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Quantized Fields in External Field

II. Existence Theorems

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Abstract. This is the second part of an article devoted to the study of quantized fields interacting with a smooth classical external field with fast space time decrease. The case of a charged scalar field is considered first. The existence of the corresponding Green's functions is proved. For weak fields, as well as pure electric or scalar external fields, the Bogoliubov S-operator defined in Part I of this work is shown to be unitary, covariant, causal up-to-a-phase. Its perturbation expansion is shown to converge on a dense set in Fock space. These results are generalised to a class of higher spin quantized fields, "nicely" coupled to external fields, which includes the Dirac theory, and, in the case of minimal and magnetic dipole coupling, the spin one Petiau-Duffin-Kemmer theory. It is not known whether this class contains examples of physical interest involving quantized fields carrying spins larger than one.

I. Introduction

In the first part [1] of this paper, we have described some general facts concerning the Green functions and the Bogoliubov S-operator for the problem of a spin zero quantized field interacting with an external field. The main result was that the S-operator constructed according to the perturbation scheme is covariant, unitary, and causal up-to-a-phase if and only if there exists a "non-perturbative" solution (N-P in [1], Definition II.2.3) of the following integral equation:

$$I = A + A\Delta_F I = A + I\Delta_F A \tag{I.1a}$$

with

$$A(x, y) = [v(x) + A_\mu A^\mu(x)]\delta(x-y) + i[A_\mu(x) + A_\mu(y)]\partial^\mu \delta(x-y). \tag{I.1b}$$

v and A_μ denoting respectively the scalar and vector parts of the external field, and Δ_F the usual Feynman propagator.

Assuming the existence of a N-P solution the main result of [1] mentioned can easily be extended from the case of a spin zero quantized field to arbitrary integer spin. For half integer spin the algebraic properties of the Green functions undergo some well-known modifications [4a, 38] as explained in [1]. The method used in [1] to construct S have to be changed [2, 6a, 6b].

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Of course, the difficult part is now to find models which yield the N-P character required in [1] for the Green functions. As we have shown, this can be achieved either by solving the field equation with external field and Cauchy data, or equivalently [6a] by directly finding a weakly causal fundamental solution of this equation fulfilling the N-P property.

Concerning the existence of the fundamental solution, the following is actually known:

— for spin zero, there exists a tempered and causal solution, as it follows from standard theorems about strictly hyperbolic partial differential operators [3, 28, 34];

— for spin one-half, the same result holds although the Dirac system is not strictly hyperbolic. The existence follows from the fact that this system is symmetric with respect to a positive quadratic form [34, 29].

— for the spin one Petiau-Duffin-Kemmer equation [7a–c] with scalar electromagnetic or magnetic dipole coupling with an external field, the existence of such a solution follows from an algebraic trick (cf. [24] and Svensson in [33]). In the case of a weak symmetric quadrupole coupling with the external field a fundamental tempered solution has been recently shown to exist by Velo [35], but it is non causal [8].

The existence of such a fundamental solution then allows us to construct the kernel of the classical S-operator S_{cl}, which, in principle, describes the scattering theory in the one-particle space. This kernel defines two operators T_R and T_A on the space of test functions $\mathscr{S}(\mathbb{R}^4)$ [6a, b, 33]. They are actually shown to be isomorphisms of $\mathscr{S}(\mathbb{R}^4)$ into itself by adapting the Capri arguments [6b] to the case of spin zero, one half and one. Hence the first defining property for N-P solutions holds in this case.

Less is actually known about the second one, i.e., the boundedness of the operators on the one-particle Hilbert space with kernel defined by restricting Green functions to the mass-shell. The boundedness of S_{cl} as defined on the one particle Hilbert space was proved by Schroer, Seiler, and Swieca [4a] for spin zero, and Seiler [4b] for spin one-half.

On the other hand, the point which is crucial in order to deduce the existence of the out vacuum, is to show that the part of S_{cl} that connects one-particle states with one antiparticle states is a Hilbert-Schmidt operator. This was proved by the same authors only in the case in which the magnetic part A of the external electromagnetic field is zero, for spin zero [4a][1] and spin one-half [4b]. These results were announced in another form by Wightman in [6b][2].

As one can see from the previous check-list of results, a number of holes remain to be filled in. For this reason, we find it useful to give a reasonably self contained description which covers three cases of physical interest: spin zero, one half and one quantized fileds. We push the analysis to the uttermost generality, hoping for further applications. No new light will be a priori shed on models which

[1] The method used in this article relies on the proof of existence of an evolution operator, which unfortunatly fails in the presence of magnetic field, in spite of the existence of an elementary solution.

[2] It is unfortunate that the general discussion given there leaves one point out of control (cf. the comment following last lemma of Section 4 in [6a]), without which the existence of a Bogoliubov S-operator cannot be ascertained.

involve quantized fields carrying spins greater than one. There, acausalities are expected to occur [8] and non strict hyperbolicity may require a weakening of the notion of locality [24, 32], at least when perturbations are not too singular.

The present work is therefore divided into two main parts. In section II we study only the spin zero case to complete the field theory described in [1]. The essential new results are contained in the following:

Theorem I.1. *Let v and A_μ belong to the space $\mathscr{C}_0^\infty(\mathbb{R}^4)$ of smooth functions with compact support in $\mathbb{R}^4$* [3, 9]. *Then for some $\varepsilon>0$, the equation*

$$I(\lambda)=A(\lambda)+A(\lambda)\Delta_F I(\lambda)=A(\lambda)+I(\lambda)\Delta_F A(\lambda)\,, \tag{I.2}$$

where $A(\lambda)$ corresponds to $(\lambda v, \lambda A_\mu)$, has a unique N-P *solution which is analytic in $|\lambda|<\varepsilon$, in the* N-P *norm topology sense.*

The N-P norm topology is defined in Section II [cf. Formula (II.1.4)].

Here we have no longer any restriction about the magnetic part of the external field, but only on its strength. On the other hand, if we replace N-P by W-N-P (cf. [1], Definition II.2.2), then the same result holds for any λ in the complement of a discrete subset Σ of $\mathbb{C}$. Moreover, if v, and A_μ are real valued, then Σ does not not intersect the real axis. To our knowledge, nothing was known before about analyticity. On the other hand, the result of Schroer, Seiler, and Swieca [4a] shows that $I(\lambda)$ is also N-P for $\lambda\in\mathbb{R}$, if v and A_0 are real and $A=0$.

These results yield Theorem II.3.2 according to which the S-operator constructed in Part I is strongly continuous with respect to the external field, in a neighbourhood of zero, and expandable into a convergent series on a dense domain in Fock's space, including vectors with finite number of particles. This last theorem seems to be new.

Section III is an attempt to extend Theorem I.1 to quantized fields with higher spins. Let ψ be a Wightman free field defined as the solution of the first order covariant differential system [18–22, 37]

$$\psi[(-i\beta_\mu\partial^\mu-m)f]=0 \qquad \forall f\in\mathscr{S}(\mathbb{R}^4)\,, \tag{I.3}$$

where β_μ are $N\times N$ matrices and $m>0$. Then, in order to exploit the algebraic trick discovered for the spin one field [24, 33] we have been led to give the following definition:

Definition I.2. Let Φ be an $N\times N$ matrix valued $\mathscr{C}_0^\infty(\mathbb{R}^4)$ function. Φ will be called a *nice interaction*, if one can find four $N\times N$ matrix valued partial differential operators A_R, A_L, B_R, B_L satisfying the following conditions:

i) The coefficients of A_R, A_L, B_R, B_L are polynomials in Φ and its derivatives.

ii) A_R and A_L are first order partial differential operators.

iii) If $d(-i\partial)$ is the Klein-Gordon divisor [5, 6] of $L=-i\beta_\mu\partial^\mu-m$ then

$$(d+B_L)(L-\Phi)=-(\square+m^2)\mathbb{1}+A_L\,, \tag{I.4a}$$

$$(L-\Phi)(d+B_R)=-(\square+m^2)\mathbb{1}+A_R\,. \tag{I.4b}$$

This definition covers all the known examples of interactions giving rise to causal fundamental solutions. All interactions are nice in the Dirac equation. Scalar, electromagnetic as well as magnetic dipole couplings are nice in the spin one

Petiau-Duffin-Kemmer equation [7]. It would be interesting to search for nice interactions in other first order covariant differential systems.

Such a definition allows us to treat nice interactions almost as in the spin zero case, although the right hand side of (I.4) is not strictly hyperbolic. It leads to the following new result:

Theorem I.3. *Let Φ be a nice interaction. Then the equation:*

$$I(\lambda)=\Phi(\lambda)+\Phi(\lambda)S_F I(\lambda)=\Phi(\lambda)+I(\lambda)S_F\Phi(\lambda)\,, \tag{I.5a}$$

with

$$\Phi(\lambda)(x,y)=\lambda\Phi(x)\delta(x-y)\,, \tag{I.5b}$$

$$S_F(x,y)=d(-i\partial)\Delta_F(x-y)\,, \tag{I.5c}$$

has a unique N-P *solution if λ is real and small enough, and if Φ is hermitian with respect to the invariant form (the definition is given in Chapter III).*

Moreover, for the Dirac equation, the hermitian character of Φ can be relaxed and $I(\lambda)$ is analytic in a neighbourhood of $\lambda=0$ in the sense of the N-P-norm topology. The result of Seiler [4b] shows that $I(\lambda)$ is, in this case, N-P for any real λ as long as only scalar and purely electric couplings are involved.

