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Unbounded Derivations and Invariant Trace States

Ola Bratteli* **

Centre de Physique Théorique, CNRS Marseille, F-13274 Marseille Cedex 2, France

Derek W. Robinson

Université d'Aix-Marseille II, U.E.R. de Luminy and Centre de Physique Théorique, CNRS Marseille, F-13274 Marseille Cedex 2, France

Abstract. Let $\mathfrak{M}$ be a von Neumann algebra with cyclic trace vector Ω. Let $\delta(A)=i[H,A]$ be a spatial derivation of $\mathfrak{M}$ implemented by an operator H such that $H\Omega=0$ and H is essentially self-adjoint on $D(\delta)\Omega$.
It follows that:

$$e^{itH}\mathfrak{M}e^{-itH}=\mathfrak{M}, \qquad t\in\mathbb{R}.$$

1. Introduction

In a previous paper [1] we discussed the general theory of unbounded derivations of a von Neumann algebra $\mathfrak{M}$ on a Hilbert space $\mathscr{H}$ and, in particular, introduced the notion of a spatial derivation. This latter form of derivation is defined in terms of a symmetric operator H, on $\mathscr{H}$, and a weakly dense *-subalgebra $D(\delta)$ of $\mathfrak{M}$, which leaves the domain $D(H)$ of H invariant. The derivation δ is defined to be a mapping

$$A\in D(\delta)\rightarrow\delta(A)\in\mathfrak{M}$$

with the property that

$$\delta(A)\psi=i[H,A]\psi, \qquad \psi\in D(H).$$

It is of particular interest to study the case that H is self-adjoint and has an eigenvector Ω such that $D(\delta)\Omega$ is a core of H. In [1] it was conjectured that if Ω is also cyclic and separating for $\mathfrak{M}$ then

$$e^{itH}\mathfrak{M}e^{-itH}=\mathfrak{M}, \qquad t\in\mathbb{R}.$$

This conjecture was verified in various special cases. If $\mathfrak{M}$ is abelian then it is essentially a theorem of Gallavotti and Pulvirenti [2]. In this note we extend the abelian result by verifying the conjecture whenever Ω is a trace vector.

2. Main Theorem

Theorem 1. *Let $\mathfrak{M}$ be a von Neumann algebra on a Hilbert space $\mathscr{H}$ and let Ω be a cyclic normalized vector defining a trace on $\mathfrak{M}$, i.e.*

$$(\Omega, AB\Omega)=(\Omega, BA\Omega), \qquad A,B\in\mathfrak{M}.$$

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** Present address: ZIF der Universität Bielefeld, D-4800 Bielefeld, Federal Republic of Germany

Let δ be a spatial derivation of $\mathfrak{M}$ implemented by a self-adjoint operator H such that $H\Omega=0$.

If H is essentially self-adjoint on $D(\delta)\Omega$ then

$$e^{itH}\mathfrak{M}e^{-itH}=\mathfrak{M}, \qquad t\in\mathbb{R}.$$

The proof of the theorem will be divided into three Lemmas.

Lemma 1. *Let $\mathfrak{M}$ be a von Neumann algebra with a normalized cyclic trace vector Ω. Assume that there exists a sequence $B_n=B_n^*\in\mathfrak{M}$ such that $B_n\Omega\to\psi$.*

It follows that there exists a self-adjoint operator B affiliated with $\mathfrak{M}$ such that $B_n\to B$ in the strong resolvent sense. In particular if $\chi\in\mathcal{S}(\mathbb{R})$ then $\chi(B_n)$ converges strongly to $\chi(B)$.

