the present work. To the same field theoretical context belong the works by

Williams and Hepp [6], in which these authors derive certain macrocausality conditions of scattering amplitudes (for special space-time configurations) directly from the axioms of field theory. However in the latter works, macrocausality was expressed through conditions of rapid and not exponential decrease, and the links between macrocausality and analyticity properties of the scattering amplitudes were therefore not considered there.

In the present work, our purpose will be to derive *precise* macrocausality conditions of the type introduced in [4] which correspond to the analyticity of the scattering kernels in global regions of the mass-shell, obtained from field theory. As a matter of fact, the equivalence which had been proved in [4] did not provide a precise correspondence between the shape and size of analyticity domains on the one hand, and detailed conditions of exponential decrease on the other hand. However the mathematical ideas used in [4] [1] were developed in [5] under the name of "generalized Fourier transformation" and a generalization of the Laplace transform theorem could be proved in this context. This theorem states the equivalence between the fact that a function f is the boundary value of an analytic function F in a certain (specified) domain (called a local tube) and corresponding exponential decrease properties of the generalized Fourier transform of f.

The necessity of such a generalization is suggested by the fact that the ordinary Laplace transform theorem [2] only applies to domains which are invariant by translations in the real space (namely "tubes").

In the present work we restrict ourselves to considering the analyticity properties of two-body scattering amplitudes except in the case of momentum transfer analyticity where $2 \to n$ collision amplitudes are also treated.

In Section 2 we recall what a "macrocausality condition" means in the sense of the Iagolnitzer-Stapp [4] formalism, and we present two versions of the generalized Laplace transform theorem of [5] which apply to the case of distributions; the proof of the second one will be indicated in the Appendix.

In Section 3 we apply the above theorem to the $2 \to n$ collision amplitudes, by considering the analyticity of the latter with respect to the relative momentum of the two incoming particles (as obtained in field theory [8–10]). There we obtain an exponential decrease of transition amplitudes in the space variables which expresses the short range character of forces. The results obtained are then compared with those of [2].

In Section 4 we exploit the analyticity of $2 \to 2$ scattering amplitudes with respect to the squared total energy s. Application of the same theorem yields:

i) a causality condition corresponding to analyticity in the physical sheet of s, namely in bounded subdomains of the s-cut plane [3] obtained in axiomatic field theory [11, 12],

ii) a condition of the type "relaxation" corresponding to the analyticity domain in the elastic unphysical sheet of s (as it was obtained in [13]).

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[1] The main ingredient of this method was in fact the use of Gaussian wave packets as introduced by Omnès in [1]. Such an idea had also been proposed independently by Froissart [7].

[2] See for example: Streater and Wightman [14] and [5].

[3] For simplicity, we have restricted ourselves to the case of one scalar field producing particles (of a single type) of mass m.

2. Causality and Analyticity

2.1. The I.S. Macrocausality Conditions on Connected Scattering Amplitudes [4]

Let:

$$S^c_{IJ}(\{p_i\},\{p_j\}) = \delta(\textstyle\sum_{i\in I} p_i - \sum_{j\in J} p'_j)\, T_{IJ}(p_i, p'_j) \tag{2.1}$$

be the kernel of the connected S-matrix for a process $I \to J$ involving m (resp. $n-m$) incoming (resp. outgoing) particles with masses m_i (resp. m'_j) and momenta: $\{p_i; i\in I, |I|=m\}$ (resp. $\{p'_j; j\in J, |J|=n-m\}$) such that:

$$p_i^2 = m_i^2\,; \qquad p_i^0 > 0\,, \tag{2.2}$$

$$p_j'^2 = m_j'^2\,; \qquad p_j^{0\prime} > 0\,, \tag{2.3}$$

$$\textstyle\sum_{i\in I} p_i = \sum_{j\in J} p'_j\,. \tag{2.4}$$

The conditions (2.2), (2.3), (2.4) express that the point $p = (\{p_i\}\{p'_j\})$ belongs to the mass shell manifold $\mathscr{M}_{IJ}$ of the process $I \to J$. For arbitrary test-function $\varphi_i, \varphi'_j \in \mathscr{S}(\mathbb{R}^3)$, with $i\in I$, $j\in J$, the corresponding scattering amplitudes are defined by:

$$S^c_{IJ}(\{\varphi_i\},\{\varphi'_j\}) = \int S^c_{IJ}(\{p_i\},\{p'_j\}) \textstyle\prod_{i\in I} \varphi_i(\boldsymbol{p}_i) \prod_{j\in J} \varphi'_j(\boldsymbol{p}'_j)(d_3p_i/2\omega_i)(d_3p'_j/2\omega_j) \tag{2.5}$$

where

$$\omega_i = \sqrt{\boldsymbol{p}_i^2 + m_i^2}\,, \qquad \omega'_j = \sqrt{\boldsymbol{p}_j^{2\prime} + m_j'^2}\,.$$

We also introduce the space $\mathbb{R}^4$ of "multiple displacements in space-time" $u = (\{u_i\}, i\in I; \{u'_j\}, j\in J)$ where the u_i's and u'_j's are arbitrary four-vectors in Minkowski space. Let $P = (\{P_i; i\in I\}; \{P'_j; j\in J\})$ be a fixed point in $\mathscr{M}_{IJ}$ and u an arbitrary displacement. With every configuration (P, u) one can associate in space-time the set of displaced "trajectories" [in the sense of classical relativistic kinematics] of the particles (i) (resp. (j)) whose equations are

$$\begin{aligned} x_i &= u_i + t_i p_i/m_i\,, \\ x'_j &= u'_j + t'_j p'_j/m'_j\,, \end{aligned} \tag{2.6}$$

t_i and t'_j being the "proper time" parameters.

A configuration (P, u) is said to be "normalized" if (in a chosen Lorentz frame L) $\|u\|_L^2 \equiv \sum_i ((u_i^{(0)})^2 + \boldsymbol{u}_i^2) + \sum_j ((u_j'^{(0)})^2 + \boldsymbol{u}_j'^2) = 1$ where $u_i = (u_i^{(0)}, \boldsymbol{u}_i)$ $u'_j = (u_j'^{(0)}, \boldsymbol{u}'_j)$.

Following [4], we call the configurations (P, u) *causal* if the initial and final displaced trajectories form a "causal set"; that means, if there exists at least one multiple-collision classical process with possible exchange of (stable) physical intermediate particles, connecting the initial and final particles. If such a process does not exist, (P, u) is called "non causal".

Being given an arbitrary normalized configuration (P, u), we say that the quantum scattering process $I \to J$ satisfies an "*I.S. condition*" $C(P, u, \alpha, \gamma_0)$ if the following bound holds: $\forall \gamma$ with $0 < \gamma < \gamma_0$:

$$\begin{aligned} &\left|\int T(\{p_i\},\{p'_j\})\chi(\{p_i\},\{p'_j\}) e^{-\gamma\tau\Phi(\{\boldsymbol{p}_l - \boldsymbol{P}_l\})} e^{-i\tau\sum_l \varepsilon_l p_l u_l} \right. \\ &\left. \delta(p_I - p_J) \textstyle\prod_{l\in I\cup J} \delta(p_l^2 - m_l^2)\,\theta(p_l^0) d^4p_l \right| \leqq C_N e^{-\alpha\gamma\tau}/(1+\tau^N) \end{aligned} \tag{2.7}$$

where:

$$l \in I \cup J, \quad p_I = \sum_{i \in I} p_i, \quad p'_J = \sum_{j \in J} p'_j; \quad \tau = \|x\|_L$$

$\varepsilon_l = -1$ (resp. $+1$) if l is initial (resp. final).

$\Phi(\{\boldsymbol{p}_l - \boldsymbol{P}_l\})$ is an analytic function which vanishes at $p_l = P_l$, has there a "critical point of spherical type" (it is strictly positive at all other real points and is such that the level surfaces $\Phi(\{p_l - P_l\}) = C$ keep a nested structure for sufficiently small values of C).

The simplest example of such a function Φ (which we shall use later), is:

$$\Phi(\{\boldsymbol{p}_l - \boldsymbol{P}_l\}) = \sum_{\substack{l \in I \cup J \\ l \neq l_0}} (\boldsymbol{p}_l - \boldsymbol{P}_l)^2 . \tag{2.8}$$

In this definition the bound (2.7) is assumed to hold for every test function χ in $\mathscr{S}$ which is *locally analytic* at P (i.e. analytic in a suitable neighbourhood of P). Furthermore, we note that the scattering amplitude[4] is expressed in terms of a localizing "sequence" of states labelled by the parameter τ, which localize the particles in the classical configuration when $\tau \to \infty$. The actual displacements of the wave packet χ occurring in (2.7) are $x_l = \tau u_l$, with the dilatation parameter $\tau \to \infty$.

The macrocausality requirements of [4] amount to state that for every *non causal* configuration (P, u) a certain condition $C(p, u, \alpha, \gamma_0)$ is fulfilled (without specification of α and γ_0).

In the following such *precise* conditions $C(p, u, \alpha, \gamma_0)$ (i.e. with α, γ_0 specified) will be derived from the analyticity properties of the scattering kernels in the field theoretical framework.