Therefore, the quantized theory can be made in these cases. Presumably as an immediate consequence the S-operator is again strongly continuous. However as indicated in Remark III.4.5, it seems that except for the Dirac field we lose the essential of analyticity; therefore we do not expect the perturbation expansion to converge. Would this be the manifestation in the present context of the non renormalizability of such interactions?

At last an Appendix is devoted to a summary of some relevant results about Klein-Gordon-like equations.

In spite of these results many problems remain open.

i) Prove that in the spin zero case $I(\lambda)$ is in fact N-P for any λ in the complement of a discrete subset Σ of C.

ii) Complete in the case of fermion fields the analysis done in Part I [1].

iii) Prove that in the case of the Dirac equation, the subset Σ does not intersect the real line, in order to eliminate the case in which the vacuum expectation value of the S-operator is zero [36].

iv) Do there exist field equations describing spins greater than or equal to 3/2 admitting nice interactions?

v) Following Velo [35] does the non causal tempered solution of the Petiau-Duffin-Kemmer theory with symmetric quadrupole coupling give rise to a weakly causal distribution? Is it possible to complete the quantized theory by exhibiting the N-P character? What about the causality of the Bogoliubov S-operator?

vi) Is it possible to extend Velo's result to the case of spin 3/2 with electromagnetic or scalar coupling?

II. Existence Theorems for Spin Zero Quantized Fields

II.1. Weak External Fields

In our preceding paper [1] we have introduced the family $\mathcal{N}_{E_N}(m,\alpha,M)$ of Banach spaces, whose elements are $\mathscr{C}^m$ functions defined in $\mathbb{R}^4\times\mathbb{R}^4$ with values in a finite dimensional Banach space E_N, and normed by:

$$\|K\|_{m,\alpha,M} = \sum_{|\mu|,|\nu|\leqq m} \sup_{(p,q)\in\mathbb{R}^8} \{([1+(p^0-q^0)^2]^\alpha[1+(\boldsymbol{p}-\boldsymbol{q})^2]^\alpha/[1+|p|^2]^M[1+|q|^2]^M) \cdot \|(\partial^\mu/\partial p^\mu)(\partial^\nu/\partial q^\nu)K(p,q)\|_{E_N}\}\,. \tag{II.1.1}$$

Let us recall that the Fourier transform $\tilde{A}$ of A:

$$\tilde{A}(p,q)=\tilde{v}(p-q)+\widetilde{A_\mu A^\mu}(p-q)+i\tilde{A}_\mu(p-q)[p^\mu+q^\mu] \tag{II.1.2}$$

is in $\mathcal{N}_{\mathbb{C}}(m,\alpha,1/4)$ $\forall m\geqq 0$ $\forall\alpha\geqq 0$. ([1], Appendix 5), if v, A_μ are in $\mathcal{S}(\mathbb{R}^4)$ [9].

We have also associated to a kernel K the operator $\hat{K}$ on the one-particle space $\mathcal{H}_+\oplus\mathcal{H}_-$ of our theory defined by restricting the Fourier transform $\tilde{K}$ on the positive or negative mass-shell. We will denote by $\mathfrak{S}_{\text{N.P.}}$ the subspace in $\mathcal{L}(\mathcal{H}_+\oplus\mathcal{H}_-)$ of operators of the type

$$\hat{K}=\begin{bmatrix} K_{++} & K_{+-} \\ K_{-+} & K_{--} \end{bmatrix} \tag{II.1.3}$$

such that K_{+-} and K_{-+} are in the Hilbert Schmidt class [10]. We shall define in $\mathfrak{S}_{\text{N.P.}}$ the N-P norm by:

$$\|\hat{K}\|_{\text{N.P.}}=\text{Max}\,\{\|K_{++}\|_{\text{op}}, \|K_{--}\|_{\text{op}}, \|K_{+-}\|_{\text{H.S.}}, \|K_{-+}\|_{\text{H.S.}}\}\,. \tag{II.1.4}$$

The following results are shown in [1].

K1) if $m\geqq 1$, $\beta\geqq\alpha$, $\beta>3/2$ then there exists a constant $C>0$ such that:

$$\|K\Delta_{\text{ex}}L\|_{m,\alpha,1/4}\leqq c\|K\|_{m,\alpha,1/4}\|L\|_{m,\beta,1/4} \tag{II.1.5}$$

$$\forall K\in\mathcal{N}_{E_N}(m,\alpha,1/4), \qquad \forall L\in\mathcal{N}_{E_N}(m,\beta,1/4)\,,$$

where Δ_{ex} is any of the two point functions of the free field ([1], Lemma A.5.6).

K2) The mapping $K\to\hat{K}$ is linear and bounded from $\mathcal{N}_{E_N}(m,\alpha,1/4)$ to $\mathfrak{S}_{\text{N.P.}}$ if $\alpha>3/2$ ([1], Lemma A 5.4).

Therefore we are able to show the first main theorem:

Theorem II.1.1. *Let v, A_μ be in $\mathcal{S}(\mathbb{R}^4)$ and λ be a complex number. Then the equation:*

$$K(\lambda)=A(\lambda)+A(\lambda)\Delta_{\text{ex}}K(\lambda)=A(\lambda)+K(\lambda)\Delta_{\text{ex}}A(\lambda) \tag{II.1.6}$$

(where $A(\lambda)$ is defined in Theorem I.1) has a unique solution with $\tilde{K}\in\mathcal{N}_{\mathbb{C}}(m,\alpha,1/4)$ for $\alpha\geqq 0$ and $m\geqq 0$, which is analytic with respect to λ in a neighbourhood of zero.

Remark II.1.2. This theorem also applies if v and A_μ are matrix-valued functions.

Proof. Let $m\geqq 1$ and $\beta\geqq\alpha$, $\beta>3/2$. Using the result K1) one finds:

$$\|(\tilde{A}(\lambda)\Delta_{\text{ex}})^n\tilde{A}(\lambda)\|_{m,\alpha,1/4}\leqq C^n\|\tilde{A}(\lambda)\|^{n+1}_{m,\beta,1/4}\,. \tag{II.1.7}$$

Because (cf. [1], Proposition II.2.4), $\|\tilde{A}(\lambda)\|_{m,\beta,1/4}$ is finite, the Neumann series:

$$K(\lambda)=\sum_{n\geqq 0} A(\lambda)[\Delta_{\text{ex}}A(\lambda)]^n \tag{II.1.8}$$

converges in $\mathcal{N}_{\mathbb{C}}(m, \alpha, 1/4)$, if λ is small enough since $A(\lambda) \underset{\lambda \to 0}{=} O(\lambda)$. Since $A(\lambda)$ is a polynomial in λ, $K(\lambda)$ is analytic for small λ.

Corollary II.1.3. *The solution $K(\lambda)$ of* (II.1.6) *defines an analytic family $\hat{K}(\lambda)$ of elements of $\mathfrak{S}_{\text{N.P.}}$ for λ small enough.*

Proof. This can be seen by using K2).

II.2. Strong External Fields

Let us denote, as in [1] the kernel $K(\lambda)$ defined by Eq. (II.1.6) by $J_r(\lambda)$ [resp. $J_a(\lambda)$, $I(\lambda)$, $\bar{I}(\lambda)$] if $\Delta_{\text{ex}} = \Delta_r$ (resp. Δ_a, Δ_F, $\Delta_{\bar{F}}$) and let $\hat{K}(\lambda) = \Gamma J_R(\lambda) \Gamma^{-1}$ [resp. $\Gamma J_A(\lambda) \Gamma^{-1}$, $I_S(\lambda)$, $\bar{I}_S(\lambda)$], where

$$\Gamma = \begin{bmatrix} \mathbb{1}_+ & 0 \\ 0 & -\mathbb{1}_- \end{bmatrix}. \tag{II.2.1}$$

The main result is then the following (see [1], Definition II.2.2).

Theorem II.2.1. *Let v, A_μ be in $\mathscr{C}_0^\infty(\mathbb{R}^4)$* [3]. *Then there exists a unique* W-N-P *solution for $J_r(\lambda)$ which is analytic in λ, in the whole complex plane in the following sense:*

i) *$\lambda \to J_r(\lambda)$ is analytic if one chooses the simple convergence topology* [30] *of linear continuous operators from $O_M(\mathbb{R}^4)$ to $\mathscr{S}(\mathbb{R}^4)$* [9].

ii) *$\lambda \to J_R(\lambda)$ is an analytic family of bounded operators, such that $J_{+-}(\lambda)$ and $J_{-+}(\lambda)$ are compact* [10].

Remark II.2.2. The same theorem holds for $J_a(\lambda)$.

