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Classical KMS Condition and Tomita-Takesaki Theory

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Abstract. Relationships between the classical KMS condition and the time evolution for classical systems are discussed.

1. Introduction

The well known theorem of Tomita-Takesaki [1] on automorphisms of Von Neumann algebras becomes trivial in the case of a commutative algebra.

On the other hand the physical ideas related to the Tomita's theorem suggest that there should be some non trivial version of the theorem in the commutative case.

In this paper we analyze a possible "commutative" version of the Tomita-Takesaki's theorem.

2. Non Commutative ("Quantum") Theory

It is believed that states in quantum statistical mechanics can be described by some positive linear functional on an involutive normed algebra $\mathfrak{A}$, with identity.

It is somehow clear that there is not a unique algebra which is useful for the description of a given statistical mechanical system. Usually the algebra $\mathfrak{A}$ is an union of an increasing family of concrete (local) C^*-algebras of bounded operators on Hilbert spaces.

We shall call an algebra $\mathfrak{A}$ of the type just described "algebra of strictly local quantum observables" [2]. We shall drop in what follows, the words "strictly local" when referring to this concept.

A "state" ϱ is a positive normalized linear functional on $\mathfrak{A}$ [2].

Given a state ϱ on $\mathfrak{A}$ we can find a Hilbert space $\mathscr{H}_\varrho$, a representation π of $\mathfrak{A}$ as algebra of bounded operators in $\mathscr{H}_\varrho$, and a cyclic vector $\xi \in \mathscr{H}_\varrho$, such that $\varrho(A) = (\xi, \pi(A)\xi)\ \forall A \in \mathfrak{A}$.

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A state ϱ over an algebra $\mathfrak{A}$ of quantum observables will be called "admissible" if

1) $\varrho(B^*B)=0 \Leftrightarrow \varrho(BB^*)=0$,
2) the quadratic form Ψ defined on $\pi(\mathfrak{A})\xi$ by

$$\Psi(\pi(A)\xi, \pi(B)\xi)=\varrho(BA^*)=(\pi(B^*)\xi, \pi(A)^*\xi), \qquad A, B\in\mathfrak{A}$$

is closable and hence, its closure defines an (unbounded) operator $\sqrt{\Delta}:\mathscr{H}_\varrho \to \mathscr{H}_\varrho$ such that:

$$\Psi(\pi(A)\xi, \pi(B)\xi)=(\sqrt{\Delta}\,\pi(A)\xi, \sqrt{\Delta}\,\pi(B)\xi)$$

which is essentially selfadjoint on $\pi(\mathfrak{A})\xi$.

The first admissibility condition implies the separating character of ξ for $\pi(\mathfrak{A})$ [i.e. if $\pi(A)\xi=0$, $A\in\mathfrak{A}$, then $\pi(A)=0$].

The following theorems can be shown to be equivalent.

Theorem 1. *Let $\mathfrak{A}$ be an algebra of quantum observables and let ϱ be an admissible state on $\mathfrak{A}$. Then the operator Δ is strictly positive and Δ^{it} is a group of unitary operators on $\mathscr{H}_\varrho$ such that:*

$$\Delta^{it}\mathscr{R}(\mathfrak{A})\Delta^{-it}\subset\mathscr{R}(\mathfrak{A}) \qquad \forall t\in(-\infty, +\infty),$$

where $\mathscr{R}(\mathfrak{A})$ is the Von Neumann algebra generated by $\pi(\mathfrak{A})$.

Theorem 2. (Tomita-Takesaki's theorem). *If $\mathscr{R}$ is a Von Neumann algebra and ξ is a cyclic and separating vector for $\mathscr{R}$, then there is a unique positive operator Δ such that $\mathscr{D}_{\sqrt{\Delta}}\supset\mathscr{R}\xi$ and:*

i) $(B^*\xi, A^*\xi)=(\sqrt{\Delta}A\xi, \sqrt{\Delta}B\xi)$ $\forall A, B\in\mathscr{R}$ (KMS condition),
ii) $\Delta^{it}\mathscr{R}\Delta^{-it}\subset\mathscr{R}$ $\forall t\in(-\infty, +\infty)$.

