Commun. math. Phys. 45, 159—165 (1975)

Communications in
Mathematical
Physics

Mathematics of Noncanonical Quantum Theory

G. Braunss

Fachbereich Mathematik der Justus-Liebig-Universität,
D-6300 Giessen, Federal Republic of Germany

Abstract. The Weyl relation $e^{-isQ}e^{itP}e^{isQ}=e^{it(P+sI)}$ is generalized so as to hold for noncanonical couples (P, Q) implying the commutation relation $i[P,Q]=C$ where C is arbitrary bounded self-adjoint. It is shown that if 0 is not in the closure of the numerical range of C then both P and Q are spectrally continuous and neither bounded from below nor above. The dynamical equations in noncanonical theory are established. It is shown that H (which is no longer given by correspondence) cannot be bounded from below (above) if $C\leqq 0$ $(C\geqq 0)$, $C\neq 0$.

Introduction

Consider a one-dimensional possibly nonlinear quantum mechanical oscillator,

$$i[H,Q]=P\,, \tag{0.1a}$$

$$i[H,P]=-F(Q)\,, \tag{0.1b}$$

where P, Q, and H are the operators of momentum, position, and energy, respectively, and where F is a nice function, for example, a polynomial. In addition, P and Q satisfy the canonical commutation relation

$$i[P,Q]=I \quad (=\text{identity operator})\,. \tag{0.1c}$$

Some years ago it was suggested by Heisenberg (cf. [1]) to replace (0.1c) by a "noncanonical" commutation relation,

$$i[P,Q]=C\,, \tag{0.2}$$

where C is some self-adjoint linear operator. The motivation for this proposal has to do with the removal of divergencies in quantum field theory. But there are also other reasons to study noncanonical quantum theories, for example, general relativistic models. Clearly, H cannot be in correspondence with the classical Hamiltonian of (0.1a, b) if C is not a multiple of the identity operator. However, this does not matter as long as we can expect to have the operator H the desired spectral properties. In a paper to follow it will be shown that, for example, the dynamical equations (0.1a, b) have solutions for a large class of functions F (such as $F(Q)=Q^n$, n odd) in which P and Q are bounded whereas H coincides spectrally with the Hamiltonian of the corresponding canonical system (same F but $C=I$).

In this paper we shall investigate some mathematical properties of one-dimensional noncanonical quantum systems. Section 1 deals with the commutation relation (0.2), Section 2 with the dynamical equations (0.1a, b).

1. Generalized Weyl Relations

In the following the self-adjoint operator C in the commutator equation (0.2) will be assumed to be bounded. Let $\sigma_f(A)=\|[A-(f,Af)]f\|$ be the standard deviation of a symmetric operator A with respect to a state f. Then (cf. [2])

$$\sigma_f(P)\sigma_f(Q)\geqq \tfrac{1}{2}|(Cf,f)|\,.$$

If $0\in W_C$ (= numerical range of C) then $\sigma_f(P)\sigma_f(Q)=0$ could happen for some f, i.e. one could perform a simultaneous measurement of P and Q with respect to a certain state f. Since such distinguished states are not known we shall demand $0\notin W_C$.

As is well known the commutation relation (0.1c) without further specification does not necessarily give the quantum mechanical operators P and Q. Therefore it has been replaced by the Weyl relation,

$$e^{-isQ}e^{itP}e^{isQ}=e^{it(P+sI)}\,,\qquad s,t\in\mathbb{R}\,, \tag{1.1}$$

which involves only bounded (namely unitary) operators. This relation implies (0.1c) and gives the correct quantum mechanical operators P and Q. It would be desirable to find a relation which in a similar manner replaces (0.2). Motivated by the following proposition we shall establish such a relation.

Proposition. *Let A, B, and C be bounded operators on a Banach space such that*

$$[A,B]=C \tag{1.2}$$

and let

$$C_B(s)=\int_0^s e^{-\sigma B}Ce^{\sigma B}d\sigma\,,\quad s\in\mathbb{R}\,,$$
$$C_A(t)=\int_0^t e^{-\tau A}Ce^{\tau A}d\tau\,,\quad t\in\mathbb{R}\,.$$

Then (1.2) *implies*

$$e^{-sB}e^{tA}e^{sB}=e^{t(A+C_B(s))}\,,\qquad s,t\in\mathbb{R}\,, \tag{1.3}$$

$$e^{-tA}e^{sB}e^{tA}=e^{s(B-C_A(t))}\,,\qquad s,t\in\mathbb{R}\,. \tag{1.4}$$

Conversely, (1.3) *and* (1.4) *each imply* (1.2).