We first need some partial results:

Lemma II.2.3. *If v, $A_\mu \in \mathscr{C}_0^\infty(\mathbb{R}^4)$ and $\lambda_0 \in \mathbb{C}$, then* $(1 - A(\lambda)\Delta_r)$ *defines an analytic family of automorphisms of $\mathscr{S}(\mathbb{R}^4)$ for the simple convergence topology in a neighbourhood of λ_0.*

The proof of this lemma is the main content of the appendix. This result can be extended without change to the case in which v, A_μ are in $\mathscr{S}(\mathbb{R}^4)$ with support in $[t_0, t_1] \times \mathbb{R}^3$ for some $t_0, t_1 \in \mathbb{R}$. Capri [6b] and Wightman [6a] have already proved than $(1 - A\Delta_r) = T_r$ is an automorphism of $\mathscr{S}(\mathbb{R}^4)$.

Lemma II.2.4. *If v, $A_\mu \in \mathscr{C}_0^\infty(\mathbb{R}^4)$ and $\lambda_0 \in \mathbb{C}$, then $J_R(\lambda)$ defines an analytic family of bounded operators on $\mathscr{H}_+ \oplus \mathscr{H}_-$ in a neighbourhood of λ_0.*

The proof of this lemma is also in the appendix.

Lemma II.2.5 [39]. *Let D be a domain in $\mathbb{C}$, $\mathscr{H}$ be a Hilbert space, and $\hat{K}(\lambda)$ an analytic family of bounded operator on $\mathscr{H}$. If there exists a subdomain D' in D, such that $\hat{K}(\lambda)$ is compact for any λ in D', then $\hat{K}(\lambda)$ is compact for any λ in D.*

Proof of Theorem II.2.1. i) Because $J_r(\lambda) = (1 - A(\lambda)\Delta_r)^{-1} A(\lambda)$ the first part of Theorem II.2.1 follows from Lemma II.2.3.

ii) From Corollary II.1.3 we know that $J_R(\lambda)$ is N-P for small λ. Therefore if λ is small, $J_{r+-}(\lambda)$ and $J_{r,-+}(\lambda)$ are Hilbert-Schmidt and thus compact. From Lemmas II.2.4 and II.2.5 the other part of Theorem II.2.1 follows.

II.3. Existence and Properties of the Bogolioubov S-Operator

We are now able to prove Theorem I.1.

Theorem II.3.1. *Let* v *and* A_μ *be in* $\mathscr{C}_0^\infty(\mathbb{R}^4)$, *and* $\lambda \in \mathbb{C}$.

a) *Then the equation*

$$I(\lambda) = A(\lambda) + I(\lambda)\Delta_F A(\lambda) = A(\lambda) + A(\lambda)\Delta_F I(\lambda) \tag{II.3.1}$$

has a unique W-N-P *solution except if* λ *belongs to some discrete subset* Σ *of* $\mathbb{C}$.

b) $\lambda \to I(\lambda)$ *is analytic in* $\mathbb{C}\backslash\Sigma$ *for the simple convergence topology of linear continuous operators from* $\mathcal{O}_M(\mathbb{R}^4)$ *to* $\mathscr{S}(\mathbb{R}^4)$.

c) $\lambda \to I_S(\lambda)$ *is analytic in* $\mathbb{C}\backslash\Sigma$ *for the topology of* $\mathscr{L}(\mathscr{H}_+ \oplus \mathscr{H}_-)$.

d) $\exists \varepsilon > 0$, *such that* $I(\lambda)$ *is* N-P *and* $\lambda \to I_S(\lambda)$ *is analytic for the* N-P *norm topology, if* $|\lambda| < \varepsilon$.

e) *If* v, A_μ *are real,* Σ *does not intersect the real line.*

f) *If* v, A_μ *are real and* $A_i = 0$ $(i = 1, 2, 3)$ *then* $I_S(\lambda)$ *is* N-P *for all real* λ*'s.*

Proof. From Theorem II.2.1 one knows the existence and unicity of $J_r(\lambda)$ as a W-N-P kernel. The equivalence theorem ([1], Theorem II.4.1) gives the existence and unicity of $I(\lambda)$ if and only if, $\mathbb{1}_{++} + J_{r,+-}(\lambda)J_{a,-+}(\lambda)$ is invertible. Since $\lambda \to J_{r,+-}(\lambda)$ [resp. $\lambda \to J_{a,-+}(\lambda)$] is an analytic family of compact operators, then, for all λ, by a slight extension of the classical Fredholm alternative [10–12] there exists a discrete subset Σ of $\mathbb{C}$ such that $\mathbb{1}_{++} + J_{r,+-}(\lambda)J_{a,-+}(\lambda)$ is invertible in $\mathbb{C}\backslash\Sigma$. This proves a)–c).

d) Follows from Theorem II.1.1.

e) Follows from ([1], Theorem II.4.1).

f) Follows from a theorem given by Schroer-Seiler-Swieca [4].

Using the results given in [1] (Chapter III) about the construction of a Bogoliubov S-operator up-to-a-phase one finds:

Theorem II.3.2. *Let* v, A_μ *be real functions in* $\mathscr{C}_0^\infty(\mathbb{R}^4)$ *and let* $S_0(\lambda v, \lambda A_\mu)$ *be the quantized S-operator* [1]*:*

$$S_0(\lambda v, \lambda A_\mu) = \det(\mathbb{1}_+ - I_{+-}(\lambda)\bar{I}_{-+}(\lambda))^{1/2} : e^{i\int \varphi^* I(\lambda)\varphi} : . \tag{II.3.2}$$

Then, if $\Phi_{f,G}$ *is the coherent state with wave functions* f, G, *defined by* $\Phi_{f,G} = e^{a^+(f) + b^+(G)}\Omega$, $f \in \mathscr{H}_+$, $G \in \mathscr{H}'_-$ *:*

B0) $(\lambda, f, G) \to S_0(\lambda v, \lambda A_\mu)\Phi_{f,G}$ *is analytic in the domain* $|\lambda| < \varepsilon$, $f \in \mathscr{H}_+$, $G \in \mathscr{H}'_-$, *in the topology of Fock space, for some* $\varepsilon > 0$ [3].

B1) $(v, A_\mu) \to S_0(v, A_\mu)$ *is continuous in a neighbourhood of zero in the space of real valued* $\mathscr{C}_0^\infty(\mathbb{R}^4)$ *functions.*

B2)

$$S_0(\lambda v, \lambda A_\mu) = \mathbb{1}_{\mathscr{F}} + i\lambda \int_{\mathbb{R}^4} \{ :\varphi^*\varphi:(x)v(x) + i :\varphi^* \overleftrightarrow{\partial^\mu} \varphi:(x)A_\mu(x)\} d^4x + O(\lambda^2) \tag{II.3.3}$$

on the dense domain spanned by vectors with finite number of particles.

Moreover if $\lambda \in \mathbb{R}$ *is small enough, or if* $A_i = 0$ $(i = 1, 2, 3)$. *Then*

B3) $S_0(\lambda v, \lambda A_\mu)$ *is relativistically covariant.*

B4) $S_0(\lambda v, \lambda A_\mu)$ *is unitary.*

B5) $S_0(\lambda v, \lambda A_\mu)$ *is causal up-to-a-phase.*

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[3] This implies the analyticity in λ on the domain of vectors with a finite number of particles, which we believe to hold also in the spin 1/2 case. Note that $\mathscr{H}'_-$ is the dual of $\mathscr{H}_-$ (see [1]).

Proof. Properties B3)–B5) follow from Theorem II.4.1 of [1], and the fact that $I(\lambda)$ is N-P under the hypothesis of the theorem:

Property B0. One can write [[1], Eq. (III.3.1)]

$$S_0(\lambda v, \lambda A_\mu)\Phi_{f,G} = \det(\mathbb{1}_{++} - I_{+-}(\lambda)\bar{I}_{-+}(\lambda))^{1/2} \cdot e^{i\langle G, I_{-+}(\lambda)f\rangle + ia^+ I_{+-}(\lambda)b^+} \Phi_{(\mathbb{1}_{++} + iI_{++}(\lambda))f, G(\mathbb{1}_{--} + iI_{--}(\lambda))} \cdot \tag{II.3.4}$$

Note in particular that

$$\bar{I}_{-+}(\lambda) = I_{+-}(\bar{\lambda})^* \tag{II.3.5}$$

since v and A_μ are real functions.

The analyticity of:

$$(K, f, G) \to e^{ia^+ K b^+}\Phi_{f,G} \quad \text{in the domain} \quad \|K\|_{\mathrm{H.S.}} < \infty$$
$$\|K\|_{\mathrm{op}} < 1 \quad \text{and} \quad f \in \mathscr{H}_+, G \in \mathscr{H}'_- \tag{II.3.6}$$

and of:

$$K \to \det(1-K)^{1/2} \quad \text{if } K \text{ is trace class with} \quad \|K\|_{\mathrm{op}} < 1 \tag{II.3.7}$$

yields: B0).

Property B2. It is a simple consequence of B0) and of the fact that $I(\lambda)$ is $O(\lambda)$ as $\lambda \to 0$. In view of the analyticity with respect to f and G, one can generate vectors with a finite number of particles by taking derivatives with respect to f and G in Eq. (II.3.4).