The reason why we state Tomita-Takesaki theorem in the non conventional form of Theorem 1 is twofold. Its formulation has a content which, in some way, is physically clearer than that of Theorem 2: the separability condition for $\mathscr{R}(\mathfrak{A})$ is replaced by the separability for $\pi(\mathfrak{A})$ (which seems easier to check in the applications, even though there are few cases [3], in which it can be really checked directly) and by the essential selfadjointness of $\sqrt{\Delta}$ over $\pi(\mathfrak{A})\xi$ which seems to be the really hard question whose understanding is probably equivalent to the understanding of the time evolution of the quantum system [3]. This first reason seems not objective enough and the above argument should receive more support from the fact that Theorem 1 has a word by word non trivial analogue in the case the algebra $\mathfrak{A}$ is an algebra of functions over a phase space $\mathscr{K}$ and ϱ is a probability measure on $\mathscr{K}$ ("state" on $\mathfrak{A}$) such that $\mathfrak{A}\subset L_2(\mathscr{K}, \varrho)$.

Remark. The equivalence of Theorems 1 and 2 is well known: in fact Theorem 1 follows from Theorem 2 via the remark that, by the Kaplanski density theorem (for instance), the closability of Ψ implies that ξ is separating for $\mathscr{R}(\mathfrak{A})$ vice versa an examination of the first steps of the proof of Theorem 2 easily shows that it would be a consequence of Theorem 1 [1].

We thank O. Bratteli and D. W. Robinson for this remark which replaces an earlier more complicate argument.

3. Commutative ("Classical") Theory

It is believed that the states of classical statistical mechanics are described by some positive linear functional on an algebra $\mathfrak{A}$, with identity, of continuous functions on a topological space $\mathscr{K}$, "the phase space" [2].

There is no unique algebra $\mathfrak{A}$ which is useful for the above mentioned description. Usually $\mathfrak{A}$ is a selfadjoint algebra of bounded functions (i.e. if $f \in \mathfrak{A}$ also $\bar{f} \in \mathfrak{A}$). A property on $\mathfrak{A}$ is assumed. Let $C(\mathscr{K})$ be the space of continuous, possibly unbounded, functions on $\mathscr{K}$; then a bilinear mapping $\{\cdot,\cdot\}: \mathfrak{A} \times \mathfrak{A} \to C(\mathscr{K})$ ("Poisson bracket" [4]) is defined, with the properties:

i) $\{f,g\} = -\{g,f\} \quad \forall f, g \in \mathfrak{A}$.

ii) $\overline{\{f,g\}} = \{\bar{f},\bar{g}\} \quad \forall f, g \in \mathfrak{A}$.

iii) $\{fg,h\} = \{g,h\}f + \{f,h\}g \quad \forall f, g, h \in \mathfrak{A}$.

iv) There exists an automorphism I for the algebraic structure of $\mathfrak{A}$ such that

$$I^2 = 1, \quad I\bar{f} = \overline{If}, \quad I\{f,g\} = -\{If, Ig\}$$

("time reversal").

An algebra $\mathfrak{A}$ of functions with the above structure and properties will be called an "algebra of (strictly local) classical observables".

A state ϱ is a probability measure on $\mathscr{K}$ such that $\mathfrak{A} \subset L_2(\mathscr{K}, \varrho)$ and furthermore $\mathfrak{A}$ is dense in $L_2(\mathscr{K}, \varrho)$.

We shall say that ϱ is "admissible" if $(\mathscr{K}, \varrho)$ is a Lebesgue space and:

α) $\{g,f\} \in L_1(\mathscr{K}, \varrho) \quad \forall g, f \in \mathfrak{A}$,

β) the quadratic form $\varrho(\{\bar{f}, g\})$ defines an antisymmetric operator $\mathscr{L}$ on $\mathfrak{A}$ such that:

$$\varrho(\{\bar{f}, g\}) = (\mathscr{L}f, g) \quad \text{(KMS condition [4])}.$$

[Here $(\cdot,\cdot)$ denotes the scalar product in $L_2(\mathscr{K}, \varrho)$ and $\varrho(f) = \int f d\varrho$] and $i\mathscr{L}$ is essentially selfadjoint on $\mathfrak{A}$.