2.2. The Generalized Laplace Transform Theorem

Let us recall some geometrical definitions given in [5a] and [5b]:

i) Given a fixed point P in $\mathbb{R}^n$, we consider in the complex space $\mathbb{C}^n_{(k)}$ $(k = p + iq)$ the analytic function[5]

$$\Phi_P(k) = \sum_{i=1}^{n} (k_i - P_i)^2 \tag{2.9}$$

and call Ω_α the real open ball:

$$\{p; 0 \leqq (p - P)^2 \equiv \sum_{i=1}^{n} (p_i - P_i)^2 < \alpha\} . \tag{2.10}$$

In the following, we drop the subscript P of Φ_p for simplicity.

ii) Let B be a bounded convex domain in an auxiliary n-dimensional real space, such that the closure of B contains the origin. B is supposed to be described in polar coordinates (ϱ, ω) by an inequality of the form $\varrho < r(\omega)$. In the following we always assume that $0 < r(\omega) < 1/2\sqrt{\alpha}$ for all points ω in the unit sphere s^{n-1}. We now consider the set $\mathscr{E}$ of points $k = p + iq$ such that:

$$\begin{aligned} &|q| + r(\omega)(\mathrm{Re}\,\Phi(p+iq) - \alpha) < 0 \\ &q = |q|\omega . \end{aligned} \tag{2.11}$$

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[4] We still use this term, although χ is not necessarily factorized as in (2.5).

[5] In [5] more general functions Φ, local tubes $T_{B\Phi\alpha}$, and generalized Fourier transforms $\mathscr{F}^{(\Phi)}$ are considered and yield similar results.

We define a *local tube* $T_{B\Phi\alpha}$ with basis B as the interior of the connected component of $\mathscr{E}$ whose edge on the real is Ω_α; since $r(\omega) < 1/2\sqrt{\alpha}$ it is easily checked that this domain is bounded.

iii) The $\mathscr{F}^{(\Phi)}$ transform of a function g (or distribution) is defined in the $(n+1)$-dimensional space of the variables $x = (x_1, \ldots, x_n)$, x_0, through the following formula:

$$\mathscr{F}^{(\Phi)}(g)(x, x_0) = (2\pi)^{-n/2} \int e^{-ipx - \Phi(p)x_0} g(p)dp\,. \tag{2.12}$$

iv) For a distribution f which is the boundary value of an analytic function F in the local tube $T_{B\Phi\alpha}$ it is convenient to introduce the following set S_B in (x, x_0)-space which is called the *essential support of f over Ω_α* (this name being justified by the theorem below). S_B is the convex cone with apex at the origin, whose basis is the set of points $\{x \in \tilde{B}, x_0 = 1\}$ where $\tilde{B}$ is the polar set of the basis B, defined by:

$$\tilde{B} = \{x \in \mathbb{R}^n; x \cdot \omega + r^{-1} \geqq 0 \quad \text{for every point} \quad (r, \omega) \in B\}\,. \tag{2.13}$$

v) For every (α, α') with $\alpha' < \alpha$, we define $\mathscr{D}_{\alpha'}(\Omega_\alpha)$ as the space of all functions χ in $\mathscr{D}(\Omega_\alpha)$,[6] which are moreover analytic in $\bar{\Omega}_{\alpha'}$ [7] and have an analytic continuation in the closed local tube $\bar{T}_{B\Phi\alpha'}$ [for instance the latter condition is satisfied if $\chi(p) = 1$ for p in $\bar{\Omega}_{\alpha'}$].

The following theorem has been proved in (5b). (A slightly weaker version of it can be found in (5a).)

Theorem 1. *There is equivalence between the two following properties of the distribution f:*

i) *f/Ω_α is the boundary value of a function F, which is analytic in the local tube $T_{B\Phi\alpha}$.*

ii) *The generalized Fourier transform $\mathscr{F}^\Phi(\chi \cdot f)(x, x_0)$ of $f(p)\chi(p)$ satisfies the set of exponential bounds:*

$$\forall \alpha' < \alpha: |\mathscr{F}(\chi \cdot f)(x, x_0)| < C_{N,\alpha'} e^{-\alpha' x_0}/(1 + \|x\|^N), \quad \textit{for} \quad (x, x_0) \notin S_B, x_0 \geqq 0. \tag{2.14}$$

More precisely these bounds hold with uniform constants $C_{N,\alpha'}$ in every closed cone of the half space $\{x_0 \geqq 0\}$ lying outside S_B.

Let us notice that the left-hand side of the "I.S. condition" (2.7) $C(P, u, \alpha, \gamma_0)$ is precisely the generalized Fourier transform $\mathscr{F}^\Phi$ of:

$$\prod_l (2\omega_l)^{-1} \delta(p_I - p_J) T_{IJ}(\{p_i\}_{i \in I}; \{p'_j\}_{j \in J}) \chi(\{p_i\}, \{p'_j\}) \tag{2.15}$$

evaluated at $(x = \tau u, x_0 = \gamma\tau)$.

Therefore I.S. conditions (2.7) have exactly the same form as the inequalities (2.14) of the above theorem; in view of the latter, there will be equivalence between appropriate sets of I.S. conditions and the analyticity of the T_{IJ} kernel in corresponding local tubes.

In the following, it will be convenient to make use of the analyticity of the scattering kernel T_{IJ}, with respect to a subset (p_1) of the energy-momentum variables p. According to the above theorem, this "partial analyticity" yields conditions of the type (2.14) written in terms of a "partial" $\mathscr{F}^\Phi$ transform corresponding to the variables p_1.

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[6] $\mathscr{D}(\Omega_\alpha)$ is the space of C^∞ functions with compact support in Ω_α.

[7] The closure of $\Omega_{\alpha'}$.

However these conditions can also be equivalently formulated in terms of a "total" $\mathscr{F}^{\Phi}$ transform, i.e. with respect to all variables. Indeed, two equivalent sets of conditions can be given, either one of which expresses the fact that a distribution $f(p_1, p_2)$ is the boundary value of an analytic function of p_1 in a fixed local tube $T_{B\Phi\alpha}$, when p_2 lies in the neighbourhood of an average position P_2. More precisely, we can state:

Theorem 2. *Let f be a distribution in the space of the variables (p_1, p_2) defined in an open neighbourhood of the set:*

$$\{(p_1, p_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}; p_1 \in \Omega_\alpha; p_2 = P_2\} \tag{2.16}$$

Ω_α being the open ball: $\{p_1; (p_1 - P_1)^2 < \alpha\}$.

The following properties of f are equivalent:

i) *For every positive number $\alpha' < \alpha$, there exists an open neighbourhood ω of P_2 in $\mathbb{R}^{n_2}$, such that: $\forall \varphi \in \mathscr{D}(\omega)$, the distribution:*

$$f_\varphi(p_1) = \int f(p_1, p_2)\varphi(p_2)dp_2 \quad \text{is defined in } \Omega_{\alpha'}. \tag{2.17}$$

Moreover, it is the boundary value on $\Omega_{\alpha'}$ of a function F_φ which is analytic in the local tube $T_{B\Phi\alpha'}$ [with $\Phi(p_1) = (p_1 - P_1)^2$].

ii) *For every (α', α'') with $0 < \alpha' < \alpha'' < \alpha$, there exists an open neighbourhood of P_2 in $\mathbb{R}^{n_2}$ such that:*

$\forall \chi_1 \in \mathscr{D}_{\alpha'}(\Omega_{\alpha''})$, $\forall \varphi \in \mathscr{D}(\omega)$, the "partial" generalized Fourier transform:

$$\begin{aligned}&\mathscr{F}_{\text{part}}(\chi_1 \cdot f_\varphi)(x_1, x_0)\\ &= (2\pi)^{-(n_1+n_2)/2} \int e^{-ip_1x_1 - (p_1 - P_1)^2 x_0} f(p_1p_2)\chi_1(p_1)\varphi(p_2)dp_1dp_2\end{aligned} \tag{2.18}$$

satisfies the following set of bounds:

$$\forall N > 0 \ |\mathscr{F}_{\text{part}}(\chi_1 \cdot f_\varphi)(x_1, x_0)| \leqq C_N \|\varphi\|_m (1 + \|x_1\|^N)^{-1} e^{-\alpha' x_0} \tag{2.19}$$

valid in the region:

$$\{(x_1, x_0) \notin S_B, \quad x_0 \geqq 0\}. \tag{2.20}$$

In (2.19) m is a fixed integer which expresses the order of the distribution f with respect to the variables p_2.