Property B1. Recall that ([1], Appendix 5) by Theorem II.1.1 and by property K2), of Section II.1 the map

$$(v, A_\mu) \to I_S \tag{II.3.8}$$

is continuous in the N-P norm topology at least in a neighbourhood of zero in $\mathscr{C}_0^\infty(\mathbb{R}^4)$. The analyticity property [Eq. (II.3.6)] for generalized coherent states implies that:

$$(v, A_\mu) \to S_0(v, A_\mu)\Phi_{f,G} \tag{II.3.9}$$

is continuous if v, A_μ are small enough.

If we now restrict ourselves to real valued functions then $S_0(v, A_\mu)$ is unitary [1], hence uniformly bounded, and

$$(v, A_\mu) \to S_0(v, A_\mu) \tag{II.3.10}$$

is strongly continuous in a neighbourhood of zero in the subspace of real valued $\mathscr{C}_0^\infty(\mathbb{R}^4)$ functions.

III. Higher Spin Quantized Fields in External Fields

III.1. A Special Class of Free Fields

We now consider a charged Wightman [13, 14] free field describing an irreducible system of mass $m > 0$, spin "s" and charge 1, transforming under the restricted Poincaré group as follows:

$$U(a, A)\psi_\alpha(x)U(a, A)^{-1} = \textstyle\sum_\beta L_{\beta\alpha}(A^{-1})\psi_\beta(Ax + a), \tag{III.1.1}$$

where L is some finite dimensional representation of the group $\mathrm{SL}(2, \mathbb{C})$.

The classification of such fields was given by several authors [15a, 15b, 16], at least for fields without internal symmetries. If one introduces the $U(1)$-symmetry which specifies the charge, these results can be trivially extended. The main result is the following:

Proposition III.1.1 [16]. *A necessary and sufficient condition for the Wightman charged field, with transformation law given by Eq.* (III.1.1), *to be an irreducible free field is that:*

i) *There exists a positive λ, and irreducible* SU(2)*-invariant spin "s" orthogonal projector P_0 in the space E_N of the representation L.*

ii) *The non zero two point Wightman functions are:*

$$\begin{aligned} &\langle\psi(f)^*\psi(g)\rangle=\lambda\int d\Omega_m(p)\langle\tilde{f}(p), L[p]^{*-1}P_0L[p]^{-1}\tilde{g}(p)\rangle \\ &\langle\psi(g)\psi(f)^*\rangle=\lambda\int d\Omega_m(p)\langle\tilde{f}(-p), L[p]^{*-1}RP_0RL[p]^{-1}\tilde{g}(-p)\rangle\,. \end{aligned} \tag{III.1.2}$$

Here $d\Omega_m$ is the Lorentz invariant measure on the positive mass-shell, $[p]$ is a solution of

$$[p][p]^*=\frac{p^0+\boldsymbol{p}\cdot\boldsymbol{\sigma}}{m}\qquad p^0=(m^2+\boldsymbol{p}^2)^{1/2}\,, \tag{III.1.3}$$

$$R=L(\mathbb{1},-\mathbb{1}) \tag{III.1.4}$$

($\sigma_1,\sigma_2,\sigma_3$ are the Pauli matrices),

where $L(A,B)$ is the analytic continuation of L to $\mathrm{SL}(2,\mathbb{C})\times\mathrm{SL}(2,\mathbb{C})$ [14, 16, 17].

$\tilde{f}$ is the Fourier transforms of the function f [which will be assumed to belong to $\mathscr{S}(\mathbb{R}^4)\otimes E_N$].

We shall restrict ourselves to a class of fields defined by a first order system of partial differential equations [5, 6]. Recalling that [16] it is always possible to choose an inner product $\langle,\rangle$ in E_N such that

$$L(A)^*=L(A^*) \tag{III.1.5}$$

we shall assume the following axioms ([5]):

S1) There exists a linear mapping $p\to\beta(p)$ from $\mathbb{R}^4$ to $\mathscr{L}(E_N)$ which is covariant

$$L(A)\beta(p)L(A)^{-1}=\beta(Ap)\qquad\forall A\in\mathrm{SL}(2,\mathbb{C}),\ \forall p\in\mathbb{R}^4\,. \tag{III.1.6}$$

S2) There exists an invertible element η of $\mathscr{L}(E_N)$ such that:

$$\eta\beta(p)\eta^{-1}=\beta(p)^*\,, \tag{III.1.7a}$$

$$\eta L(A)\eta^{-1}=L(A^{*-1})\,. \tag{III.1.7b}$$

η can be chosen [18, 19] up to a sign such that,

$$\eta=\eta^*=\eta^{-1}\,. \tag{III.1.8}$$

S3) Let $T(p)$ be an $\mathscr{L}(E_N)$-valued tempered distribution such that:

$$L(A)T(p)L(A)^{-1}=T(Ap)\,, \tag{III.1.9a}$$

$$T(p)(\beta(p)-m)=0=(\beta(p)-m)T(p)\,. \tag{III.1.9b}$$

Then $\eta T(p)$ is a linear combination of the Fourier transform of the two-point Wightman functions of the irreducible Wightman field.

Before characterizing this class of fields, let us remark that axioms S1 and S2 restrict the choice of the representation L [18–20]. However, for an arbitrary spin "s" it is possible to construct such a field [21, 22, 40]. For instance, the Dirac field and the Petiau-Duffin-Kemmer field are in this class.

Now we have the following preliminary result:

Lemma III.1.2. *Let ψ be a charged Wightman field satisfying Eq.* (III.1.1), *and axioms S1–S3. Then ψ is a free field with non negative mass.*

Proof. Let $\tilde{W}(p)$ be the Fourier transform of a non zero two-point Wightman function for ψ. By S3 it is solution of

$$\tilde{W}(p)(\beta(p)-m)=0\,. \tag{III.1.10}$$

By a classical theorem on distributions [3, 9], the support of $\tilde{W}(p)$ is contained in the subset σ of $\mathbb{R}^4$ in which $\beta(p)-m$ is non invertible.

Because E_N is finite dimensional and $\beta(p)$ is covariant by S1, σ is a finite union of hyperboloids (with one or two shells). In view of S3, the only possible solutions of (III.1.10) must describe an irreducible Wightman field. Therefore, σ is reduced to a two-shell hyperboloid, since zero mass as well as imaginary mass solutions are excluded from the Wightman theory, and the decomposability into several hyperboloids is excluded by the irreducibility hypothesis. Therefore $\exists m'>0$ with

$$\sigma=\{p\in\mathbb{R}^4;\ p^2=m'^2\}\,. \tag{III.1.11}$$

Now [13, 14]

i) $\operatorname{supp}\tilde{W}=\sigma\cap V^+$ by covariance and spectral condition.

ii) $\tilde{W}$ is a covariant positive measure, and therefore it satisfies the Klein-Gordon equation

$$(p^2-m'^2)\tilde{W}(p)=0\,. \tag{III.1.12}$$

iii) ψ is a generalized free field by the Borchers-Greenberg theorem [23].

i)–iii) prove that ψ is a solution of the Klein-Gordon equation corresponding to mass m'.

Lemma III.1.3. *Let β be a linear mapping from $\mathbb{R}^4$ to E_N satisfying* S1–S3. *Then the minimal equation of the matrix $\beta_0=\beta(\mathring{p}/m)$ where $\mathring{p}=(m',\mathbf{0})$ is*

$$\beta_0^n(\beta_0^2-\mathbb{1})=0 \tag{III.1.13}$$

for some integer n.

Proof. We have shown that σ is the hyperboloid $p^2=m'^2$. Therefore the spectrum of β_0 is reduced to $\{+1,-1,0\}$. Because of the covariance law one finds $R\beta_0R^{-1}=-\beta_0$; thus, $+1$ and -1 have the same multiplicity.

On the other hand, if $T(p)$ is a covariant distribution solution of (III.1.9b), its support is contained in σ, and one can decompose T as a covariant linear combination of transverse derivatives of the invariant measure $d\Omega_m$:

$$T(p)=\sum_{\varepsilon=\pm 1}\sum_{n=0}^{N_\varepsilon}\{(\partial/\partial\sqrt{p^2})^n(\delta(p^0-\varepsilon\omega_p)/2\omega_p)\}L[p]C_n^\varepsilon L[p]^{-1}\,. \tag{III.1.14}$$

Putting $z = \pm\sqrt{p^2}$ if $p \in V^\pm$, Eq. (III.1.9b) becomes by a suitable redefinition of $C_n^\pm$:

$$\sum_{n=0}^{N_\varepsilon} \delta^{(n)}(z/m' - \varepsilon)(\beta_0 z - \mathbb{1})C_n^\varepsilon = 0 \tag{III.1.15}$$

which can be written in the form:

$$(\beta_0 - \varepsilon)C_n^\varepsilon = 0, (\beta_0 - \varepsilon)C_{n-1}^\varepsilon = \beta_0 C_n^\varepsilon, \ldots, (\beta_0 - \varepsilon)C_0^\varepsilon = \beta_0 C_1^\varepsilon . \tag{III.1.16}$$

A necessary and sufficient condition to have

$$C_1^\varepsilon = 0 = \ldots = C_n^\varepsilon \tag{III.1.17}$$

is that the eigennilpotents corresponding to the eigenvalues ± 1, be zero [12]. This proves the lemma.