γ) I [see iv)] extends to a unitary operator on L_2.

The following theorem holds:

Theorem 3. *Let $\mathfrak{A}$ be an algebra of classical observables and let ϱ be an admissible state on $\mathfrak{A}$, then there exists a family of measure preserving automorphisms $(T_t, t \in (-\infty, \infty))$ of $(\mathscr{K}, \varrho)$ such that $(e^{\mathscr{L}t} f)(x) = f(T_t x)$, $x \in \mathscr{K}$ or, equivalently, $e^{\mathscr{L}t}$ maps $L_\infty(\mathscr{K}, \varrho)$ onto itself in a multiplicative way $e^{\mathscr{L}t}(fg) = (e^{\mathscr{L}t} f)(e^{\mathscr{L}t} g)$.*

We shall not insist on the analogy between Theorems 1 and 3.

We notice only that the the condition that the $\varrho(\{\bar{f}, g\})$ defines an antisymmetric operator on $\mathfrak{A}$ is in some sense analogue to the separability condition $\varrho(BB^*) = 0 \Leftrightarrow \varrho(B^*B) = 0$ in the non commutative case.

In the classical case however this condition can be easily checked in many interesting cases [4]. The operator $\mathscr{L}$ has automatically equal defect indices in consequence of time reversal invariance of ϱ. The essential selfadjointness of $i\mathscr{L}$ on $\mathfrak{A}$ seems, also in the classical case, to be the really deep and difficult thing to check in the applications and is probably equivalent, in the applications, to the problem of understanding the existence of the dynamics [4].

4. Proof of Theorem 3

Theorem 3 is an immediate consequence of the following result:

Theorem 4. *Let $i\mathscr{L}$ be an operator essentially selfadjoint on a domain $\mathfrak{A}\subset L_\infty(\mathscr{K},\varrho)$, which is a selfadjoint algebra with identity (i.e. if $f\in\mathfrak{A}$ then $\bar{f}\in\mathfrak{A}$) and such that:*

$$\mathscr{L}(fg)=f\mathscr{L}g+g\mathscr{L}f\,,\qquad \overline{\mathscr{L}f}=\mathscr{L}\bar{f}\qquad \forall f,g\in\mathfrak{A}\,.$$

Then there exists a family of measure preserving automorphisms $T_t:\mathscr{K}\to\mathscr{K}$ such that: $(e^{\mathscr{L}t}f)(x)=f(T_tx)$ a.e. for $x\in\mathscr{K}$, $t\in(-\infty,\infty)$.

Proof. Let $\tilde{C}_0\equiv\{\varphi|\varphi:(-\infty,\infty)\to(-\infty,\infty)$ bounded and infinitely differentiable, with bounded derivatives, and such that: $|x|\,|\varphi'(x)|\to 0$ for $x\to\pm\infty\}$.

Let $\mathscr{D}(\mathscr{L})$ be the domain of $\mathscr{L}$.

By the hypothesis the following property is easily verified: if $\psi\in\tilde{C}_0$ then $\psi(f)\in\mathscr{D}(\mathscr{L})$ if f is real, $f\in\mathfrak{A}$, and furthermore $\mathscr{L}\psi(f)=\psi'(f)\mathscr{L}f$.

This can be obtained by uniformly approximating ψ' with polynomials in $[-\|f\|_\infty,\|f\|_\infty]$.

If $\lambda\neq 0$ is real then $(\mathscr{L}-\lambda)\mathfrak{A}$ is dense in L_2, because $i\mathscr{L}$ is essentially selfadjoint on $\mathfrak{A}$.

Let a be real, $a\in L_\infty$, and let $\{b_n\}$ be a sequence of real elements in $\mathfrak{A}$ such that: $(\mathscr{L}-\lambda)b_n\equiv a_n\to a$ in L_2. Then: $b_n\to(\mathscr{L}-\lambda)^{-1}a\equiv b$ in L_2 and $\mathscr{L}b_n=a_n+\lambda b_n\to a+\lambda b$ in L_2.