Proof. It is no difficulty to see that (1.2) implies

$$e^{-sB}Ae^{sB}=A+\int_0^s dt(d/dt)(e^{-tB}Ae^{tB})=A+C_B(s)\,.$$

Hence

$$e^{-sB}e^{tA}e^{sB}=e^{t(A+C_B(s))}$$

which proves (1.3). The proof that (1.2) implies (1.4) is completely analogous. To prove that (1.3) implies (1.2) differentiate first with respect to t and then with respect to s and set $t=s=0$. The proof that (1.4) implies (1.2) is completely analogous. □

Theorem 1.1. *Let A, B, and C be self-adjoint linear Hilbert space operators with C bounded and let (in the strong topology)*

$$C_B(s)=\int_0^s e^{-i\sigma B}Ce^{i\sigma B}d\sigma\,, \quad s\in\mathbb{R}\,,$$

$$C_A(t)=\int_0^t e^{-i\tau A}Ce^{i\tau A}d\tau\,, \quad t\in\mathbb{R}\,.$$

Assume that

$$e^{-isB}e^{itA}e^{isB}=e^{it(A+C_B(s))}, \quad s\in\mathbb{R}\,, \tag{1.5}$$

$$e^{-itA}e^{isB}e^{itA}=e^{is(B-C_A(t))}, \quad t\in\mathbb{R}\,. \tag{1.6}$$

Then (1.5) *and* (1.6) *each imply on* $\mathscr{D}_{[A,B]}$

$$i[A,B]=C\,. \tag{1.7}$$

Equations (1.5) *and* (1.6) *are mutually equivalent.*

For the proof we need the following

Lemma 1.1. *Let F and $G(s)$ be self-adjoint linear Hilbert space operators with $G(s)$ bounded and at least once bounded differentiable with respect to $s\in\mathbb{R}$. Then*

$$(d/ds)e^{i(F+G(s))}=ie^{i(F+G(s))}\int_0^1 e^{-it(F+G(s))}(dG(s)/ds)e^{it(F+G(s))}dt\,. \tag{1.8}$$

Proof. Let $T=F+G(s)$ and $\Delta T=G(s+\varepsilon)-G(s)$. Then the Duhamel formula gives

$$e^{it(T+\Delta T)}-e^{itT}=\int_0^t e^{i(t-\tau)(T+\Delta T)}(i\Delta T)e^{i\tau T}d\tau\,.$$

Dividing by ε, taking the limit $\varepsilon\to 0$, and setting $t=1$ the assertion follows. □

Proof of Theorem 1.1. From (1.5) it follows immediately that A and $A+C_B(s)$ are unitarily equivalent for every $s\in\mathbb{R}$, that is,

$$(*) \quad e^{-isB}Ae^{isB}=A+C_B(s)\,, \quad s\in\mathbb{R}\,.$$

Since $(d/ds)(e^{isB}f)|_{s=0}=iBf$ for all $f\in\mathscr{D}_B$ and $(d/ds)(C_B(s)f)|_{s=0}=Cf$, Eq. (1.7) follows from (*) by strong differentiation. The proof that (1.6) implies (1.7) is similar.

To prove that (1.5) implies (1.6) note that from (1.5) it follows

$$e^{-itA}e^{-isB}e^{itA}=e^{-itA}e^{it(A+C_B(s))}e^{-isB}\,.$$

The left hand side is $\exp[-is(e^{-itA}Be^{itA})]$. Let the right hand side be $F(s)$. For any f in $\mathscr{D}_B$, we shall prove that $F(s)f$ is strongly differentiable at $s=0$ and the result is $-i(B-C_A(t))f$. By Stone's theorem, f is in the domain of $e^{-itA}Be^{itA}$ and we have $e^{-itA}Be^{itA}\supseteq B-C_A(t)$, where $B-C_A(t)$ has the same domain as B due to the boundedness of $C_A(t)$. Since both sides of this equation are self-adjoint, they are equal and hence we have (1.6).

To prove $(d/ds)F(s)f = -i(B - C_A(t))f$, we first note that $e^{-isB}f$ is strongly differentiable if f is in the domain of B and $(d/ds)(e^{-isB}f)|_{s=0} = -iBf$. We also have

$$(d/ds)[e^{-itA}e^{it(A+C_B(s))}]|_{s=0} = iC_A(t)$$

strongly due to Lemma 1.1.

The derivation of (1.5) from (1.6) is similar. □

Remark. If $C = I$ then both (1.5) and (1.6) coincide with the Weyl relation (1.1).