If one modifies $\beta(p)$ in such a way that $m' = m$ one finds:

Corollary III.1.4. *The matrices $\beta(p)$ satisfy:*

$$[\beta(p) - m]^{-1} = d(p)(p^2 - m^2)^{-1} \qquad \forall p \in \mathbb{R}, \tag{III.1.18}$$

where $d(p)$ is the Klein-Gordon divisor [5, 6, 22]*:*

$$d(p) = (\beta(p) + m)[\beta(p)/m]^n - ((p^2 - m^2)/m) \sum_{k=0}^{n-1} [\beta(p)/m]^{1k2} . \tag{III.1.19}$$

If $p^2 > 0$ the eigen-projectors of $\beta(p)$ corresponding to the eigenvalues $\pm\sqrt{p^2}$ are

$$\Pi_\pm(p) = \tfrac{1}{2}(\beta(p)/\sqrt{p^2})^n(\mathbb{1} \pm \beta(p)/\sqrt{p^2}) . \tag{III.1.20}$$

Proof. These results are standard consequences of the covariance and spectral theorems for finite dimensional matrices.

We are now able to characterize the free fields of interest, within the class we have considered.

Proposition III.1.5. *Let ψ be an irreducible charged Wightman field, fulfilling Eq.* (III.1.1), *and axioms* S1–S3. *Then ψ is a free field with mass $m > 0$ and spin "S", and the eigenprojector P_0 is given by*

$$P_0 = \alpha\eta\Pi_+(\mathring{p}), \ \alpha > 0 \qquad \mathring{p} = (m, \mathbf{0}) . \tag{III.1.21}$$

Proof. If ψ is such a field one can easily check [cf. Eq. (III.1.2)] that

$$P_0(\mathbb{1} - \beta_0) = (\beta_0^* - \mathbb{1})P_0 = 0 . \tag{III.1.22}$$

Therefore ηP_0 commutes with β_0 and is annihilated to the right and the left by $\beta_0 - \mathbb{1}$. Since the eigennilpotent of β_0 for the eigenvalue $+1$ is zero, we get

$$\eta P_0 = \eta P_0 \Pi_+ = \Pi_+ \eta P_0 , \tag{III.1.23}$$

where, for short, we have put $\Pi_+ = \Pi_+(\mathring{p})$.

On the other hand, since $\eta\Pi_+$ is hermitian, we get:

$$\mathrm{Im}(\eta\Pi_+) = \mathrm{Ker}(\eta\Pi_+)^\perp . \tag{III.1.24}$$

Therefore the initial and final projectors [31] of $\eta\Pi_+$ are equal to the same projector P, which is of course SU(2)-invariant and, by Eq. (III.1.23)

$$P_0 \leqq P. \tag{III.1.25}$$

If $P_0 \neq P$, there exists P'_0 orthogonal to P_0 less than $P-P_0$, SU(2)-invariant and irreducible such that (III.1.22) holds for P'_0. This would imply the existence of another solution for ψ, which is excluded by S3. Hence:

$$P_0 = P. \tag{III.1.26}$$

Thus, by definition of P, and the Schur lemma,

$$\eta\Pi_+ = \eta\Pi_+ P_0 = P_0\eta\Pi_+ = P_0\eta\Pi_+ P_0 = \alpha^{-1}P_0. \tag{III.1.27}$$

α is a real number since $\eta\Pi_+$ is hermitian, and α can be chosen positive by possibly changing the sign of η.

Definition III.1.6. *$\mathscr{H}_\pm$ will denote the Hilbert space spanned by classes of functions on the positive mass-shell with values in E_N, and defined by the inner product*

$$(\varphi|\varphi')_\pm = \int d\Omega_m(p)\langle\varphi(p), (\pm)^{2s+n}\eta\Pi_\pm(p)\varphi'(p)\rangle. \tag{III.1.28}$$

This definition is motivated by the form of the two point functions

$$L[p]^{*-1}\eta\Pi_+ L[p]^{-1} = \eta\Pi_+(p) \tag{III.1.29a}$$

$$\begin{aligned} L[p]^{*-1}R\eta\Pi_+ R^{-1}L[p]^{-1} &= (-)^{2s}\eta R\Pi_+(p)R^{-1} \\ &= (-)^{2s+n}\eta\Pi_-(p) \end{aligned} \tag{III.1.29b}$$

where use has been made of $R = L(\mathbb{1}, -\mathbb{1}) = L(-\mathbb{1})\eta R\eta^{-1}$ and Eq. (III.1.20).

III.2. Nice Interactions

Let us examine the problem where ψ is coupled to an external field through the following interaction lagrangian density:

$$\mathscr{L}_I(x;\Phi) = :\psi^*(x)\eta\Phi(x)\psi(x): \tag{III.2.1}$$

where $\Phi \in \mathscr{S}(\mathbb{R}^4) \otimes \mathscr{L}(E_N)$.

In the introduction we have defined "nice" interactions (cf. Definition I.2) and we want to show that in this case, the problem can be reduced to the treatment of a scalar quantized field interacting with an external field as far as the existence of the Schwinger kernel is concerned. In order to do so we will use the equivalence theorems ([1] Theorem II.4.1), classical theorems about the Klein-Gordon operators (see Appendix) and the techniques applied in Section II to the case of weak external fields.

First of all, one has to exhibit some non trivial nice interactions

Proposition III.2.1. *Let ψ be the Dirac field. Then any external field in $\mathscr{C}_0^\infty(\mathbb{R}^4) \otimes \mathscr{L}(E_N)$ is nice.*

Proof. Take $\beta_\mu = \gamma_\mu$ and [cf. Eq. (I.4)] $B_R = B_L = \Phi$.

Then:

$$A_R = -\Phi^2 - 2m\Phi - i(\gamma\partial\Phi - \Phi\gamma\partial)\,, \tag{III.2.2a}$$

$$A_L = -\Phi^2 - 2m\Phi + i(\gamma\cdot\partial\cdot\Phi - \Phi\cdot\gamma\cdot\partial)\,, \tag{III.2.2b}$$

since

$$(-i\gamma\cdot\partial + m + \Phi)(-i\gamma\partial - m - \Phi) = -(\Box + m^2) + A_L\,, \tag{III.2.3a}$$

$$(-i\gamma\partial - m - \Phi)(-i\gamma\partial + m + \Phi) = -(\Box + m^2) + A_R\,, \tag{III.2.3b}$$

and A_R, A_L are differential operators of degree one.

Proposition III.2.2 [24]. *If one assumes*

$$\beta(p)(\beta(p)^2 - p^2\,\mathbb{1}) = 0 \qquad \forall p \in \mathbb{R}^4 \tag{III.2.4}$$

then the minimal coupling is a nice interaction.

Proof. If $A_\mu \in \mathscr{C}_0^\infty(\mathbb{R}^4)$ then the minimal coupling is obtained by replacing $-i\partial_\mu$ by $\nabla_\mu = -i\partial_\mu - A_\mu$ in the field equations. Thus $\Phi = \beta_\mu A^\mu$.
Let us choose B_R and B_L in such a way that

$$d(-i\partial) + B_R = d(\nabla) = d(-i\partial) + B_L\,. \tag{III.2.5a}$$

Because of Eq. (III.2.4), one finds:

$$d(\nabla) = m + \beta(\nabla) + \frac{\beta(\nabla)^2 - \nabla_\mu \nabla^\mu}{m}\,. \tag{III.2.5b}$$

Therefore

$$d(\nabla)(\beta(\nabla) - m) = \nabla_\mu \nabla^\mu - m^2 + \frac{\beta(\nabla)^3 - \nabla_\mu \nabla^\mu \beta(\nabla)}{m}\,. \tag{III.2.6}$$

The last term in the right hand side can be written as:

$$-1/m(\beta_{\mu_1}\beta_{\mu_2}\beta_{\mu_3} - g_{\mu_1\mu_2}\beta_{\mu_3})(i\partial^{\mu_1} + A^{\mu_1})(i\partial^{\mu_2} + A^{\mu_2})(i\partial^{\mu_3} + A^{\mu_3})\,. \tag{III.2.7}$$

Equation (III.2.4) tells us that the symmetric part of the tensor $\beta_{\mu_1}\beta_{\mu_2}\beta_{\mu_3} - g_{\mu_1\mu_2}\beta_{\mu_3}$ is zero, therefore, all symmetric terms in the expansion of $\nabla^{\mu_1}\nabla^{\mu_2}\nabla^{\mu_3}$ disappear. Using the commutation relation $[\partial^\mu, A^\nu] = (\partial^\mu A^\nu)$, one can see that (III.2.7) is a partial differential operator of degree less than or equal to one.

Remark III.2.3. One should first note that Proposition III.2.2 covers the Petiau-Duffin-Kemmer theory [26]. Using general results about the matrices β_μ (see for instance [18] or [19]) one can construct all solutions β such that Eq. (III.2.4) holds. In particular, for any spin "S", it is possible to find such β's. Unfortunately, it can be easily seen that if $s \geqq 3/2$ the projector $\eta\Pi_+$ (see Proposition III.1.5) is not irreducible, under SU(2) but reducible into a direct sum of an *even* number of SU(2)-irreducible projectors. This fact leads to difficulties [25] in the particle interpretation of the quantized out field.