The semi-norm $\|\varphi\|_m$ is defined by:

$$\|\varphi\|_m = \sup_{\substack{1 \leqq j \leqq m \\ p_2 \in \mathbb{R}^{n_2}}} |D^{(j)}\varphi(p_2)|$$

the constants C_N can depend on α', α'' and $\|x_2\|$.

iii) *For every (α', α'') with $0 < \alpha' < \alpha'' < \alpha$ and every $\chi_2(p_2)$ analytic in a neighbourhood of P_2 with $\chi_2(p_2) \neq 0$ at $p_2 = P_2$, $\chi_1(p_1)$ as in* ii) *the total generalized Fourier transform:*

$$\begin{aligned}&\mathscr{F}_{\text{tot}}(\chi_1 \cdot \chi_2 \cdot f)(x_1, x_2, x_0)\\ &= (2\pi)^{-(n_1+n_2)/2} \int e^{-i(p_1x_1 + p_2x_2) - [(p_1-P_1)^2 + (p_2-P_2)^2]x_0} f(p_1p_2)\chi_1(p_1)\chi_2(p_2)dp_1dp_2\end{aligned} \tag{2.21}$$

satisfies the following set of bounds: $\forall N > 0$

$$|\mathscr{F}_{\text{tot}}(\chi_1 \cdot \chi_2 \cdot f)(x_1, x_2, x_0)| \leqq C_N(1 + \|x_2\|^m)(1 + \|x_1\|^N)^{-1} e^{-\alpha' x_0} \tag{2.22}$$

valid in the region

$$\{(x_1, x_2, x_0) : (x_1, x_0) \notin S_B ; x_0 \geqq 0 ; x_2 \in \mathbb{R}^{n_2}\} \tag{2.23}$$

(with m the same as in ii)*).*

The proof of this theorem is given in the Appendix; here we shall rather indicate its physical interpretation when the distribution f is chosen to be the scattering kernel of a certain process; P_1 (resp. P_2) can for example refer to the set of relative (resp. absolute) four momenta of a first (resp. second) group of particles. (These two groups can be those of incoming and outgoing particles as it will be the case in Section 3, but this is not necessary.) The set of conditions (2.22) is a set of macrocausality conditions of the type I.S.; each of them requires the localization of all the particles near the average configuration $(P_1, x_1/\|x_1\|)$, $(P_2, x_2/\|x_2\|)$. Theorem 2 essentially says that the analyticity of the scattering kernel with respect to p_1 in the specified domain $T_{B\Phi\alpha}$, is equivalent to a precise exponential damping of the scattering amplitude with respect to relative space-time displacements of the first group of particles.

The equivalence of statements ii) and iii) means that for the short-range behaviour to hold it is immaterial to consider the second group of particles either in a localizing "sequence" of states in the sense of I.S., or in an arbitrary fixed state φ with momentum spectrum localized in a neighbourhood of P_2.

3. Analyticity with Respect to Momentum Transfer and Conditions of Short Range of Forces

We consider the scattering of two initial and n final identical scalar particles with mass m. Let us denote by $r = \{r_1 \dots r_n\}$ the final four-momenta which satisfy the mass-shell conditions:

$$r_j^2 = m^2, \quad r_j^{(0)} > 0 \qquad j = 1, \dots, n\,. \tag{3.1}$$

We shall represent by $k_i = p_i + iq_i$, $i = 1, 2$, $k_i \in \mathbb{C}^4$ the corresponding momenta for the two initial particles.

The total energy momentum conservation:

$$k_1 + k_2 = \textstyle\sum_{j=1}^n r_j \equiv r \tag{3.2}$$

leaves us with only $n+1$ independent four vectors. We choose the set

$$\{k = p + iq = (k_1 - k_2)/2 ; \quad r_j \quad j = 1, \dots, n\}$$

where k is the relative momentum of the incoming particles, which is considered as a complex four-vector. One also puts:

$$r^2 = s \tag{3.3}$$

(squared total energy).

In the context of axiomatic field theory [8, 9] it has been proved that when the $r_j, j=1,\dots, n$ are kept fixed and real, the scattering kernel $T(p;\{r_j\})$ is the restriction to the physical region of an analytic function $H(k)$ inside the Jost-Lehmann-Dyson domain of holomorphy D [15]. The latter has a non-empty intersection with the complex mass-shell manifold $\mathcal{M}^c$ (which is a two-dimensional complex sphere):

$$\mathcal{M}^c = \left\{ k \in \mathbb{C}^4; \quad \begin{matrix} k^2 = m^2 - s/4 \\ k \cdot r = 0 \end{matrix} \right\}. \tag{3.4}$$

This intersection $D \cap \mathcal{M}^c$ is of spherical type, described by:

$$\text{or (equivalently):} \quad \left. \begin{matrix} |\boldsymbol{q}| < C_s \\ \boldsymbol{p}^2 < s/4 - m^2 + C_s^2 \end{matrix} \right\} \tag{3.5}$$

with: $C_s = 3m^2/\sqrt{s}$.

In a more rigorous way, $T(p, r)$ is a distribution in the variables $r = \{r_j\}$ which is analytically valued in k; it means that for every test functions $\prod_j \varphi_j(\boldsymbol{r}_j) \in \mathcal{D}(\mathbb{R}^{3n})$, the function:

$$T\{\varphi_j\}(k) = \int T(k, r) \prod_j \varphi_j(r_j) d^3 r_j / 2\omega_j, \qquad \omega_j = \sqrt{r_j^2 + m^2} \tag{3.6}$$

is analytic in a J.L.D. domain of $\mathbb{C}^4_{(k)}$.

We shall always choose the functions φ_j so as to have their support in an arbitrarily small neighbourhood of a fixed configuration $\{r_j = R_j\}$ of the final momenta. Then, the corresponding domain $D \cap \mathcal{M}^c$ of the function $T\{\varphi_j\}(k)$ is defined by the constant:

$$C'_s = 3m^2/\sqrt{s} - \varepsilon \tag{3.7}$$

where the number $\varepsilon > 0$, takes into account the size of the support of the wave packets $\prod_j \varphi_j$.

The scattering amplitude for the process $A_1 + A_2 \to B_1 + \dots + B_n$ when the initial $(A_i, i = 1, 2)$ and the final $(B_j, j = 1, \dots, n)$ particles are translated by the four vectors $x_1, x_2, x'_j, j = 1, \dots, n$ respectively reads:

$$\begin{aligned} S(\{\varphi\}, \{\varphi'_j\})(x, x'_j X) = \int \textstyle\prod_{i \in J} \varphi_j^{*\prime}(\boldsymbol{r}_j)\varphi(p) e^{-i[px + \sum_{j\in J} r_j(x'_j + X)]} T(p, \{r_j\}) \\ \cdot \delta(pr)\delta(p^2 + r^2/4 - m^2) dp \textstyle\prod_{j \in J} d^3 r_j / 2\omega_j \end{aligned} \tag{3.8}$$

with $J = \{1, \dots, n\}$

$$(\boldsymbol{r}_j, p) \in \mathcal{M}^r : \begin{cases} r_j^2 = \omega_j^2 - m^2 \\ pr = 0 \\ p^2 = m^2 - s/4 \equiv -\alpha = \alpha(s) \end{cases} \tag{3.9}$$

for arbitrary test functions $\prod_i \varphi_j(\boldsymbol{r}_j)\varphi(p) \in \mathcal{D}(\mathcal{M}^r)$ with a suitable parametrization of $\mathcal{M}^r$ described in the following).

In (3.8) the energy momentum conservation $p_1 + p_2 = \sum_{j=1} r_j$, has been taken into account besides the change of variables:

$$(x_1 + x_2)/2 = X, \qquad x_1 - x_2 = x. \tag{3.10}$$

By using Theorems 1 and 2 we shall be able to write conditions $C(P, u, \alpha, \gamma)$ of the type "I.S." for the scattering amplitudes (3.8); these conditions will be equivalent to the above analyticity domain.

The average momentum configuration "P" is here specified by fixing:

$$r_j = R_j,\ r = R,\ k = P\,(\text{real}) \quad \text{with} \quad P \cdot R = 0,\ P^2 = m^2 - R^2/4\,.$$

It is convenient to choose axes of coordinates such that:

$$R = (R^{(0)}, \mathbf{0}), \qquad P = (0, 0, 0, P^{(3)}), \qquad k = (k^{(0)}, k^{(1)}, k^{(2)}, k^{(3)}) \tag{3.11}$$

we also put:

$$k_{\mathrm{tr}} = (k^{(1)}, k^{(2)}) \quad (k \text{ transverse})\,.$$

The space time (normalized) translation "u" = "x"/‖"x"‖ is here specified by fixing the set

$$\text{"}x\text{"} = \{x_j, j \in J;\ \ X = (x_1 + x_2)/2;\ \ x = x_1 - x_2 = (0, x_{\mathrm{tr}}, 0)\}\,. \tag{3.12}$$

The two dimensional vector x_{tr} is the "impact parameter" with respect to the classical momentum configuration (R, P).

We shall use the vector variables k_{tr} (resp. p_{tr}) as a parametrization of $\mathscr{M}^c$ (resp. $\mathscr{M}^r$). With the above choice of "x" the matrix element (3.8) can be written more precisely:

$$\begin{aligned} &S(\{\varphi\}, \{\varphi'_j\})(x_{\mathrm{tr}}, \{x'_j\}, X) \\ &= \int \prod_{j \in J} \varphi_j^{*\prime}(r_j) \varphi(p_{\mathrm{tr}}) e^{-i[p_{\mathrm{tr}} x_{\mathrm{tr}} - \sum r_j (x'_j + X)]} T(p_{\mathrm{tr}} \{r_j\}) \prod_{j \in J} (d^3 r_j / 2\omega_j) d^2 p_{\mathrm{tr}} \end{aligned} \tag{3.13}$$

where $\varphi \in \mathscr{D}(R^2)$ has a sufficiently small support.

We now construct a maximal domain in $\mathscr{M}^c$ whose projection in the variables k_{tr} will be a local tube with real open set Ω (centered at P) defined by:

$$\Omega = \{p_{\mathrm{tr}};\ 0 \leqq \Phi(p) = p_{\mathrm{tr}}^2 < \alpha(s)\}\,. \tag{3.14}$$

When p_{tr} varies in Ω the point $p = (p_{\mathrm{tr}}; \sqrt{\alpha - p_{\mathrm{tr}}^2})$ varies in an hemisphere of $\mathscr{M}^r$ (with pole P).