Theorem 1.2. *Let A, B, and C be as in Theorem 1.1 and let $0 \notin W_C$. Then A and B are spectrally continuous. If $0 \notin \bar{W}_C$ (= closure of W_C) then A and B are neither bounded from below nor above.*

Proof. Let $V_s = e^{isB}$, let

$$K_B(s) = \int_0^1 V_{ts}^* C V_{ts} dt\,,$$

and let $A_\lambda = A - \lambda I$, $\lambda \in \mathbb{R}$. Then (1.5) implies $V_s^* A_\lambda V_s = A_\lambda + sK_B(s)$. Assume $A_\lambda f = 0$, $f \neq 0$. Then the above equation inserted between f and $V_s^* f$ yields

$$s(V_s^* f, K_B(s) f) = 0\,.$$

Dividing by s and taking the limit $s \to 0$ it follows $(f, Cf) = 0$ which contradicts $0 \notin W_C$. Hence A is spectrally continuous. Similarly one proves that B is spectrally continuous.

Let now $0 \notin \bar{W}_C$. Then either $C \geqq cI$ or $C \leqq -cI$ for some $c > 0$. Hence, since C is bounded, we have either $(K_B(s)f, f) \geqq c$ or $(K_B(s)f, f) \leqq -c$ for all real s and all unit vectors f. Assume that A is bounded from below or above. Then both (Af, f) and $(AV_s f, V_s f)$ for all unit vectors f in $\mathcal{D}_A$ are bounded from below or above by some real a. If $(AV_s f, V_s f) \geqq a$, then $V_s^* A V_s = A + sK_B(s)$ implies $s(K_B(s)f, f) \geqq a - (Af, f)$ for all real s, which contradicts with $(K_B(s)f, f) \geqq c$ at $s \to -\infty$ and with $(K_B(s)f, f) \leqq -c$ at $s \to \infty$. Similar arguments hold for the other cases. □

In view of the foregoing results we now can replace the commutation relation (0.2) by one of the relations (1.5) or (1.6) with P instead of A and Q instead of B.

Definition. A (one-dimensional) *generalized Weyl triple* consists of self-adjoint linear operators P, Q, and C where C is bounded, such that for all $s, t \in \mathbb{R}$

$$e^{-isQ} e^{itP} e^{isQ} = e^{it(P + C_Q(s))} \tag{1.9}$$

or, equivalently,

$$e^{-itP} e^{isQ} e^{itP} = e^{is(Q - C_P(t))}\,, \tag{1.10}$$

where (in the strong topology)

$$C_Q(s) = \int_0^s e^{-i\sigma Q} C e^{i\sigma Q} d\sigma\,, \quad C_P(t) = \int_0^t e^{-i\tau P} C e^{i\tau P} d\tau\,.$$

Remark. Any pair of bounded self-adjoint linear operators belongs due to the above Proposition to a noncanonical Weyl triple.

It follows from Theorem 1.2

Theorem 1.3. *Let (P, Q, C) be a generalized Weyl triple where C is strictly definite, that is, $0 \notin \bar{W}_C$. Then P and Q are spectrally continuous and neither bounded from below nor above.*

The following two examples will show that $0 \notin W_C$, $0 \in \bar{W}_C$, can be realized in the commutator relation (0.2) for bounded as well as unbounded P and Q.

Example 1. Let P and Q be Jacobi matrices with elements p_{mn} and q_{mn}, respectively, where

(i) $p_{mn} = i2^{-1/2}(a_m \delta_{m+1,n} - a_n \delta_{m,n+1})$,

(ii) $q_{mn} = 2^{-1/2}(a_m \delta_{m+1,n} + a_n \delta_{m,n+1}) \qquad m, n = 1, 2, \ldots$.

Then $C = i[P, Q]$ is a diagonal matrix with diagonal elements

$$c_1 = a_1^2, \qquad c_n = a_n^2 - a_{n-1}^2 \qquad n = 2, 3, \ldots.$$

Let $0 < a_1 < a_2 < \ldots \leqq \text{constant} < \infty$. Then P and Q are bounded self-adjoint and (cf. [3]) absolutely continuous. Clearly, C is of trace class and $0 \notin W_C$, $0 \in \bar{W}_C$.

Example 2. Let P and Q be Jacobi matrices with elements given by (i) and (ii) in Example 1 where $a_n = n^{1/4}$. Then P and Q are both unbounded and (cf. [4]) self-adjoint. $C = i[P, Q]$ is a diagonal matrix with diagonal elements

$$c_1 = 1, \qquad c_n = n^{1/2} - (n-1)^{1/2} \qquad n = 2, 3, \ldots.$$

Clearly, C is compact but not of trace class and $0 \notin W_C$, $0 \in \bar{W}_C$.