Consider now $\psi(b)$, $\psi\in\tilde{C}_0$; clearly: $\psi(b)=\lim_n\psi(b_n)$ in L_2

$$\mathscr{L}\psi(b_n)=\psi'(b_n)(a_n+\lambda b_n)\to\psi'(b)(a+\lambda b)\ \text{ in }\ L_2\,.$$

Hence $\psi(b)\in\mathscr{D}(\mathscr{L})$ and $\mathscr{L}\psi(b)=\psi'(b)(a+\lambda b)$ (because $\psi\in\tilde{C}_0$).

Let $\lambda_1,\ldots,\lambda_{p-1}$ be $p-1$ non zero real numbers and $a_1,a_2,\ldots,a_{p-1}\in L_\infty$, a_i real, $i=1,2,\ldots,p-1$, $h\in\mathfrak{A}$ $\psi_1,\psi_2,\ldots,\psi_{p-1}\in\tilde{C}_0$ and put $b_i=(\mathscr{L}-\lambda)^{-1}a_i$. Then:

$$f=\prod_{i=1}^{p-1}\psi_i(b_i)\in\mathscr{D}(\mathscr{L})\,,\qquad g=fh\in\mathscr{D}(\mathscr{L})$$

and

$$\mathscr{L}f=\sum_{i=1}^{p-1}\Big(\prod_{j\neq i}\psi_j(b_j)\Big)\psi_i'(b_i)(\mathscr{L}b_i)\,,$$

$$\mathscr{L}g=(\mathscr{L}f)h+(\mathscr{L}h)f$$

as follows by joint induction on f and g.

Let $\mathscr{B}$ be the (complex) algebra generated by the functions of the form $f=\prod_{i=1}^{p-1}\psi_i(b_i)$.

One has:

i) $\mathscr{L}\mathscr{B}\subset L_\infty$ [and, of course, $\mathscr{B}\subset L_\infty\cap\mathscr{D}(\mathscr{L})$].

ii) $i\mathscr{L}$ is essentially selfadjoint on $\mathscr{B}$.

iii) If $g\in\mathscr{D}(\mathscr{L})$ and $h\in\mathscr{B}$ then:

$$\bar{g},gh\in\mathscr{D}(\mathscr{L})\quad\text{and}\quad\mathscr{L}(gh)=g\mathscr{L}h+h\mathscr{L}g\,.$$

iv) $L_\infty\cap\mathscr{L}^{-1}(L_\infty)$ is dense in L_p in the L_p-norm if $1\leq p<+\infty$.

i) Is obvious.

Notice first that ii) $(\mathscr{L}-\lambda)\mathscr{B}$ at fixed $\lambda\neq 0$, contains the functions of the form $(\mathscr{L}-\lambda)\psi((\mathscr{L}-\lambda)^{-1}a)$, $a=\bar{a}\in L_\infty$; choosing $\psi_n\in\tilde{C}_0$ such that:

$$\psi_n(x)=x \quad \text{if} \quad |x|\leqq n$$

$$\psi_n(x)=1+n \quad \text{if} \quad |x|\geqq n$$

$$0\leqq\psi_n'(x)\leqq 1$$

one realizes that $\psi_n((\mathscr{L}-\lambda)^{-1}a)\to(\mathscr{L}-\lambda)^{-1}a\equiv b$ in L_2, and $\mathscr{L}\psi_n((\mathscr{L}-\lambda)^{-1}a)=\psi_n'((\mathscr{L}-\lambda)^{-1}a)(a+\lambda b)\to(a+\lambda b)$ in L_2 and hence $L_\infty\subset\overline{(\mathscr{L}-\lambda)\mathscr{B}}$.

iii) If $l\in\mathscr{B}$ then:

$$(gh,\mathscr{L}l)=(g,\bar{h}\mathscr{L}l)=(g,\mathscr{L}(\bar{h}l))-(g,l\mathscr{L}\bar{h})$$

$$=-(h\mathscr{L}g,l)-(g\mathscr{L}h,l)$$

(where we have used the obvious fact that $\overline{\mathscr{L}h}=\mathscr{L}\bar{h}$) which proves iii).

iv) If $b=\bar{b}\in\mathscr{B}$ and $\psi\in\tilde{C}_0$ also $\psi(b)\in L_\infty\cap\mathscr{L}^{-1}(L_\infty)$ (as above) hence the density of $\mathscr{B}$ in L_2 implies that any function in L_∞ can be approximated in L_2-norm by a sequence of uniformly bounded functions in $L_\infty\cap\mathscr{L}^{-1}(L_\infty)$: hence this manifold is dense in L_p for all $1\leqq p<+\infty$ in L_p-norm. A further result that will be needed is the following basic lemma which is proven in Appendix to avoid a diversion from the main line of the proof: if $f\in L_\infty$, then $\forall\lambda$ real and non zero, $(\mathscr{L}-\lambda)^{-1}f\in\bigcap_{p\geqq 1}^{\infty}L_p$.