In an auxiliary space with dimension two, we define the basis B of the local tube $T_{B\Phi\alpha}$ as the set of points (ϱ, ω) with $\varrho < r(\omega)$.

Then (see Section 2 and [5]) $T_{B\Phi\alpha}$ is the union for all values of $\varrho < r(\omega)$ of the bounded components (with edge $\partial\Omega$) of the following "cycles":

$$\begin{aligned} \Gamma_\varrho : \{&k_{\mathrm{tr}} = p_{\mathrm{tr}} + i q_{\mathrm{tr}} \in \mathbb{C}^2 \\ &|q_{\mathrm{tr}}| + \varrho(|p_{\mathrm{tr}}|^2 - |q_{\mathrm{tr}}|^2 - \alpha) = 0\,; \\ &q_{\mathrm{tr}} = |q_{\mathrm{tr}}|\omega,\ |p_{\mathrm{tr}}|^2 < \alpha\}\,. \end{aligned} \tag{3.15}$$

Such a definition is meaningful provided that $r(\omega) < 1/2\sqrt{\alpha}$. We shall now fix $r(\omega) = r_{\max}$ (independent of ω) in such a way that $T_{B\Phi\alpha}$ be maximal and contained in the axiomatic domain $D \cap \mathscr{M}^c$.

Looking for a local tube in which the maximal value of $|\boldsymbol{q}|$ is the one given by the domain (3.5) [or (3.7)], we must impose that when $|\boldsymbol{p}| = 0$

$$|q_{\mathrm{tr}}| \equiv 1/2\varrho - (1/4\varrho^2 - \alpha)^{1/2} \leqq C_s\,. \tag{3.16}$$

One then easily verifies the following two possibilities:

i) for energies $\sqrt{s}$ such that: $C_s^2 < \alpha$, the inequality (3.16) is satisfied when $\varrho \leqq C_s/(C_s^2 + \alpha)$;

ii) for energies such that $C_s^2 \geqq \alpha$ the condition $\varrho \leqq 1/2\sqrt{\alpha}$ is sufficient for (3.16) to be true.

We conclude that the maximal local tube $T_{B\Phi\alpha}$ with image in $\mathscr{M}^c$ contained in the domain (3.5) (or (3.7)) (where $T(p, r)$ has an analytic extension) admits as basis:

$$B = \left\{ \varrho : \begin{array}{ll} \varrho \leqq 1/2\sqrt{\alpha} & \text{for } C_s \geqq \sqrt{\alpha} \\ \varrho \leqq C_s/(C_s^2 + \alpha) & \text{for } C_s < \sqrt{\alpha} . \end{array} \right. \tag{3.17}$$

Let us now choose arbitrary functions $\chi(p_{\text{tr}})$, $\prod_j \chi'_j(\boldsymbol{r}_j)$ $j = 1, \ldots, n$ with the appropriate properties described in Section 2 and consider the generalized Fourier transform:

$$\mathscr{F}^{\Phi}(\chi, \{\chi'_j\})(x_{\text{tr}}, x_0) \equiv \int e^{-p_{\text{tr}}^2 x_0} e^{-ip_{\text{tr}} x_{\text{tr}}} T(p_{\text{tr}}, \{\boldsymbol{r}_j\}) \chi(p_{\text{tr}}) \prod_j \chi'_j(\boldsymbol{r}_j)(d^3 r_j/2\omega_j) dp_{\text{tr}} . \tag{3.18}$$

Following Theorem 1, $\mathscr{F}_1^{\Phi}$ satisfies bounds of type (2.14) for $\{x_{\text{tr}}, x_0 \notin S_B\}$; where S_B is the convex cone in the space (x_{tr}, x_0) with apex at the origin, (with $x_0 = \gamma \|x_{\text{tr}}\|$, $\gamma = \text{const}$) whose intersection with the plane $x_0 = 1$ is the polar set $\hat{B}$ of B (3.17):

$$\tilde{B} = \left\{ x_{\text{tr}} : \begin{array}{ll} \|x_{\text{tr}}\| \leqq 2\sqrt{\alpha} & \text{for } C_s \geqq \sqrt{\alpha} \\ \|x_{\text{tr}}\| \leqq (C_s^2 + \alpha)/C_s & \text{for } C_s < \sqrt{\alpha} \end{array} \right\} . \tag{3.19}$$

Let us suppose that in the expression (3.13) of the scattering amplitude the wave functions are gaussian type wave packets of the form:

$$\varphi'_j(\boldsymbol{r}_j) = e^{-(\boldsymbol{r}_j - \boldsymbol{R}_j)^2 x_0} \chi'_j(\boldsymbol{r}_j), \qquad \varphi(p_{\text{tr}}) = e^{-p_{\text{tr}}^2 x_0} \chi(p_{\text{tr}}) . \tag{3.20}$$

Application of Theorem 2 then leads to the following bounds of the scattering amplitude:

$$|S(\{\varphi\}, \{\varphi'_j\})(x_{\text{tr}}, \{x_j\}, X)| \leqq \begin{cases} c_N e^{-\frac{\sqrt{\alpha}}{2}\|x_{\text{tr}}\|} / (1 + \|x\|^N); & C_s \geqq \sqrt{\alpha(s)} \\ c_N e^{-\frac{\alpha C_s}{C_s^2 + \alpha}\|x_{\text{tr}}\|} / (1 + \|x\|^N); & C_s < \sqrt{\alpha(s)} \end{cases} \tag{3.21}$$

$$\forall N, \qquad \forall (x_{\text{tr}}, x_0) \notin S_B, \ (X, x'_j) \in \mathbb{R}^{4(n+1)} .$$

We notice that the intersection of S_B with the $x_0 = 0$ plane is only a point, that means, the origin in x space.

The choice of P (centre of the local tube $T_{B\Phi\alpha}$) being arbitrary on the sphere $\mathscr{M}^r$ (3.9) we can repeat the same procedure for all points P in $\mathscr{M}^r$. It follows, that by taking the union of all these local tubes, we reconstruct the total domain of analyticity (3.5). The set of conditions of type (3.21) obtained, represent then the "I.S. conditions" corresponding to this domain.

The directions x_{tr} for each of these conditions, lie always on the plane orthogonal to each vector P; it follows that the bounds of type (3.21) are valid in every

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[8] Remark: The rigorous application of Theorem 2 says that α has to be replaced by some $\alpha' = \alpha - \varepsilon$ with $\varepsilon \to 0$ when the size of the support of φ_j contained in $\mathbb{R}_j^3$ tend to zero.

direction in $\mathbb{R}^3$. The exponential decrease of the transition amplitude is thus expressed with respect to the relative distance of the translated trajectories of the two initial particles (or impact parameter) independently of the translation $X+x_j$ of the final particles. The latter behaviour expresses the short range character of strong interactions and we have proved that this behaviour is equivalent to the J.L.D. domain of analyticity of the scattering kernel, with respect to the variable of relative momentum. The precise rate of exponential decrease is given by the size of this domain.

Because of the invariance under the Lorentz group, the analytic extension $F(k, r)$ of the scattering kernel defines an analytic function in the space of the invariants $k \cdot r_j\, j=1,\ldots, n$ (the invariant k^2 being a function of s in the manifold $\mathcal{M}^c$).

In the case of the scattering $A_1+A_2 \to B_1+B_2$ (two final particles), the remaining independent invariant is $k\cdot(r_1-r_2)$ (at fixed total energy) which by a linear transformation is related to the scattering angle $z=\cos\theta$. The image of the domain $D \cap \mathcal{M}^c$ in the space of this variable is the Lehmann ellipse [8]. Taking into account unitarity Martin [9] obtained a larger ellipse whose inverse image[9] is a spherical domain analogous to (3.5) (or (3.7)), with corresponding constant:

$$C_M^2 = m^2 s/(s/4-m^2) > C_s^2 \,. \tag{3.22}$$

What follows then is that "I.S. conditions" of type (3.21) equivalent to Martin's ellipse may be obtained in the same way as described above. By inserting the value of the constant C_M (3.22) in the formulae (3.21) we have explicitly the following bounds:

a) For $4m^2 \leqq s \leqq m^2(12+8\sqrt{2})$

$$\begin{aligned} &|S(\{\varphi\}, \{\varphi_j'\})(x_{\mathrm{tr}}, \{x_j'\}, X)| \\ &\quad \leqq c_N(1+\|x\|^N)^{-1} \exp[-(s/4-m^2)^{1/2}\|x_{\mathrm{tr}}\|/2] \,. \end{aligned} \tag{3.23}$$

b) For $m^2(12+8\sqrt{2}) \leqq s < +\infty$

$$\begin{aligned} &|S(\{\varphi\}, \{\varphi_j'\})(x_{\mathrm{tr}}, \{x_j\}, X)| \\ &\leqq c_N(1+\|x\|^N)^{-1} \exp[-m(s/(s/4-m^2))^{1/2}(m^2 s/(s/4-m^2)^2+1)^{-1}\|x_{\mathrm{tr}}\|] \end{aligned} \tag{3.24}$$

$$\forall N\,; \quad \forall \{x_{\mathrm{tr}}, x_0\} \notin S_B, (X, x_j') \in \mathbb{R}^{4(n+1)},\ \|x\| = \{|x_{\mathrm{tr}}|^2 + x_0^2\}^{1/2} \,.$$

One then verifies that for $s \to \infty$ the exponential decrease reads:

$$|S(\{\varphi\}, \{\varphi_j'\})|_{s\to\infty} < c_N e^{-2m\|x_{\mathrm{tr}}\|}/(1+\|x\|^N) \tag{3.25}$$

which expresses the physical idea that for very large energies the exponential damping is independent on the energy.