Proposition III.2.4. *The dipole coupling in the Periau-Duffin-Kemmer equations for spin 1 fields is nice.*

Remarks III.2.5. It has been remarked by Velo and Zwanziger that combining the minimal and dipole interactions the P.D.K. theory does not produce any acausality. The nice character of these interactions confirms this result.

Proof. The P.D.K. field can be looked at as the 10-component field [26] $\psi=(V_\nu, G_{\mu\nu})$ which fulfills the system

$$\partial^\nu G_{\mu\nu}+mV_\mu=0, \quad \partial_\mu V_\nu-\partial_\nu V_\mu+mG_{\mu\nu}=0. \tag{III.2.8}$$

The dipole coupling is defined by:

$$\Phi\psi=(iF_{\nu\varrho}V_\varrho, 0), \tag{III.2.9}$$

where $F^{\nu\varrho}$ is some skew-symmetric tensor with $\mathscr{C}_0^\infty(\mathbb{R}^4)$ coefficients. The choice $B_R=B_L=B$ with

$$B\psi=\left(\frac{i}{2m}F^{\varrho\sigma}\partial_\nu G_{\varrho\sigma}, 0\right) \tag{III.2.10}$$

leads to first order partial differential operators for A_R, A_L.

III.3. The Retarded Kernel

One can extend the result of Capri [6b] to nice interactions:

Proposition III.3.1. *Let Φ be a nice interaction. Then $1-\Phi S_r$ (resp. $1-\Phi S_a$) is an isomorphism of $\mathscr{S}(\mathbb{R}^4)\otimes E_N$, and $\lambda\to(1-\lambda\Phi S_r)^{-1}$ (resp. $\lambda\to(1-\lambda\Phi S_a)^{-1}$) is an entire analytic function, in the simple convergence topology.*

Proof. First of all, we recall that (see Lemma II.2.3 and Appendix) $1-A_R(\lambda)\Delta_{r,a}$, $1-A_L(\lambda)\Delta_{r,a}$ (where $A_{R,L}(\lambda)$ is computed by multiplying Φ by λ) are isomorphisms of $\mathscr{S}=\mathscr{S}(\mathbb{R}^4)\otimes E_N$. On the other hand, by a classical theorem on distributions, $1-\lambda\Phi S_r$ is a linear continuous map from $\mathscr{S}$ to $\mathscr{S}$.

If we prove that $1-\lambda\Phi S_r$ is one-to-one then by the closed graph theorem ([27], Theorem 2.15) $1-\lambda\Phi S_r$ is an isomorphism.

Since Φ is nice one has $\forall f\in\mathscr{S}$:

$$\begin{aligned}(d(-i\partial)+B_L)(1-\Phi S_r)f&=(d(-i\partial)+B_L)(\beta(-i\partial)-m-\Phi)S_rf\\&=(-(\Box+m^2)+A_L)d(-i\partial)\Delta_rf=-(1-A_L\Delta_r)d(-i\partial)f\end{aligned} \tag{III.3.1}$$

by repeated use of the associativity of convolutions [9]. Therefore, since $1-A_L\Delta_r$ is an isomorphism of $\mathscr{S}$, $(1-\Phi S_r)f=0$, $\forall f\in\mathscr{S}$, implies:

$$(d(-i\partial)f=0\ \forall f\in\mathscr{S})\Leftrightarrow\mathscr{F}=0 \tag{III.3.2}$$

because d is a polynomial.

In much the same way, if $h \in \mathcal{S}$

$$
\begin{aligned}
(1-A_R\Delta_r)h &= (\Box + m^2 - A_R)\Delta_r h \\
&= -(i\beta\partial - m - \Phi)(d(-i\partial) + B_R)\Delta_r h \\
&= -(1-\Phi S_r)(-i\beta\partial - m)\ (d(-i\partial) + B_R)\Delta_r h \\
&= -(1-\Phi S_r)[1-(i\beta\partial + m)B_R\Delta_r]h\,.
\end{aligned}
\tag{III.3.3}
$$

Therefore if $g \in \mathcal{S}$ the function:

$$f = -[1-(i\beta\partial + m)B_R\Delta_r](1-A_R\Delta_r)^{-1}g \tag{III.3.4}$$

is a solution of

$$(1-\Phi S_r)f = g\,. \tag{III.3.5}$$

Remark that here, the existence and associativity of the convolution is due to the fact that both S_r and Δ_r in (III.3.3) have their supports in $\overline{V^+}$.

Finally the analyticity in λ of $(1-A_R(\lambda)\Delta_r)^{-1}$, leads to the analyticity of $(1-\lambda\Phi S_r)^{-1}$.

Corollary III.3.2. *The kernel* $J_r = (1-\Phi S_r)^{-1}\Phi$ *(resp.* $J_a = (1-\Phi S_a)^{-1}\Phi$*) is a continuous map from* $\mathcal{O}_M = \mathcal{O}_M(\mathbb{R}^4) \otimes E_N$ *to* $\mathcal{S}$*. It is therefore a regular kernel (see* [1], *Definition II.2.1).*

Corollary III.3.3. *If* Φ *is a nice interaction,* $1-S_r\Phi$ *(resp.* $1-S_a\Phi$*) is an isomorphism of* $\mathcal{O}_M$*, and* $\lambda \to (1-\lambda S_r\Phi)^{-1}$ *is an entire analytic function in the simple convergence topology.*

Proof. Indeed

$$(1-S_r\Phi)^{-1} = 1 + S_r J_r\,. \tag{III.3.6}$$

III.4. N-P *Property of the Schwinger Kernel*

The main theorem is the following:

Theorem III.4.1. *Let* Φ *be a hermitian nice interaction. Then for small enough real* λ *the kernel*

$$K_{\mathrm{ex}}(\lambda) = (1-\lambda\Phi S_{\mathrm{ex}})^{-1}\Phi \tag{III.4.1}$$

is N-P *if* $S_{\mathrm{ex}} \in \{S_r, S_a, S_F, S_{\bar{F}}\}$.

Moreover in the Dirac case, $\lambda \to K_{\mathrm{ex}}(\lambda)$ *is analytic for* $|\lambda| < \varepsilon$ *in the* N-P *norm topology.*

To prove this theorem, we need some preliminaries.

Lemma III.4.2. *Let* Φ *be a nice interaction, and* $J_r(\lambda)$ *be the kernel* $(1-\lambda\Phi S_r)^{-1}\Phi$*. Then there exists* $P \geqq 0$ *and* $\forall \alpha \geqq 0$, $\exists \varepsilon > 0$ *such that* $\lambda \to \tilde{J}_r(\lambda)$ *(where* $\tilde{}$ *denotes the Fourier transform) is analytic, in the topology of* $\mathcal{N}_{E_N}(m, \alpha, P)$ $\forall m \geqq 0$ *for* $|\lambda| \leqq \varepsilon(\alpha)$.

Proof. With the help of Eq. (III.3.4) one finds:

$$J_r(\lambda) = -\lambda(1-(i\beta\partial + m)B_R(\lambda)\Delta_r)(1-A_R(\lambda)\Delta_r)^{-1}\Phi\,. \tag{III.4.2}$$

By Theorem III.1.1, $(1-A_R(\lambda)\Delta_r)^{-1}\Phi$ is analytic in $|\lambda|<\varepsilon(\alpha)$ for the topology of $\mathcal{N}_{E_N}(m,\alpha,1/4)$ $\forall m\geqq 0$ $\forall\alpha\geqq 0$. Using Lemma A.5.6 in [1], $(i\beta\partial+m)B_R(\lambda)\Delta_r$ maps linearly and continuously $\mathcal{N}_{E_N}(m,\alpha,1/4)$ into $\mathcal{N}_{E_N}(m,\alpha,P)$ for some $P\geqq 0$ which depends on the degree of the differential operator $B_R(\lambda)$. This is true $\forall m\geqq 0$, $\forall\alpha\geqq 0$ because B_R has $\mathscr{C}_0^\infty(\mathbb{R}^4)$ coefficients. The analyticity is also preserved because $B_R(\lambda)$ is at most a polynomial in λ.

Corollary III.4.3. *If in Lemma III.4.2, β_μ are the Dirac matrices, then P can be chosen equal to* 1/4.

Proof. Indeed in this case $B_R(\lambda)=\lambda\Phi$. P can be computed using Lemma A.5.6 of [1] which yields the value 1/4.

In particular in this case if $J_R(\lambda)$ is the operator on $\mathscr{H}_+\oplus\mathscr{H}_-$ associated with $J_r(\lambda)$ (see § II.2), then

$$\lambda\to J_R(\lambda)$$

is analytic if $|\lambda|<\varepsilon$ in the N-P norm topology (see Corollary II.1.3 or Appendix 5 of [1]).

Lemma III.4.4. *Let Φ be a nice interaction. Then the components $J_{r+-}(\lambda)$ and $J_{r-+}(\lambda)$ are Hilbert-Schmidt operators, and analytic in λ for $|\lambda|<\varepsilon(\alpha)$ for α large enough.*

Proof. The proof is the same as in Lemma A.5.5 of [1]. We have only to recall that the inner product in $\mathscr{H}$ depends on the spin through the projector $\Pi_+(p)$ which is, on the mass shell, a polynomical (see § III.1).