Let $f\in L_\infty$ and put $g=(\mathscr{L}-\lambda)^{-1}f$, $\lambda\in(-\infty,\infty)$, $\lambda\neq 0$. Let $k=(\mathscr{L}+2\lambda)h$, $h\in\mathscr{B}$. Then $k\in L_\infty$ and $g^2k\in L_1$. One has

$$|\textstyle\int g^2k\,d\varrho|=|(\bar{g},g(\mathscr{L}+2\lambda)h)|$$

$$=|(\bar{g},\mathscr{L}(gh))+(\bar{g},2\lambda gh-h\mathscr{L}g)|$$

$$=2|-(\mathscr{L}\bar{g},gh)+\lambda(\bar{g},gh)|$$

$$=2|-((\mathscr{L}-\lambda)\bar{g},gh)|$$

$$=2|-(\bar{f},gh)|\leqq 2\|f\|_\infty\|g\|_2\|h\|_2$$

but

$$\|h\|_2=\|(\mathscr{L}+2\lambda)^{-1}k\|_2\leqq(1/2|\lambda|)\|k\|_2$$

and hence:

$$|\textstyle\int g^2k\,d\varrho|\leqq(\|f\|_\infty/|\lambda|)\|g\|_2\|k\|_2\,.$$

The above chain of inequalities implies that $\|g^2\|_2\leqq\|f\|_\infty\|g\|_2/|\lambda|$ and $g^2\in\mathscr{D}(\mathscr{L})$ and, taking away the moduli signs where possible, that $\mathscr{L}g^2=2g\mathscr{L}g$. The inequality

$$\|g^2\|_2\leqq\|f\|_\infty\|g\|_2/|\lambda|$$

may be iterated by induction. One obtains:

$$\|g^n\|_2\leqq(\|f\|_\infty/|\lambda|)^n$$

or, in other words:

$$\|g\|_\infty\leqq\|f\|_\infty/|\lambda|$$

and therefore:

$$\|(\mathscr{L}-\lambda)^{-1}f\|_\infty \leqq \|f\|_\infty/|\lambda| \qquad \forall f\in L_\infty .$$

This easily implies [6], that $\|(\mathscr{L}-\lambda)^{-1}\|_p\leqq 1/|\lambda|$ and $\|e^{\mathscr{L}t}\|_p\leqq 1$, $2\leqq p<+\infty$ (because if $2\leqq p<+\infty$ the restriction of $\mathscr{L}$ to $\mathscr{D}(\mathscr{L})\cap L_p\cap\mathscr{L}^{-1}(L_p)$, as an operator on L_p, can be seen to be densely defined and is closed) [7]. Therefore also $\|e^{\mathscr{L}t}\|_\infty\leqq 1$.

Let now $f, g\in L_\infty\cap\mathscr{D}(\mathscr{L})$; then $e^{\mathscr{L}t}f$, $e^{\mathscr{L}t}g\in L_\infty\cap\mathscr{D}(\mathscr{L})$ and it is easily verified that the product $(e^{\mathscr{L}t}f)(e^{\mathscr{L}t}g)\in\mathscr{D}(\mathscr{L})\cap L_\infty$ and

$$\mathscr{L}((e^{\mathscr{L}t}f)(e^{\mathscr{L}t}g))=(e^{\mathscr{L}t}g)\mathscr{L}(e^{\mathscr{L}t}f)+(e^{\mathscr{L}t}f)\mathscr{L}(e^{\mathscr{L}t}g).$$