Proof. For each $\lambda\in\mathbb{C}\backslash\mathbb{R}$ one has

$$\begin{aligned}
&\|((\lambda-B_n)^{-1}-(\lambda-B_m)^{-1})\Omega\| \\
&\quad=\|(\lambda-B_n)^{-1}(B_m-B_n)(\lambda-B_m)^{-1}\Omega\| \\
&\quad\leqq|Im\lambda|^{-1}\,\|(B_m-B_n)(\lambda-B_m)^{-1}\Omega\| \\
&\quad=|Im\lambda|^{-1}\,\|(\bar{\lambda}-B_m)^{-1}(B_m-B_n)\Omega\| \\
&\quad\leqq|Im\lambda|^{-2}\,\|(B_m-B_n)\Omega\|\,,
\end{aligned}$$

where we have twice used

$$\|(\lambda-B_n)^{-1}\|\leqq|Im\lambda|^{-1}$$

and, at the third stage, used the trace property. This demonstrates that the resolvents $(\lambda-B_n)^{-1}$ converge strongly on Ω. But the resolvents are uniformly bounded in n and Ω is cyclic for the commutant $\mathfrak{M}'$ of $\mathfrak{M}$. Hence the resolvents converge strongly to some element R_λ of $\mathfrak{M}$. We next prove that R_λ is the resolvent of a self-adjoint operator B.

Define ψ_n by

$$\psi_n=(\lambda-B_n)\Omega$$

and hence

$$\lim_{n\to\infty}\psi_n=\lambda\Omega-\psi\,.$$

Now

$$\begin{aligned}
&\|\Omega-R_\lambda(\lambda\Omega-\psi)\| \\
&\quad=\|(\lambda-B_n)^{-1}\psi_n-R_\lambda(\lambda\Omega-\psi)\| \\
&\quad\leqq\|(\lambda-B_n)^{-1}(\psi_n-(\lambda\Omega-\psi))\| \\
&\qquad+\|((\lambda-B_n)^{-1}-R_\lambda)(\lambda\Omega-\psi)\| \\
&\quad\leqq|Im\lambda|^{-1}\,\|\psi_n-(\lambda\Omega-\psi)\| \\
&\qquad+\|((\lambda-B_n)^{-1}-R_\lambda)(\lambda\Omega-\psi)\|\xrightarrow[n\to\infty]{}0\,.
\end{aligned}$$

Hence one concludes that

$$R_\lambda(\lambda\Omega-\psi)=\Omega\,.$$

Thus for $C \in \mathfrak{M}'$

$$R_\lambda C(\lambda\Omega - \psi) = CR_\lambda(\lambda\Omega - \psi) = C\Omega .$$

But as Ω is cyclic for $\mathfrak{M}'$ this demonstrates that the range of R_λ is dense. By the Kato-Trotter theorem [3] there exists a unique self-adjoint operator B such that

$$R_\lambda = (\lambda - B)^{-1} .$$

Moreover

$$e^{itB_n}\psi \to e^{itB}\psi$$

for all ψ, uniformly for t in compacts.

Since

$$(\lambda - B)^{-1}(\lambda\Omega - \psi) = \Omega$$

one immediately concludes that

$$B\Omega = \psi .$$

Finally for $\chi \in \mathcal{S}(\mathbb{R})$

$$\chi(B_n)\psi = \int dp\, e^{ipB_n}\psi\hat{\chi}(p)$$

and

$$\chi(B)\psi = \int dp\, e^{ipB}\psi\hat{\chi}(p) .$$

Hence $\chi(B_n)$ converges strongly to $\chi(B)$.

Lemma 2. *Adopt the assumptions of Theorem 1. If* $B = B^* \in D(\delta)$ *and*

$$A = (1 + \alpha\delta)(B)$$

with $\alpha \in \mathbb{R}\backslash\{0\}$ *then*

$$(\Omega, \chi(B)B\Omega) = (\Omega, \chi(B)A\Omega)$$

for all $\chi \in \mathcal{S}(\mathbb{R})$.

Proof. As $A - B = \alpha\delta(B)$ the statement of the Lemma is equivalent to

$$(\Omega, \chi(B)\delta(B)\Omega) = 0$$

for all $\chi \in \mathcal{S}(\mathbb{R})$.