Now, we can prove the rest of Theorem III.4.1. Since J_r is a regular kernel (Corollary III.3.2), J_{r++} (resp. J_{r--}) is densely defined on a domain $\mathscr{D}_+$ (resp. $\mathscr{D}_-$) such that $J_{\varepsilon\varepsilon}\mathscr{D}_\varepsilon\subset\mathscr{D}_\varepsilon$, $\varepsilon=\pm 1$ (see [1], Proposition II.2.5). On the other hand, the unitarity relations give us ([1], Table 1 and § II.3):

$$(\mathbb{1}_{++}+iJ_{r++})(\mathbb{1}_{++}-iJ_{r++}^*)=\mathbb{1}_{++}+J_{r+-}J_{r-+}^* \quad (2s\text{ even}), \tag{III.4.3a}$$

$$(\mathbb{1}_{++}+iJ_{r++})(\mathbb{1}_{++}-iJ_{r++}^*)=\mathbb{1}_{++}-J_{r+-}J_{r-+}^* \quad (2s\text{ odd}). \tag{III.4.3b}$$

Therefore, J_{r++} (resp. J_{r--}) can be continued as a bounded operator on $\mathscr{H}_+$ (resp. on $\mathscr{H}_-$). Thus J_r is N-P.

In much the same way J_a is N-P. Since $\lambda\to J_{r+-}(\lambda)$ is analytic in the Hilbert-Schmidt norm, for $|\lambda|<\varepsilon$, $\mathbb{1}_{++}+J_{r+-}(\lambda)J_{a-+}(\lambda)$ (2S even) or $\mathbb{1}_{++}-J_{r+-}(\lambda)J_{a-+}(\lambda)$ (2S odd) is invertible for $|\lambda|<\varepsilon'$, ε' small enough. Therefore, the Feynman analogs $I=K_F$ and $\bar{I}=K_{\bar{F}}$ can be constructed by the equivalence theorem ([1], Theorem II.4.1).

This remark achieves the proof.

Remark III.4.5. If the spin "s" is larger than or equal to one, the argument concerning the analyticity property does not go through because $\mathbb{1}+iJ_R(\lambda)$ could be unbounded when λ is not real:

Lemma II.4.2 only establishes analyticity in the sense of $\mathcal{N}_{E_N}(m,\alpha,P)$. Now III.4.3 only proves that $J_R(\lambda)$ is bounded if λ is real. On the other hand, Condition K2) of Section II.1 only insures boundedness on $\mathscr{H}_+\oplus\mathscr{H}_-$ if $P\leqq 1/4$; but as explained in the proof of Lemma II.4.2, P depends on the degree of the differential operator $B_R(\lambda)$, hence upon spin. It cannot be proved to be smaller than or equal to 1/4 except in the spin 0 and 1/2 cases.

Thus for spins larger than or equal to one, the present method does not allow to derive any analyticity property for the Bogoliubov S-operator.

Conclusion

We can summarize our analysis from the physical point of view:

Spin 0 is nicely under control: in this case the Bogoliubov S-operator is even analytic for weak fields as stated in the Theorem II.3.2.

In our opinion a complete analysis of the spin 1/2 case should easily lead to identical results.

For spin 1, the Bogoliubov S-operator exists for real weak fields, but analyticity is probably lost.

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Appendix

Study of the Retarded Kernel

A.1. The Operator $(1-A\Delta_r)$

Theorem A.1.1. *Let* $(a_\alpha)_{\alpha=0,1,2,3,4}$ *be functions in* $\mathscr{C}_0^\infty(\mathbb{R}^4)\otimes\mathscr{L}(E_N)$ *where* E_N *is some finite dimensional space. Then, the operator* $(1-A\Delta_r)$ *with*

$$A(x,\partial)=\sum_{\alpha=0}^{3} a_\alpha(x)\frac{\partial}{\partial x^\alpha}+a_4(x) \tag{A.1.1}$$

is an automorphism of $\mathscr{S}=\mathscr{S}(\mathbb{R}^4)\otimes E_N$, *and* $\lambda\to(1-\lambda A\Delta_r)^{-1}$ *is an entire analytic function in the simple convergence topology.*

In order to prove this theorem one introduces the following family of seminorms: if $f\in\mathscr{S}$ then

$$\|f\|^2_{m,t_0,t_1}=\sum_{|\mu|\leqq m}\int_{t_0}^{t_1} ds\int_{\mathbb{R}^3} d^3\boldsymbol{x}\,\|\partial^\mu f(\boldsymbol{x},s)\|^2_{E_N}=\sum_{|\mu|\leqq m}\|\partial^\mu f\|^2_{0,t_0,t_1}, \tag{A.1.2a}$$

$$p_{m,t_0,t_1}(A)=\sup_\alpha\cdot\sup_{\substack{s\in[t_0,t_1]\\ \boldsymbol{x}\in\mathbb{R}^3}}\sup_{|\mu|\leqq m}\|\partial^\mu a_\alpha(s,\boldsymbol{x})\|_{E_N}. \tag{A.1.2b}$$

Here

$$\mu=(\mu_0,\dots,\mu_s)\in\mathbb{N}^4,\qquad \partial^\mu=\prod_{\alpha=0}^{3}(\partial_\alpha)^{\mu_\alpha},\qquad |\mu|=\sum_{\alpha=0}^{3}\mu_\alpha. \tag{A.1.3}$$

Let $\mathscr{H}_{m,t_0,t_1}$ be the Hilbert space obtained by completion of $\mathscr{C}_0^\infty(]t_0,t_1[\times\mathbb{R}^3)\otimes E_N$ with respect to the norm $\|\cdot\|_{m,t_0,t_1}$.

Lemma A.1.2. *Let* $(f_n)_{n\geqq 0}$ *be a converging sequence in the space*

$$\bigcup_{t_0,t_1}\bigcap_{m\geqq 0}\mathscr{H}_{m,t_0,t_1}$$

with the topology defined by the family $\|\cdot\|_{m,t_0,t_1}$ *(inductive limit topology with respect to* t_0, t_1*). If there exist* $N\in\mathbb{N}$ *and a compact subset* K *of* R^4 *such that*

$$\operatorname{supp} f_n \subset K \quad \forall n\in\mathbb{N}, \quad n\geqq N \tag{A.1.4}$$

then f_n *converges in* $\mathscr{C}_0^\infty(\mathbb{R}^4)\otimes E_N$.

This lemma follows from the definition of convergence in $\mathscr{C}_0^\infty$ ([3, 9]).

Lemma A.1.3. *The following estimate holds:*

$$\|Af\|_{m,t_0,t_1} \leqq C p_{m,t_0,t_1}(A)\|f\|_{m+1,t_0,t_1}, \tag{A.1.5}$$

where the positive constant C *depends only on* m.

Lemma A.1.4. *(A priori bound [28, 29]). Let* f *be in* $\mathscr{H}_{m,t_0,t_1}$. *Then*

$$\|\Delta_r f\|_{m+1,t_0,t_1} \leqq \int_{t_0}^{t_1} ds \|f\|_{m,t_0,s}, \tag{A.1.6a}$$

$$\|\Delta_a f\|_{m+1,t_0,t_1} \leqq \int_{t_0}^{t_1} ds \|f\|_{m,s,t_1}. \tag{A.1.6b}$$

Proof. These inequalities are well-known if $f\in\mathscr{C}_0^\infty$ with support contained in $]t_0, t_1[\times\mathbb{R}^3$ [28]. By completion, they hold for any f in $\mathscr{H}_{m,t_0,t_1}$.

Lemma A.1.5. *Let* $t_0, t_1\in\mathbb{R}$ *be such that*

$$\bigcup_\alpha \operatorname{supp}(a_\alpha) \subset]t_0, t_1[\times\mathbb{R}^3. \tag{A.1.7}$$

Then, $\forall f\in\mathscr{H}_{m,s_0,s_1}$, $s_0, s_1\in\mathbb{R}$

$$\|(A\Delta_r)^n f\|_{m,s_0,s_1} \leqq K^n p_{m,s_0,s_1}(A)^n \frac{(t_1-t_0)^n}{n!}\|f\|_{m,s_0,s_1}. \tag{A.1.8}$$

The last inequality is obtained by recursion on (A.1.5) and (A.1.6). Theorem A.1.1 is then proved since

$$a_\alpha\in\mathscr{C}_0^\infty(\mathbb{R}^4)\otimes\mathscr{L}(E_N) \Rightarrow \operatorname{supp}(A\Delta_r)^n f \subset \bigcup_\alpha \operatorname{supp}(a_\alpha) \quad \forall n\geqq 1.$$

A.2. Properties of the Classical S-Operator (Spin Zero Case)

We are now in the case $E=\mathbb{C}$, and $\mathscr{H}_\pm$ coincide with the Hilbert space

$$\mathscr{L}^2(1/2(m^2+p^2)^{-1/2}d^3p, \mathbb{R}^3).$$

Then we want to show the following theorem.