If we put $F(t)=(e^{\mathscr{L}t}f)(e^{\mathscr{L}t}g)$ then:

$$\begin{aligned}(t-t_0)^{-1}(F(t)-F(t_0))&=(t-t_0)^{-1}(e^{\mathscr{L}t}f-e^{\mathscr{L}t_0}f)\,(e^{\mathscr{L}t}g)\\&\quad+(t-t_0)^{-1}(e^{\mathscr{L}t}g-e^{\mathscr{L}t_0}g)\,(e^{\mathscr{L}t_0}f)\xrightarrow[t\to t_0]{}\mathscr{L}((e^{\mathscr{L}t_0}f)(e^{\mathscr{L}t_0}g))\end{aligned}$$

in L_2.

Therefore $F(t)$ is norm differentiable in L_2 and:

$$dF(t)/dt=\mathscr{L}F(t) \qquad F(0)=f\cdot g$$

which implies that:

$$e^{\mathscr{L}t}(fg)=(e^{\mathscr{L}t}f)(e^{\mathscr{L}t}g) \qquad \forall f, g\in L_\infty\cap\mathscr{D}(\mathscr{L}).$$

By density argument the above equality can be proven for all $f, g\in L_\infty$.

So we have shown that $e^{\mathscr{L}t}$ is a multiplicative map of L_∞ into itself, i.e. $e^{\mathscr{L}t}$ defines an automorphism of the equivalence class (mod. 0) of measurable sets of $\mathscr{K}$, i.e. a mapping which preserves countable unions, and measure, [8].

Since the measure space $(\mathscr{K},\varrho)$ is a Lebesgue space, it follows [9], that there is a family $(T_t, t\in(-\infty,\infty))$ of automorphisms (mod. 0) of $(\mathscr{K},\varrho)$ such that

$$(e^{\mathscr{L}t}f)(x)=f(T_tx) \qquad \forall f\in L_2(\mathscr{K},\varrho)$$

$\forall t\in(-\infty,+\infty)$, $\forall x\in\mathscr{K}$.

5. Concluding Remarks

The Theorem 3 is clearly related, in the case of physical interest, to the problem of the existence of dynamics for infinite (or finite), hamiltonian systems. Its usefulness is however limited because the essential selfadjointness of $\mathscr{L}$ is too difficult to be checked.

We notice that, in the case of an algebra of classical observables and of a state ϱ on it such that I extends to a unitary operator on $L_2(\mathscr{K},\varrho)$, the operator $i\mathscr{L}$, when defined at all, has selfadjoint extensions. Hence a natural question would be whether there are selfadjoint extensions $i\tilde{\mathscr{L}}$ of $i\mathscr{L}$ such that $(e^{\tilde{\mathscr{L}}t}f)(x)$ is of the form $f(\tilde{T}_tx)$ where $(\tilde{T}_t, t\in(-\infty,+\infty))$ is a family of measure preserving automorphisms of the measure space $(\mathscr{K},\varrho)$.

Generally every selfadjoint extension $\tilde{\mathscr{L}}$ of this kind corresponds intuitively to different ways of resolving the "catastrophes" which may occur along the trajectories of $\tilde{T}_t x$; hence the essential selfadjointness should mean that, with probability one, there is no real ambiguity in solving the equations of motions for $\tilde{T}_t x$ which are generated on $\mathscr{K}$ by $\mathscr{L}$ thought as a vector field on $\mathscr{K}$ (when possible, e.g. in all the applications to classical statistical mechanics).

It would be interesting to make more precise the above intuitive considerations.

Identical considerations are possible for the quantum case.

Notice also, that the available existence theorems [4, 10] for time evolution of classical systems can be thought, in our context, as theorems on the existence of good selfadjoint extensions of $i\mathscr{L}$.

Another problem which is closely related to the one investigated in this paper is the following: given a densily defined derivation δ on a C^*-algebra $\mathfrak{A}$, does it define a strongly continuous group of automorphisms of $\mathfrak{A}$?

This problem is analyzed in detail in the papers [11, 12]; we are indebted to S. Doplicher for bringing some of these works to our attention.

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Appendix

Lemma. *If $f\in L_\infty$ then, for all non zero real λ, $(\mathscr{L}-\lambda)^{-1}f\in\bigcap_{p\geqq 1}L_p$.*

Proof. It is enough to consider real functions $f\in L_\infty$. Put $g=(\mathscr{L}-\lambda)^{-1}f$.