The corresponding bounds (3.22) for the case of the Lehmann ellipse give for $s \to \infty$:

$$|S(\{\varphi\}, \{\varphi_j'\})|_{s\to\infty} < c_N/(1+\|x\|^N) \exp(-(\mathrm{const}/\sqrt{s})\|x_{\mathrm{tr}}\| \to 0) \,. \tag{3.26}$$

We should expect these different results as a direct consequence of the qualitative difference between the two domains of analyticity: the Martin ellipse has always $t_{\max} = 4m^2$ while the Lehmann ellipse shrinks to zero as $t_{s\to\infty} \sim \mathrm{const}/s \to 0$.

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9 In k-space.

Concerning the behaviour at the threshold energy $s \to 4m^2$, it is, as one expects, similar in both cases: the rate of exponential decrease vanishes as:

$$\sim \sqrt{\alpha} = (s/4 - m^2)^{1/2} . \tag{3.27}$$

Some Remarks about the O.K.R. Formalism

In the works of Omnès, Kugler-Roskies, Finley [1, 2], one starts from the short range hypothesis expressed in terms of exponential decrease property of the scattered wave for any production process (α). Such a property is immediately rewritten in terms of the exclusive absorptive part A_α of the two-body scattering amplitude associated with the channel α as follows:

$$\int A_\alpha(E, \hat{p} \cdot \hat{p}') \varphi_a(p) \varphi_a^*(p') d\Omega_p d\Omega_{p'} < ce^{-2\sigma a} \tag{K.R.}$$

with $\varphi_a(p) = (\lambda a/\pi)^{3/4} e^{-(p-P)^2 \lambda a/2} e^{-ipa}$ similar to our $\varphi e^{-ip \cdot x}$ in (3.8), (3.20).

The parameter a is the impact parameter (in semi-classical approximation) and the rate of decrease σ in this formalism is justified from intuitive potential theory considerations. By exploiting conditions (K.R.), these authors derive analyticity properties of the absorptive parts A_α. The analyticity of the scattering amplitude is yielded afterwards by unitarity. In these studies the principal scope was the understanding of analyticity properties on the basis of the short-range character of the interaction in space-time. It prefigured in a special situation the general S-matrix approach of Iagolnitzer and Stapp [3, 4].

Our method and results differ from those of O.K.R.F. in the following respects:

a) The method of the generalized Laplace transform allows us to state a precise equivalence between the size (and the shape) of the analyticity domains and the corresponding rate of exponential decrease in space-time.

b) We applied our method to the derivation of I.S. conditions for the $(2 \to n)$ scattering amplitudes. However it could have been applied as well to the corresponding scattered waves (or A_α) and would have then led to inequalities of the type (K.R.).

c) In our formulation, the fixed total energy is not necessarily fixed as in (K.R.). Testing with wave packets in all variables, including the total four momentum, is a little more satisfactory from both (physical and mathematical) points of view (see the general I.S. formulation [3, 4]).

d) Concerning the physical interest of the above mathematical equivalence, we preferred to choose the point of view according to which space-time asymptotic properties can be derived from the analyticity properties of field theory, rather than the converse. In fact, without discussing here the general motivations which can lead one to prefer either the field theoretical approach or the S-matrix approach to the analyticity properties on the mass shell, we can make the following remark: in the S-matrix approach it seems difficult to justify with physical arguments a precise "quantitative" formulation of the macrocausality conditions (including all types of "short range assumptions"). For example in [1, 2] the rate of exponential decrease σ which was chosen was only justified by an analogy with potential theory. On the contrary, if one accepts the global analyticity domains of axiomatic field theory as a firm basis, the above method yields a derivation of "quantitative" macrocausality laws in the field theoretical framework.

4. Analyticity with Respect to the Total Energy

4.1. Causality Conditions

Let us consider now the scattering

$$A_1 + A_2 \to A_3 + A_4 \tag{4.1}$$

of identical scalar particles with mass m. The four momenta p_1, p_3 are real and satisfy the mass shell conditions: $p_i^2 = m^2$, $p_i^{(0)} > 0$, $i = 1, 3$ while the corresponding momenta for A_2, A_4 particles are:

$$k_2, k_4 \in \mathbb{C}^4 .$$

The total energy-momentum conservation: $p_1 + k_2 = p_3 + k_4$ leaves us with three independent four-vectors: we choose the following ones:

$$\left.\begin{aligned} r &= p_1 - p_3 \\ r_{13} &= p_1 + p_3 \end{aligned}\right\} \in \mathbb{R}^4 \qquad k = (k_2 + k_4)/2 \in \mathbb{C}^4 (k = p + iq) . \tag{4.2}$$

The mass-shell conditions expressed in terms of those variables are the following:

$$\mathscr{M}^r \cdot \begin{cases} 4p^2 + r^2 = 4m^2 & r^2 + r_{13}^2 = 4m^2 \\ p \cdot r = 0 & r \cdot r_{13} = 0 . \end{cases} \tag{4.3}$$

We suppose that the wave packets $\{\varphi\}$ of all the particles have been translated by corresponding four-vectors x_i, $i = 1, \ldots, 4$ and we define:

$$\begin{aligned} X &= (x_4 + x_2)/2, \qquad \xi = x_2 - x_4 , \\ \tilde{X} &= (x_3 + x_1)/2, \qquad \tilde{\xi} = x_3 - x_1 . \end{aligned} \tag{4.4}$$

The scattering amplitude for this process then reads:

$$\begin{aligned} S(\{\varphi\} e^{-i(p_1 x_1 + p_2 x_2 - p_3 x_3 - p_4 x_4)}) &= \int T(p, r_{13}, r) \{\tilde{\varphi}\}(p, r_{13}, r) e^{-i\left[p\xi - \frac{r_{13}}{2}\tilde{\xi} + r(\tilde{X} - X)\right]} \\ &\quad \delta(r_{13}^2 + r^2 - 4m^2) \delta(pr) \delta(r_{13} r) \delta(4p^2 + r^2 - 4m^2) dp dr_{13} dr \end{aligned} \tag{4.5}$$

where we have put $\{\tilde{\varphi}\}(p, r_{13}, r) = \{\varphi\}(p_1, p_2, p_3, p_4)$ [with (p, r_{13}, r) satisfying (4.3)] $(\{\tilde{\varphi}\} \in \mathscr{D}(\mathscr{M}^r))$.

Let us now define the average momentum configuration "P" of the scattering process by fixing:

$$r_{13} = R_{13}, \qquad r = R, \qquad k = P \text{ (real)}$$

$$\text{with:} \quad \begin{aligned} P \cdot R &= 0 & P^2 &= m^2 - R^2/4 \\ R \cdot R_{13} &= 0 & R^2 + R_{13}^2 &= 4m^2 \end{aligned} \tag{4.6}$$

$\{\tilde{\varphi}\}$ is then assumed to have its compact support containing (P, R_{13}, R) and such that its projection in the space $\mathbb{R}^4_{(r_{13})}$ lies in an arbitrary small neighborhood of R_{13}.

We now choose for convenience axes of coordinates such that:

$$\begin{aligned} R_{13} &= (R_{13}^{(0)}, \mathbf{0}) \\ R &= (0, \boldsymbol{P}_r) \end{aligned} ; \qquad P = (P^0, \boldsymbol{P}) \tag{4.7}$$

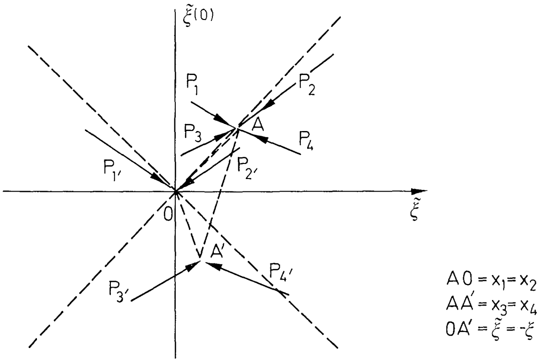

Fig. 1. Space time displacements. The relative translation ξ lies inside the light cone (see text)

and we restrict our set of "normalized translations" "u" = "x"/‖"x"‖, with "x" = $(X, \tilde{X}, \xi, \tilde{\xi})$ to be of the following form: $(x_1 = x_2, x_3 = x_4$ with $(x_2 - x_4)^2 > 0)$ or:

$$X = \tilde{X} = (X^{(0)}, \mathbf{X}), \qquad \xi = -\tilde{\xi} = (\xi^{(0)}, \boldsymbol{\xi}) \tag{4.8}$$

with $\xi^{02} > \boldsymbol{\xi}^2$.