Let f be a function such that $f' = \chi$. The Fourier transforms then satisfy

$$ip\hat{f}(p) = \hat{f'}(p) = \hat{\chi}(p) .$$

Thus by Lemma 2 of [4] one has $f(B) \in D(\delta)$ and

$$\delta(f(B)) = i \int_{-\infty}^{\infty} dp\, p\hat{f}(p) \int_0^1 dr\, e^{iprB}\delta(B)e^{ip(1-r)B} .$$

The trace property of Ω then yields

$$(\Omega, \delta(f(B))\Omega) = i\left(\Omega, \int_{-\infty}^{\infty} dp p \hat{f}(p) e^{ipB} \delta(B)\Omega\right)$$
$$= (\Omega, \chi(B)\delta(B)\Omega).$$

Hence as $H\Omega = 0$ one has

$$(\Omega, \chi(B)\delta(B)\Omega) = 0.$$

Lemma 3. *Adopt the assumptions of Theorem 1. If* $A = A^* \in \mathfrak{M}$ *and* $\alpha \in \mathbb{R} \backslash \{0\}$ *then there exists a self-adjoint* B *affiliated with* $\mathfrak{M}$ *such that*

$$B\Omega = (1 + i\alpha H)^{-1} A\Omega$$

and, furthermore,

$$(\Omega, \chi(B)B\Omega) = (\Omega, \chi(B)A\Omega)$$

for all $\chi \in \mathscr{S}(\mathbb{R})$.

Proof. As $D(\delta)\Omega$ is a core for H there exists a sequence $A_n = (1 + \alpha\delta)(B_n)$ such that $A_n\Omega$ converges to $A\Omega$. But

$$A\Omega = A^*\Omega$$
$$= \lim_{n\to\infty} A_n^*\Omega$$
$$= \lim_{n\to\infty} (1 + \alpha\delta)(B_n^*)\Omega,$$

where the second step uses the trace property of Ω. Replacing B_n by $(B_n + B_n^*)/2$ we may assume the B_n self-adjoint.

Because

$$(1 + \alpha\delta)(B_n)\Omega = (1 + i\alpha H)B_n\Omega$$

and the resolvent of H is bounded we conclude that

$$B_n\Omega = (1 + i\alpha H)^{-1} A_n\Omega$$

converges to $(1 + i\alpha H)^{-1} A\Omega$. The existence of B now follows from Lemma 1. Further B_n converges to B in the strong resolvent sense.

Next from Lemma 2

$$(\Omega, \chi(B_n)B_n\Omega) = (\Omega, \chi(B_n)A_n\Omega)$$

and the desired result follows by limiting.

Proof of Theorem 1. From Theorem 6 of [1] it suffices to show that

$$(1 + i\alpha H)^{-1}\mathfrak{M}_+\Omega \subseteq \overline{\mathfrak{M}_+\Omega}, \qquad \alpha \in \mathbb{R} \backslash \{0\}.$$

In order to show this take $A \geqq 0$ in Lemma 3 and χ also positive. One then has

$$(\Omega, \chi(B)B\Omega) = (\Omega, \chi(B)A\Omega) \geqq 0$$

by the trace property. Since Ω is separating for $\mathfrak{M}$ it follows that $\chi(B)B$ can never be negative for χ positive. Hence $B \geqq 0$ and the proof is complete.

Remark. As Ω is a trace vector for $\mathfrak{M}$ it follows that $\mathfrak{M}$ is a finite von Neumann algebra. Let $\mathfrak{N}$ be the set of operators affiliated with $\mathfrak{M}$ and having Ω in their domain. It follows from [5] that $\mathfrak{N}$ is a self-adjoint space and $\mathfrak{N}\mathfrak{M}\subseteqq\mathfrak{N}$. This last statement follows because $\mathfrak{M}\mathfrak{N}\subseteqq\mathfrak{N}$ and $\mathfrak{N}\mathfrak{M}=(\mathfrak{M}\mathfrak{N})^*$. If the definition of a spatial derivation is generalized to allow a mapping

$$A\in D(\delta)\subseteqq\mathfrak{M}\rightarrow\delta(A)\in\mathfrak{N}$$

then the result of Theorem 1 is still valid. The proof of this more general result needs a slight extension of Lemma 5 of [1] to establish that the automorphism property is equivalent to the positivity preserving property

$$(1+i\alpha H)^{-1}\mathfrak{M}_+\Omega\subseteqq\overline{\mathfrak{M}_+\Omega}\,,\qquad \alpha\in\mathbb{R}\backslash\{0\}$$

and in the proof of Lemma 2 above $\delta(f(B))$ must be calculated directly in the vector state given by Ω.

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