Let $\chi_{n,M}(x)$ be a C^∞-function such that

$$\chi_{n,M}(x)=0 \quad \text{if} \quad x\leqq n$$

$$\chi_{n,M}(x)=0 \quad \text{if} \quad x\geqq 2M$$

$$\chi_{n,M}(x)=1 \quad \text{if} \quad 2n\leqq x\leqq M \quad (\text{assume } 2n<M)\,.$$

$\chi_{n,M}(x)$ is increasing between n and $2n$ and decreasing between M and $2M$ with slope bounded by $2/n$ and $2/M$, respectively. Then $x^2\chi_{n,M}(x^2)$ is C^∞ with compact support and

$$|x^2\chi'_{n,M}(x^2)|\leqq 8 \qquad \forall x\in(-\infty,\infty)\,.$$

The following relation holds, since $g^2\chi_{n,M}(g^2)\in\mathscr{B}$:

$$\begin{aligned}\int g^2\chi_{n,M}(g^2)d\varrho&=(g^2\chi_{n,M}(g^2),1)\\&=((\mathscr{L}-2\lambda)g^2\chi_{n,M}(g^2),(-\mathscr{L}-2\lambda)^{-1}1)=-(1/2\lambda)((\mathscr{L}-2\lambda)g^2\chi_{n,M}(g^2),1)\\&=-(1/2\lambda)([2(\mathscr{L}-\lambda)g]g\chi_{n,M}(g^2),1)-(1/2\lambda)(2g^3\chi'_{n,M}(g^2)\mathscr{L}g,1)\\&=-(1/\lambda)(fg\chi_{n,M}(g^2),1)-(1/\lambda)(g^3\chi'_{n,M}(g^2)f,1)-(g^4\chi'_{n,M}(g^2),1)\end{aligned}$$

letting $M\to\infty$ and using $|g^2\chi'_{n,M}(g^2)|\leqq 8$, $g, g^2\in L_1$ we find, by dominated convergence if $\chi_n(x)=\lim_{M\to\infty}\chi_{n,M}(x)$:

$$\begin{aligned}\int g^2\chi_n(g^2)d\varrho=&-(1/\lambda)\int fg\chi_n(g^2)d\varrho-(1/\lambda)\int g^3\chi'_n(g^2)d\varrho\\&-\int g^4\chi'_n(g^2)d\varrho\,.\end{aligned}$$

Since $\chi_n'(g^2) \geqq 0$ we obtain

$$\int (g^2 + fg/\lambda) d\varrho \leqq -(1/\lambda) \int g^3 \chi_n'(g^2) d\varrho \, .$$

If n is large enough this means that

$$\int_{E(2n \leqq g^2 \leqq 4n)} (g^2 + fg/\lambda) d\varrho \leqq (1/|\lambda|) \int_{E(n \leqq g^2 \leqq 2n)} |g| g^2 \chi_n'(g^2) d\varrho$$

hence

$$(2n - (\|f\|_\infty/|\lambda|) \sqrt{4n}) \varrho(E(2n \leqq g^2 \leqq 4n)) \leqq (8\sqrt{2}/|\lambda|) \sqrt{n} \varrho(E(n \leqq g^2 \leqq 2n))$$

provided n is large enough.

Put $\mu_k = \varrho(E(2^k \leqq g^2 \leqq 2^{k+1}))$ then for $n = 2^k$ we find that $\exists C > 0$ such that

$$\mu_{k+1} \leqq (C/2^{k/2}) \mu_k$$

for k large enough.

Hence $\exists \tilde{C} > 0$ such that

$$\mu_k \leqq \tilde{C}/2^{k^2/4} \qquad \forall k \geqq 0 \, .$$

Let $p \geqq 1$ then

$$\int g^{2p} d\varrho \leqq 1 + \sum_{k=0}^{\infty} \int_{E(2^k \leqq g^2 \leqq 2^{k+1})} g^{2p} d\varrho \leqq 1 + \sum_{k=0}^{\infty} 2^{2(k+1)p} \mu_k < \infty \, .$$

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