Such a choice of "x" ensures that the trajectories of the two final particles intersect each other at a point which lies in the forward (for $\xi^{(0)} < 0$) or backward cone (for $\xi^{(0)} > 0$) with apex at the intersection of the trajectories of the two initial particles. In this subsection, we stick to the case $\xi^{(0)} < 0$ (see Fig. 1) whose physical interpretation will be a test of causality.

(The detection apparatus, represented by the average trajectories of the outgoing wave packets should not detect any scattering in this situation.) With this choice of "x", the scattering amplitude (4.5) can be rewritten:

$$S(\{\varphi\})(\xi) = \int [T\{\tilde{\varphi}\}](p, r_{13}, r) e^{-i(p + r_{13}/2)\xi} \delta(pr)\delta(r_{13}r)\delta(4p^2 + r^2 - 4m^2) \delta(r_{13}^2 + r^2 - 4m^2) dp\, dr_{13}\, dr\,.$$

Now using the Lorentz covariance of T, the integration over dr_{13} and $dr^{(0)}$ can be performed and yields [with a new test function $\{\varphi_1\}$ $(p^{(0)}, \mathbf{p}, r_{13}^{(0)}, \mathbf{r})$]:

$$S(\{\varphi\})(\xi) = \int [T\{\varphi_1\}](p^{(0)}, \mathbf{p}, r_{13}^{(0)}, \mathbf{r}) e^{-i[(p^{(0)} + r_{13}^{(0)}/2)\xi^{(0)} - \mathbf{p}\boldsymbol{\xi}]} \delta(\mathbf{pr}) \delta(4p^2 - \mathbf{r}^2 - 4m^2)\delta(r_{13}^{(0)2} - \mathbf{r}^2 - 4m^2) d\mathbf{p}\, dr_{13}^{(0)} d\mathbf{r}\,. \tag{4.9}$$

Let us introduce the usual invariants:

$s = (p_1 + k_2)^2 = (p_3 + k_4)^2 = (r_{13}/2 + k)^2$: squared total energy in the C.M. system,

$t = (p_1 - p_3)^2 = (k_4 - k_2)^2 = r^2$: squared energy-momentum transfer C.M. system. (4.10)

In $\mathbb{C}^4_{(k)}$ the complex mass-shell manifold $\mathscr{M}^c$ is defined by:

$$\mathscr{M}^c : \begin{Bmatrix} k^2 = m^2 - t/4 \\ kr = 0 \end{Bmatrix}. \tag{4.11}$$

Taking into account the definitions (4.10), (4.11) we notice that the variable s is related to the four-vector $k \in \mathcal{M}^c$ by the following mapping π_t (r_{13}^0 and $\boldsymbol{r}$ being kept fixed):

$$k \in \mathcal{M}^c \xrightarrow{\pi_t} s = 2m^2 + 2\sqrt{m^2 - t/4}\, k^{(0)} - t/2 . \tag{4.12}$$

It has been proved in Ref. [11] that when r, r_{13} are kept fixed, real, and such that [10]:

$$-28m^2 \leqq t \leqq 0 \tag{4.13}$$

then the scattering kernel $T(p^0, \boldsymbol{p}, \boldsymbol{r}, r_{13}^0)$ considered as a function of $k = (k^0, \boldsymbol{k})$ has an analytic continuation in $\mathcal{M}^c$; its analyticity domain [11] in $\mathcal{M}^c$ is the inverse image by π_t of the s-cut-plane with cuts $s \in [4m^2, +\infty)$ $s \in [-t, -\infty)$ and t fixed in (4.13).

It follows that the intersection of the manifold $\mathcal{M}^c$ with the domain of analytic continuation of the scattering kernel $T(k^0, \boldsymbol{k}, \boldsymbol{r}, r_{13}^0)$ when $\boldsymbol{r}$, r_{13}^0 are kept fixed, is represented by the k_0-cut-plane with the following cuts:

$$\begin{aligned} &p^{(0)} \in [(m^2 + t/4)/(m^2 - t/4)^{1/2}, +\infty) \\ &p^{(0)} \in [-(m^2 + t/4)/(m^2 - t/4)^{1/2}, -\infty) . \end{aligned} \tag{4.14}$$

Let us call $p_0^{(0)} = \sqrt{m^2 - t/4}$ the threshold of the physical region of (4.1). We shall exploit partially this analyticity domain of T by considering the maximal local tube inside $\{\operatorname{Im} k^0 > 0\}$ which is bounded on the real by the open interval $\Omega =]p_0^{(0)}, p_m^{(0)}[$ whose center is P.

More precisely, let:

$$P^{(0)} \equiv (p_0^{(0)} + p_m^{(0)})/2, \qquad \alpha = ((p_m^{(0)} - p_0^{(0)})/2)^2 \qquad \Phi(P^{(0)}) = (p^{(0)} - P^{(0)})^2 \tag{4.15}$$

$$\Omega = \{p^{(0)} : 0 \leqq \Phi(p^{(0)} - P^{(0)}) < \alpha\} .$$

From the equation of "cycles":

$$\left.\begin{aligned} &q^{(0)} > 0; \quad q^{(0)} + \varrho\{(p^{(0)} - P^{(0)})^2 - q^{(0)2} - \alpha\} = 0 \\ &k^{(0)} = p^{(0)} + iq^{(0)}, \quad (p^{(0)} - P^{(0)})^2 - \alpha < 0 \end{aligned}\right\} \Gamma_\varrho \tag{4.16}$$

we conclude that the basis B of the maximal local tube $T_{B\Phi\alpha}$ is defined by:

$$B = \{\varrho : 0 < \varrho < 1/2\sqrt{\alpha}\} . \tag{4.17}$$

$T_{B\Phi\alpha}$ is the bounded connected component [12] of the union of "cycles" Γ_ϱ such that $\varrho \in B$. By taking arbitrary functions $\chi_1(\boldsymbol{p})$, $\chi_2(p^{(0)})$, $\chi_3(r_{13}^{(0)})$, $\chi_4(\boldsymbol{r})$ with the appropriate

[10] Here we stick to the case when no pseudo-threshold exists in any channel (for instance the $\pi - \pi$ scattering).

[11] More precisely this is a consequence of Ref. [11] and of the complex Lorentz invariance of the analyticity domain of the four point function, Ref. [12].

[12] Whose edge on the real is Ω.

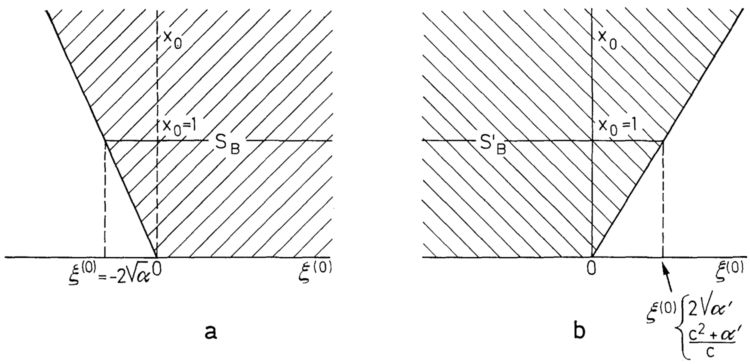

Fig. 2a and b. Essential support of $T(p^{(0)})\chi_2(p^{(0)})$ (see text). (a) For the analyticity on the first sheet; (b) For the analyticity on the second sheet

properties described in Section 2 (in particular χ_2 analytic in $T_{B\Phi\alpha}$) we define the following generalized Fourier transform $\mathscr{F}^{\Phi}_{\text{part}}$ in the space:

$$\{\xi^{(0)}, x_0\}\in \mathbb{R}^2 \tag{4.18}$$

$$\begin{aligned}&\mathscr{F}_{\text{part}}(T\{\textstyle\prod_{i=1,\dots,4}\chi_i\})(\xi^{(0)}, x_0)\\ &\quad=\int T(p^{(0)}, \boldsymbol{p}, r^{(0)}_{13}, \boldsymbol{r})\textstyle\prod_{i=1,\dots,4}\chi_i e^{-ip^{(0)}\xi^{(0)}-\Phi(p^{(0)}-P^{(0)})x_0}\\ &\qquad\delta(r^{(0)2}_{13}-\boldsymbol{r}^2-4m^2)\delta(4\boldsymbol{p}^2-\boldsymbol{r}^2-4m^2)\delta(\boldsymbol{p}\cdot\boldsymbol{r})dp^{(0)}d\boldsymbol{p}dr^{(0)}_{13}d\boldsymbol{r}\,.\end{aligned} \tag{4.19}$$