Remark. A theorem given by Putnam (cf. [3]) states that if P and Q are bounded self-adjoint linear operators and if $C = i[P, Q] \geqq 0$ then $0 \in \bar{W}_C$; if in addition $0 \notin W_C$ then P and Q are absolutely continuous.

2. The Dynamical Equations

Let for a moment P, Q, and H in (0, 1a) be bounded. Then

$$e^{-isQ} e^{irH} e^{isQ} = e^{ir(H + P_Q(s))} \qquad r, s \in \mathbb{R} \tag{2.1}$$

where ($V_s = e^{isQ}$)

$$P_Q(s) = \int_0^s V_\sigma^* P V_\sigma d\sigma = \int_0^s \left(P + \int_0^\sigma V_\tau^* C V_\tau d\tau\right) d\sigma = sP + \int_0^s C_Q(\sigma) d\sigma .$$

Equation (2.1) implies

$$V_s^* H V_s = H + P_Q(s) = H + sP + \int_0^s C_Q(\sigma) d\sigma . \tag{2.2}$$

If P, Q, and H are unbounded then (2.1) is not necessarily a consequence of (0.1a). However, in canonical theory $V_s \mathscr{D}_H \subset \mathscr{D}_H \subset \mathscr{D}_P$ for every real s and (2.2) holds on $\mathscr{D}_H$. If in addition H is self-adjoint then (2.1) follows from (2.2). Since the condition of H being self-adjoint is too restrictive (one might be willing to consider essentially self-adjoint operators H) we shall replace (0.1a) by (2.2) [and not by (2.1)].

Consider now (0.1b). Again, if P, Q, and H were bounded (and F, say, a polynomial) then

$$e^{-itP} e^{irH} e^{itP} = \exp ir\{H - [F(Q)]_P(t)\} \qquad r, t \in \mathbb{R} \tag{2.3}$$

where ($U_t = e^{itP}$)

$$[F(Q)]_P(t) = \int_0^t U_\tau^* F(Q) U_\tau d\tau = \int_0^t F(U_\tau^* Q U_\tau) d\tau = \int_0^t F(Q - C_P(\tau)) d\tau .$$

Equation (2.3) implies

$$U_t^* H U_t = H - [F(Q)]_P(t)\,. \tag{2.4}$$

If P, Q, and H are unbounded then (0.1b) does not necessarily imply (2.3). In canonical theory one has

$$[F(Q)]_P(t) = G(Q) - G(Q - tI)\,, \qquad G(\alpha) = \int_0^\alpha F(\beta)d\beta\,,$$

and $H = \frac{1}{2}P^2 + G(Q)$ (with a rest-mass $m = 1$). Hence

$$H - [F(Q)]_P(t) = \tfrac{1}{2}P^2 + G(Q - tI) = U_t^*(\tfrac{1}{2}P^2 + G(Q))U_t = U_t^* H U_t\,.$$

This agrees with (2.4). If in addition H is self-adjoint then (2.4) implies (2.3). For reasons which have led us to establish (2.2) we replace (0.1b) by (2.4) and not by (2.3).

Definition. A (one-dimensional) *general quantum mechanical system (GQMS)* consists of a generalized Weyl triple (P, Q, C), an essentially self-adjoint operator H, and a function F such that (2.2) and (2.4) hold.

Theorem 2.1. *If for a GQMS $C \leqq 0$, $C \neq 0$, then H cannot be bounded from below.*

Proof. Let

$$s^2 L_Q(s) = s^2 \int_0^1 \int_0^1 V_{\sigma\tau s}^* C V_{\sigma\tau s} d\sigma d\tau = \int_0^s \int_0^\sigma V_\sigma^* C V_\sigma d\sigma = \int_0^s C_Q(\sigma) d\sigma\,.$$

Then it follows from (2.2)

$$(HV_s f, V_s f) = (Hf, f) + s(Pf, f) + s^2(L_Q(s)f, f)\,. \tag{*}$$

If $C \leqq 0$ then $(L_Q(s)f, f) \leqq 0$ for all real s. It is possible to choose f so that $\beta \equiv (Pf, f) \neq 0$ because otherwise the closed operator P satisfying $(Pf, f) = 0$ for a dense linear subset $\mathscr{D}_{H+sP} = \mathscr{D}_H \cap \mathscr{D}_P$ must vanish contradicting with $C \neq 0$. Hence the right-hand side of (*) tends to $-\infty$ either for $s \to -\infty$ if $\beta > 0$ or for $s \to \infty$ if $\beta < 0$. □

Corollary. *If $C \geqq 0$, $C \neq 0$, then H cannot be bounded from above.*

Remark. If $C \geqq 0$, $C \neq 0$, then H cannot be bounded from above even for bounded P, Q, and $F(Q)$, which is possible if $0 \in \bar{W}_C$ (see Example 1 in Section 1).