Theorem A.2.1. *Using the notations introduced in Theorem A.1.1, the kernel* $(\Delta_+-\Delta_-)(1-\lambda A\Delta_r)^{-1}A$, $(\lambda\in\mathbb{C})$, *defines an analytic family of bounded operators on* $\mathscr{H}_+\oplus\mathscr{H}_-$.

Remark A.2.2. This bounded operator is nothing but J_R defined in [1] § II (see [1], Proposition II.5.1).

Let us introduce the following notations: For a function f on $\mathbb{R}^4$ we define $\tilde{f}$ and $\mathring{f}$ as follows:

$$f(t,\mathbf{x}) = \int_{\mathbb{R}^4} e^{i(p^0 t - \mathbf{p}\mathbf{x})} \tilde{f}(p^0,\mathbf{p}) d^0 d^3\mathbf{p} = \int_{\mathbb{R}^3} e^{-i\mathbf{p}\cdot\mathbf{x}} \mathring{f}(t,\mathbf{p}) d^3\mathbf{p} \tag{A.2.1}$$

if $\varphi \in \mathscr{H}_\pm$

$$\hat{\varphi}(t,\mathbf{x}) = \int_{\mathbb{R}^3} e^{i(\omega_p t - \mathbf{p}\cdot\mathbf{x})} \varphi(\mathbf{p}) d^3\mathbf{p}/2\omega_p ; \qquad \omega_p = (m^2 + \mathbf{p}^2)^{1/2} . \tag{A.2.2}$$

The restrictions of $\tilde{f}$ to the mass shell will be:

$$f^\pm(\mathbf{p}) = \tilde{f}(\pm\omega_p, \mathbf{p}) . \tag{A.2.3}$$

Finally, let us define:

$$q_{m,t_0,t_1}(\{a_\alpha\}) = \sup_\alpha \sup_{|\mu| \leq m} \sup_{s \in]t_0,t_1[} \int_{\mathbb{R}^3} |\overset{\circ}{\overline{\partial^\mu a_\alpha}}(s,\mathbf{p})| d^3\mathbf{p} . \tag{A.2.4}$$

Let K be the operator defined by

$$\overset{\circ}{\overline{Kf}}(t,\mathbf{p}) = (m^2 + \mathbf{p}^2)^{1/4} \mathring{f}(t,\mathbf{p}) \tag{A.2.5}$$

and

$$A' = KAK^{-1} . \tag{A.2.6}$$

Lemma A.2.3. *One has:*

i) $\Delta_\pm f = -i\pi \widehat{f^\pm}(s,\mathbf{x})$.

ii) $\|K\hat{\varphi}\|^2_{0,t_0,t_1} = (2\pi)^3 (t_1 - t_0) \|\varphi\|^2_{\mathscr{H}_\pm}$. (A.2.7)

iii) $\|K\Delta_\pm f\|^2_{0,t_0,t_1} = (2\pi)^3 \pi^2 (t_1 - t_0) \| f^\pm \|^2_{\mathscr{H}_\pm}$.

This is merely the result of a tedious calculation.

Lemma A.2.4. *If* $f^\pm, (\partial f/\partial p_\alpha)^\pm, \in \mathscr{H}_\pm$ *for* $\alpha = 0, 1, 2, 3$ *then*

$$\lim_{\substack{t_1 \to +\infty \\ t_0 \to -\infty}} (K\Delta_+ f | K\Delta_- f)_{0,t_0,t_1} / (t_1 - t_0) = 0 . \tag{A.2.8}$$

Proof. If $(\cdot|\cdot)_{0,t_0,t_1}$ is the scalar product in $\mathscr{H}_{0,t_0,t_1}$, one has:

$$(K\Delta_+ f | K\Delta_- f)_{0,t_1,t_1} = C \int_{t_0}^{t_1} ds \int_{\mathbb{R}^3} e^{is\omega_p} (d^3\mathbf{p}/2\omega_p) \overline{\tilde{f}(\omega_p,\mathbf{p})} \tilde{f}(-\omega_p,\mathbf{p}) = \int_{t_0}^{t_1} F(s) ds.$$

By Lebesgue's lemma ([27], Theorem 7.5) $\lim_{s\to\infty} s \cdot F(s) = 0$, and the result follows.

Lemma A.2.5. *The following estimate holds:*

$$\|K^{-1}AKf\|_{0,t_0,t_1} \leqq Cq_{0,t_0,t_1}(\{Ka_\alpha\}_\alpha)\|f\|_{1,t_0,t_1}. \tag{A.2.9}$$

In order to prove this lemma we have used the inequality

$$(1+p^2)/(1+q^2) \leqq 2(1+(p-q)^2) \tag{A.2.10}$$

and the properties of the Fourier transform (notice that the norm $\|f\|_{m,t_0,t_1}$ is easily computable with the Parseval equality in terms of $\mathring{f}$).

Lemma A.2.6. *If* $\bigcup_\alpha \operatorname{supp}(a_\alpha) \subset]t_0, t_1[\times \mathbb{R}^2$, *then:*

$$\|[\Delta_r(KAK^{-1})]^n f\|_{0,s_0,s_1} \leqq H(A)^n((t_1-t_0)^n/n!)\|f\|_{0,s_0,s_1}. \tag{A.2.11}$$

Proof. Let A^+ be the adjoint of the operator A in $\mathscr{H}_{0,t_0,t_1}$, A^+ is associated with a first order differential operator. Then:

$$\begin{aligned}(h|(\Delta_r KAK^{-1})^n f)_{0,s_0,s_1} &= ((K^{-1}A^+K\Delta_r)^n h, f)_{0,s_0,s_1} \\ &\leqq CH(A)^n((t_1-t_0)^n/n!)\|h\|_{0,s_0,s_1}\|f\|_{0,s_0,s_1}.\end{aligned} \tag{A.2.12}$$

Proof of the Theorem A.2.1. One can remark that, if $J_r = (1-\lambda A\Delta_r)^{-1}A$

$$(\Delta_+ - \Delta_-)J_r f = -i\pi\{(\widehat{J_r f})^+ - (\widehat{J_r f})^-\} \tag{A.2.13}$$

with the notations introduced in the beginning.

Let now $f \in \mathscr{C}_0^\infty$. Since $a_\alpha \in \mathscr{C}_0^\infty$, by Lemma A.2.4:

$$\begin{aligned}&\|[(A\Delta_r)^n Af]^+\|^2_{\mathscr{H}_+} + \|[(A\Delta_r)^n Af]^-\|^2_{\mathscr{H}_-} \\ &\quad = C \lim_{\substack{s_1\to+\infty \\ s_0\to-\infty}} \|K(\Delta_+-\Delta_-)(A\Delta_r)^n Af\|^2_{0,s_0,s_1}/(s_1-s_0).\end{aligned} \tag{A.2.14}$$

Since $\Delta_+ - \Delta_- = \Delta_r - \Delta_a$, we have also:

$$\begin{aligned}&\|K(\Delta_+-\Delta_-)(A\Delta_r)^n Af\|^2_{0,s_0,s_1} \\ &\quad \leqq 2\{\|(\Delta_r A')^{n+1}Kf\|^2_{0,s_0,s_1} + \|\Delta_a A'(\Delta_r A')^n Kf\|^2_{0,s_0,s_1}\} \\ &\quad \leqq (H(A)^n((t_1-t_0)^n/n!)\|Kf\|_{0,s_0,s_1})^2.\end{aligned} \tag{A.2.15}$$

Now let $\varphi_\pm$ be in $\mathscr{S}(\mathbb{R}^3)$. Then by Lemma A.2.3

$$\begin{aligned}&\|((A\Delta_r)^n A\hat{\varphi}_\pm)^+\|^2_{\mathscr{H}_+} + \|((A\Delta_r)^n A\hat{\varphi}_\pm)^-\|^2_{\mathscr{H}_-} \\ &\quad \leqq C \lim_{\substack{s_1\to+\infty \\ s_0\to-\infty}} (H(A)^n(t_1-t_0)^n/n!)^2 \|K\hat{\varphi}_\pm\|^2_{0,s_0,s_1}/(s_a-s_0) \\ &\quad \leqq C(H'(A)^r((t_1-t_0)^n/n!)\|\varphi_\pm\|^2_{\mathscr{H}_\pm}).\end{aligned} \tag{A.2.16}$$

Therefore, the operators from $\mathscr{H}_\varepsilon$ to $\mathscr{H}_{\varepsilon'}$ $(\varepsilon, \varepsilon' \in \{+,-\})$

$$\varphi_\varepsilon \to ((A\Delta_r)^n A\hat{\varphi}_\varepsilon)^{\varepsilon'} \tag{A.2.17}$$

are bounded on $\mathscr{S}(\mathbb{R}^3)$ which is dense in $\mathscr{H}_\varepsilon$. They can therefore be continued as a bounded operator from $\mathscr{H}_\varepsilon \to \mathscr{H}_{\varepsilon'}$, and their norms are majorized by $C^n/n!$ for some positive C. Therefore, the perturbation expansion for $(\Delta_+ - \Delta_-)\cdot(1-\lambda A\Delta_r)^{-1}A$ converges in the norm topology in $\mathscr{L}(\mathscr{H}_+ \oplus \mathscr{H}_-)$.

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