The essential support $S_B(\xi^{(0)}, x_0)$ of $T(p^0)\chi(p^0)$ is defined as the convex cone in $(\xi^{(0)}, x_0)$ space with apex at the origin, having as equations

$$\left\{x_0\geqq -\frac{1}{2\sqrt{\alpha}}\xi^{(0)};\ x_0\geqq 0\right\}$$

whose intersection with the line $x_0=1$ is the set (Fig. 2a)

$$\tilde{B}=\{\xi^{(0)}:\xi^{(0)}\geqq -2\sqrt{\alpha}\})\,. \tag{4.20}$$

Application of Theorem 1 then provides bounds of type (2.14) for $\mathscr{F}_{\text{part}}$ whenever $(\xi^{(0)}, x_0)\notin S_B$. If we further suppose that the wave function $\{\varphi_1\}$ in (4.17) contains Gaussian type wave packets, precisely:

$$\{\varphi_1\}(p, r^{(0)}_{13}, \boldsymbol{r})=\varphi_1(\boldsymbol{p})\varphi_2(p^{(0)})\varphi_3(r^{(0)}_{13})\varphi_4(\boldsymbol{r}) \tag{4.21}$$

with:

$$\left.\begin{aligned}\varphi_1(\boldsymbol{p})&=e^{-(\boldsymbol{p}-\boldsymbol{P})^2x_0}\chi_1(\boldsymbol{p})\\ \varphi_2(p^{(0)})&=e^{-(p^{(0)}-P^{(0)})^2x_0}\chi_2(p^{(0)})\\ \varphi_3(r^{(0)}_{13})&=e^{-(r^{(0)}_{13}-P^{(0)}_{13})^2x_0}\chi_3(r^{(0)}_{13})\\ \varphi_4(\boldsymbol{r})&=e^{-(\boldsymbol{r}-\boldsymbol{P}_r)^2x_0}\chi_4(\boldsymbol{r})\,,\end{aligned}\right. \tag{4.22}$$

Theorem 2 then yields the following bounds of the scattering amplitude (4.9):

$$|S\{\varphi\}(\xi)|\leqq c_N e^{-|\xi^{(0)}|\sqrt{\alpha/2}}/(1+\|\xi_1\|^N) \tag{4.23}$$

$$\forall N \quad\text{and}\quad (\xi^{(0)}, x_0)\notin S_B,\ \boldsymbol{\xi}\in\mathbb{R}^3 \qquad \|\xi_1\|=[\xi^{(0)2}+x_0^2]^{1/2}\,.$$

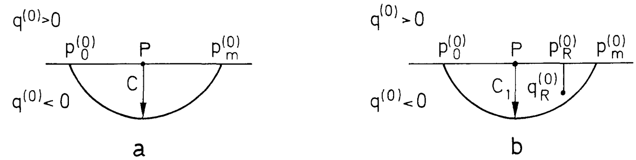

Fig. 3a and b. Local tube $T_{B\Phi\alpha'}$ on the second sheet (see text). (a) Without pole; (b) With a pole inside it

When $x_0=0$, S_B contains the positive $\xi^{(0)}$ axis, that means the bounds (4.23) are valid for all $\xi^{(0)}<0$.

We have to remark that the only space-time variable which contributes to the exponential bound is the time $\xi^{(0)}$ (conjugate variable to the squared total energy s with respect to which we have exploited the analyticity of the scattering amplitude).

In conclusion, we can interprete the bounds (4.23) as giving a precise expression of causality (in the usual sense). Qualitatively it indicates that the probability of the scattering process $A_1+A_2 \to A_3+A_4$ decreases exponentially with respect to the negative time-shift of detection of the final particles with respect to the initial ones. The constant in the exponent of (4.23) corresponds to the size of the bounded analyticity domain we have considered in the half-plane $\operatorname{Im} s>0$ for $s>4m^2$. Taking more and more analyticity into account, the exponential decrease is more and more rapid and approaches in some sense the classical limit (i.e. an infinite rate of decrease); as a matter of fact, due to the space-time spreading of wave packets, one could not expect a "sharper" expression of causality in the quantum case.

4.2. Relaxation Time Conditions

Under an additional smoothness postulate, it has been proved from axiomatic field theory [13] that the two-body scattering kernel has at least locally an analytic extension in the second Riemann sheet of the s-plane, when the real part of s is in the elastic region[13] ($4m^2 \leqq s < 16m^2$). We can then consider a maximal local tube on the second sheet: $q^0<0$ with $|q^0|<C$ (see Fig. 3).

This local tube will be defined as a union of manifolds Γ_ϱ with end points: $s_0=4m^2-t$, $s_m=16m^2$. The constant C will measure the "distance" of the nearest singularity from the real axis. (More precisely: C indicates to which cycle Γ_ϱ the nearest singularity belongs.) The equation of "cycles Γ_ϱ" for this case reads:

$$\begin{aligned} &|q^{(0)}|+\varrho[(p^{(0)}-P^{(0)})^2-q^{(0)2}-\alpha']<0 \\ &q^{(0)}<0 \qquad |q^{(0)}|<C \end{aligned} \tag{4.24}$$

where the constants: $\alpha'=((p_m^{0\prime}-p_0^0)/2)^2$, $P^{0\prime}=(p_m^{0\prime}+p_0^0)/2$ and $p_m^{0\prime}$, p_0^0 correspond to the values $s=16m^2$, $s=4m^2-t$ respectively; more explicitly from (4.12) we have:

$$\sqrt{\alpha'}=(3m^2+t/4)/(m^2-t/4)^{1/2} \qquad P^{(0)\prime}=4m^2/(m^2-t/4)^{1/2} \qquad -28m^2 \leqq t \leqq 0\,. \tag{4.25}$$

The conditions ensuring the boundedness of the domain (4.24) define the basis B of the local tube $T_{B\Phi\alpha'}$ and one verifies that:

$$B=\{\varrho : 0<\varrho<\tilde{C}\} \qquad \tilde{C}=\{C/(C^2+\alpha'),\ \text{for } C^2<\alpha'),\ 1/2\sqrt{\alpha'}\ (\text{for } C^2 \geqq \alpha')\}\,. \tag{4.26}$$

[13] Here we considered the case of an even theory.

The polar set is then given by:

$$\tilde{B} = \{\xi^{(0)} : \xi^{(0)} \leqq 1/\tilde{C}\}. \tag{4.27}$$

By analogous considerations to those of Subsection 4.1 we then define the essential support S'_B of $T(p^0)\chi(p^0)$ (see Fig. 2b) as the convex cone in $\{\xi^{(0)}, x_0\}$ space with $x_0 \geqq \tilde{C}\xi^{(0)}$; $x_0 \geqq 0$. Application of Theorem 2 then leads to the following bounds for the scattering amplitude (4.9):

$$|S\{\varphi\}(\xi)| \leqq c_N e^{-\alpha'\tilde{C}\|\xi^{(0)}\|}/(1 + \|\xi_1\|^N) \tag{4.28}$$

$\forall N$ and for $(\xi^{(0)}, x_0) \notin S_B$, $\xi \in R^3$, $\|\xi_1\| = \{\xi^{(0)2} + x_0^2\}^{1/2}$.

We obtain consequently (for $x_0 = 0$) an exponential decrease of the scattering amplitude for all times $\xi^{(0)} > 0$.

This property expresses the fact that the system of interacting particles A_1, A_2 can produce the final particles A_3, A_4 during an average time which is bounded by $1/\alpha'\tilde{C}$.

According to the above result we can therefore say that the analyticity of the scattering amplitude in a domain of the second sheet whose size is measured by C (Fig. 3) is equivalent to the existence of an upper bound $1/\alpha'\tilde{C}$ on the relaxation time of the "composite system" due to the interaction of A_1 and A_2.

We can refine the above result in the case where the extension of the two-body scattering kernel on the second Riemann sheet is known to be meromorphic in a bounded domain, with a single pole inside this domain at a point $S = S_R - i\Gamma$, $\Gamma > 0$. We can separate then the pole contribution from an analytic background in the following way:

$$T(p^{(0)}, \boldsymbol{p}, \boldsymbol{r}, r_{13}^{(0)}) = T_R(p^{(0)}; \boldsymbol{p}, \boldsymbol{r}, r_{13}^{(0)}) + T_B(p^{(0)}, \boldsymbol{p}, \boldsymbol{r}, r_{13}^{(0)}) \tag{4.29}$$

where:

$$\begin{aligned} T_R(p^{(0)}, \boldsymbol{p}, \boldsymbol{r}, r_{13}^{(0)}) &= b(\boldsymbol{p}, \boldsymbol{r}, r_{13}^{(0)})/(p^{(0)} - p_R^{(0)} + iq_R^{(0)}) \\ &= [2\sqrt{m^2 - t/4}(s - s_R + i\Gamma)]^{-1} b. \end{aligned} \tag{4.30}$$

b is a constant[14] and $T_B(p^0, p, r, r_{13}^0)$ an analytic function inside the maximal local tube that we can construct as previously discussed. That means that for $q^0 < 0$ we have: $|q^0| < C_1$ (see Fig. 3b) and the constant C_1 determines again to which cycle Γ_ϱ the nearest singularity (apart from the pole) belongs. Similar equations as (4.24,..., 4.27) then hold with the replacements $C \to C'_1$ and the generalized Fourier transform $\mathscr{F}_{\text{part}}$ reads:

$$\mathscr{F}_{\text{part}} = \mathscr{F}_{\text{part}}^R + \mathscr{F}_{\text{part}}^B. \tag{4.31}$$

The explicit evaluation of the pole contribution gives:

$$\begin{aligned} &\mathscr{F}_{\text{part}}^R\{\textstyle\prod_i \chi_i\}(\xi^{(0)}, x_0) \\ &\quad = \text{cnst}\, e^{-i(p_R^{(0)}\xi^{(0)} - 2(p_R^{(0)} - P^{(0)})q_R^{(0)}x_0)} e^{-[(p_R^{(0)} - P^{(0)})^2 - q_R^{(0)2}]x_0 - q_R^{(0)}\xi^{(0)}}. \end{aligned} \tag{4.32}$$

The scattering amplitude (4.9) can be separated in the same way:

$$S\{\varphi\}(\xi) = S^R\{\varphi\}(\xi) + S^B\{\varphi\}(\xi). \tag{4.33}$$

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[14] In $P^{(0)}$ space.