Theorem 2.2. *Let f be in the domain of $H - [F(Q)]_P(t)$ for all real t. If the function $([F(Q)]_P(.)f, f)$ has no upper (lower) bound then H cannot be bounded from below (above).*

The proof follows from (2.4) by copying the proof of Theorem 2.1.

Example. Let $F(Q) = kQ$, k real constant. Then

$$[F(Q)]_P(t) = k\int_0^t (Q - C_P(\tau))d\tau = ktQ - kt^2 L_P(t)$$

where

$$L_P(t) = \int_0^1 \int_0^1 U_{\tau\sigma t}^* C U_{\tau\sigma t} d\tau d\sigma\,.$$

Let $C \geqq 0$ so that $(L_P(t)f,f) \geqq 0$ and let

$$\varphi(t) \equiv ([F(Q)]_P(t)f,f) = kt(Qf,f) - kt^2(L_P(t)f,f).$$

where $(Qf,f) \neq 0$. Then φ has no lower (upper) bound if $k>0$ $(k<0)$. Hence by Theorem 2.2 $k \geqq 0$ is a necessary condition for H being bounded from below.

Remark. In [1] it has been shown that H and C commute. The easy proof follows from Eqs. (0.1a), (0.1b), and (0.2).

We conclude with a simple noncanonical model. Consider a system of particles with rest-masses m_k: $0 < m_1 < m_2 < \ldots$. Let Q be the many-particle position operator and assume $(\dot{Q}, Q, C)$ to be a generalized Weyl triple in a Hilbert space $\mathscr{H}$. Let $\mathscr{H}_k$ be the subspace of particles with rest-mass m_k so that for any $f^{(k)} \in \mathscr{H}_k$ which is in the domain of $[\dot{Q}, Q]$ there holds

$$i[\dot{Q}, Q] f^{(k)} = (1/m_k) f^{(k)} \qquad (k = 1, 2, \ldots).$$

Then we should have $C = (1/m_1)E_1 + (1/m_2)E_2 + \ldots$ where E_k is the projector onto $\mathscr{H}_k$. Since C must commute with H (the total Hamiltonian) it follows that H is reduced by each $\mathscr{H}_k$. This would agree with the assumption that m_k is a (lowest) eigenvalue of H for an eigenvector in $\mathscr{H}_k$ or that $\mathscr{H}_k$ is an eigenspace of H belonging to an eigenvalue m_k (of a certain multiplicity). Clearly, $C > 0$, so H cannot be bounded from above. Hence, if H were bounded on each $\mathscr{H}_k$, then the sequence $m_1, m_2, \ldots$ could not be bounded and therefore $0 \in \bar{W}_C$ (in this case $\dot{Q}$ and Q possibly could be bounded). Note, however, that if $H_k = E_k H = H E_k$ is bounded then $\dot{Q}$ and Q are not reduced by $\mathscr{H}_k$. Because if $Q_k = E_k Q = Q E_k$ then $i[\dot{Q}_k, Q_k] = (1/m_k)E_k$ and this would imply (due to Theorem 2.1) that H_k is not bounded from above.

Acknowledgements. The author would like to thank the Deutsche Forschungsgemeinschaft for supporting a stay at the Department of Mathematics at Cornell University, Ithaca, N.Y. For corrections and suggestions, in particular for the proof of the second part of Theorem 1.1, the author is very much indebted to Prof. Araki, Kyoto.

References

1. Rampacher, H., Stumpf, H., Wagner, F.: Fortschr. Phys. **13**, 385—480 (1965)
2. Prugovečki, E.: Quantum Mechanics in Hilbert Space. New York and London: Academic Press 1971
3. Putnam, C. R.: Commutation Properties of Hilbert Space Operators. Berlin-Heidelberg-New York: Springer 1967
4. Stone, M. H.: Linear Transformations in Hilbert Space, AMS, Colloqium Publications, Vol. XV

Communicated by H. Araki

Received May 27, 1974; in revised form August 1, 1975