S^B satisfies bounds of type (4.28) and on the other hand one verifies that:

$$\begin{aligned}
&\text{for } |p_R^{(0)}-P|\gg|q_R^{(0)}|: \ |S^R\{\varphi\}(\xi^{(0)},x_0)|\approx \text{cnst}\, e^{-(p_R^{(0)}-P^{(0)})^2x_0}e^{-|q_R^{(0)}||\xi^{(0)}}, \ \xi^{(0)}>0\\
&|p_R^{(0)}-P|\approx|q_R^{(0)}|: \ |S^R\{\varphi\}(\xi^{(0)},x_0)|\approx \text{cnst}\, e^{-|q_R^{(0)}|\xi^{(0)}}; \qquad \xi^{(0)}>0\\
&|p_R^{(0)}-P|\ll|q_R^{(0)}|: \ |S^R\{\varphi\}(\xi^{(0)},x_0)|\approx \text{cnst}\, e^{-|q_R^{(0)}|\xi^{(0)}(1-\gamma|q_R^{(0)}|)},\\
&\qquad\qquad (x^{(0)}=\gamma\xi^{(0)},\ \xi^{(0)}>0)\,.
\end{aligned} \tag{4.34}$$

The last case in (4.34) expresses the fact that the "relaxation time" contribution of the "resonance" is limited as the isolated pole approaches the boundary of the local tube.

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Appendix

Proof of Theorem 2. (i)$\rightleftarrows$(ii) is a direct corollary of Theorem 1 since in view of the latter it is equivalent to state that $\forall\varphi\in\mathscr{D}(\omega)$, $f_\varphi(p_1)$ is analytic in $T_{B\Phi\alpha''}$ or that:

$$\forall\alpha'<\alpha''; \qquad \forall N>0,\ \forall\chi_1 \quad \text{in} \quad \mathscr{D}_{\alpha'}(\Omega_{\alpha''})$$
$$|\mathscr{F}_{\text{part}}(\chi_1\cdot f_\varphi)|<c'_N e^{-\alpha' x_0}/(1+\|x_1\|^N)\,. \tag{A.1}$$

Let us show that each constant C'_N can be rewritten $C_N\|\varphi\|m$, with C_N independent of φ. In Theorem 1, the constants C_N are proportional to $\sup|g(p)|$ g being a bounded primitive of f; here similarly, the function $f_\varphi(p_1)$ admits a bounded primitive (of a certain order):

$$g_\varphi(p_1)=\int g(p_1,p_2)\varphi(p_2)dp_2\,, \tag{A.2}$$

where g is a distribution of order m in p_2; we thus have an inequality of the form: $\sup|g_\varphi(p_1)|<C\|\varphi\|_m$, and this accounts for the φ dependence of the right hand side of (A.1).

Conversely, this dependence is necessary to be able to prove in (ii)→(i) the fact that $\varphi\to f_\varphi$ is a *continuous* linear functional (i.e. a distribution).

A proof that (i)→(iii) could be given along the same line as the proof of Theorem 1 (by making use of a contour deformation in the variables p_1); however it is simpler to prove directly that (ii)$\rightleftarrows$(iii).

(ii)→(iii): Let β be such that $\alpha'<\beta<\alpha''<\alpha$, and such that χ_1 [chosen in $\mathscr{D}_{\alpha'}(\Omega_{\alpha''})$ also belongs to $\mathscr{D}_\beta(\Omega_{\alpha''})$].

We then choose a sphere ω centered at P_2 such that condition (2.19) be satisfied, but with the exponential $e^{-\beta x_0}$ instead of $e^{-\alpha' x_0}$. Let us now choose the test function φ as follows:

$$\varphi(p_2)=\chi_2(p_2)e^{-(p_2-P_2)^2x_0}e^{-ip_2x_2} \tag{A.3}$$

with $\chi_2(p_2)$ as in (iii) and x_0, x_2 given constants. With this choice we obviously have:

$$\mathscr{F}_{\mathrm{tot}}(\chi_1 \cdot \chi_2 \cdot f)(x_1, x_2, x_0) = \mathscr{F}_{\mathrm{part}}(\chi_1 \cdot f_\varphi)(x_1, x_0) \tag{A.4}$$

and assumption (2.19) readily entails:

$$|\mathscr{F}_{\mathrm{tot}}(\chi_1 \cdot \chi_2 \cdot f)(x_1, x_2, x_0)| \leqq c'_N \|\varphi\|_m e^{-\beta x_0}/(1 + \|x_1\|^N) \tag{A.5}$$

[valid in the region (2.23)].

In view of (A.3), evaluation of $\|\varphi\|_m$ yields the majorization:

$$\|\varphi\|_m \leqq (1 + \|x_2\|^m) \cdot \mathscr{P}(x_0) . \tag{A.6}$$

$\mathscr{P}(x_0)$ is a certain fixed polynomial in x_0 which can be absorbed in the exponential if we majorize $e^{-\beta x_0}$ by $e^{-\alpha' x_0}$; inequality (2.22) is thus established in the region (2.23).

(iii)→(ii): Let us define:

$$F(x_1, x_2, x_0) = \mathscr{F}_{\mathrm{part}}(\chi_1 \cdot f_{\varphi \cdot e^{-ip_2 x_2}})(x_1, x_0) .$$

For every test function χ_2 satisfying (iii), with $\operatorname{supp}\chi_2 \supset \operatorname{supp}\varphi$, and χ_2 positive in the interior of its support, we can write:

$$\begin{aligned} F(x_1, x_2, x_0) = \int & f(p_1 p_2)\chi_1(p_1)\chi_2(p_2) e^{-i(p_1 x_1 + p_2 x_2)} e^{-[(p_1 - P_1)^2 + (p_2 - P_2)^2]x_0} \\ & \cdot \{[\chi_2(p_2)]^{-1} \varphi(p_2) e^{(p_2 - P_2)^2 x_0}\} dp_1 dp_2 . \end{aligned} \tag{A.7}$$

Then putting:

$$G(x_2, x_0) = \int e^{-ip_2 x_2} [\chi_2(p_2)]^{-1} \varphi(p_2) e^{(p_2 - P_2)^2 x_0} dp_2 \tag{A.8}$$

it follows that

$$F(x_1, x_2, x_0) = \int \mathscr{F}_{\mathrm{tot}}(\chi_1 \cdot \chi_2 \cdot f)(x_1, x_2 - x'_2, x_0) G(x'_2, x_0) dx'_2 . \tag{A.9}$$

Because of the properties of $\chi_2(p_2)^{-1}$ and $\varphi(p_2)$, Eq. (A.8) yields the following majorization for $G(x'_2, x_0)$:

$$|G(x'_2, x_0)| < \tilde{c}_N \|\varphi\|_m e^{x_0 \varrho^2}/(1 + \|x'_2\|^N) \tag{A.10}$$

where ϱ is the radius of the sphere ω.

On the other hand by choosing x_1, x_0 inside the region (2.23) and taking into account (2.22), we have:

$$|\mathscr{F}_{\mathrm{tot}}(x_1, x_2 - x'_2, x_0)| \leqq \tilde{c}'_N(1 + \|x_2 - x'_2\|^m) e^{-\beta x_0}, \quad \text{with} \quad \alpha' < \beta < \alpha'' < \alpha \tag{A.11}$$

such that $\chi_1 \in \mathscr{D}(\Omega_{\alpha''})$.

By inserting the bounds (A.10), (A.11) into (A.9) we obtain:

$$\begin{aligned} & |F(x_1, x_2, x_0)| \\ & \leqq e^{-(\beta - \varrho^2)x_0} \tilde{c}_N \|\varphi\|_m (1 + \|x_1\|^N)^{-1} \int c'_N (1 + \|x_2 - x'_2\|)(1 + \|x'_2\|^N)^{-1} dx'_2 . \end{aligned} \tag{A.12}$$

The integral on the right hand side of (A.12) is bounded by a (finite) constant and we can always take ϱ as small as we wish and such that $\beta - \varrho^2 > \alpha'$. Noting that $F(x_1, 0, x_0) = \mathscr{F}_{\mathrm{part}}(\chi_1 \cdot f_\varphi)(x_1, x_0)$, we see that the bound (2.19) readily follows from (A.12) and holds in region (2.20).

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Note Added in Proof. In the line which follows formula (3.21) (and similarly for (3.24)) replace $\forall(x_{tr}, x_0) \notin S_B$ by:

$$\forall x_{tr} \in \mathbb{R}^2, \; x_0 = \mathrm{Inf}(C_s, \sqrt{\alpha})\,[\alpha + (\inf(C_s, \sqrt{\alpha}))^2]^{-1}\,\|x_{tr}\|$$

$$\|x\| = \{|x_{tr}|^2 + x_0^2\}^{1/2}$$

In the line which follows formula (4.23) (respectively (4.28)) replace: $\forall(\xi^{(0)}, x_0) \notin S_B$ by:

$$\forall x_0 = \frac{|\xi^{(0)}|}{2\sqrt{\alpha}}, \qquad \xi^{(0)} < 0$$

(resp. $\forall x_0 = \tilde{C}|\xi^{(0)}|$, $\xi^{(0)} > 0$).