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A Remark on a Theorem of Powers and Sakai

S. Doplicher

Istituto Matematico "G. Castelnuovo", Università di Roma, I-00185 Roma, Italy

Abstract. Given an abelian locally compact group G and a C^*-algebra with unit $\mathfrak{A}$, the set of those continuous representations of G by automorphisms of $\mathfrak{A}$ which fulfill a spectrum condition is closed.

In a recent paper [1] Powers and Sakai proved, among other things, that if a sequence of continuous one-parameter groups of automorphisms of a C^*-algebra with identy, each with a generator in the algebra, converges strongly, uniformly on compact sets of the line, the limit one parameter group has a ground state.

As any one parameter group with a generator in the algebra has a ground state [1, proof of Theorem 2.3, first paragraph] this theorem is implied by a closedness property of the set of one parameter groups having a ground state.

The purpose of this note is to remark that from the algebraic spectrum condition [2] this closedness property follows naturally for any locally compact abelian group replacing the line.

Let G be a locally compact abelian group and $\mathfrak{A}$ a C^*-algebra with identity I; let $\mathscr{A}$ be the set of all continuous homomorphisms of G into the group of $*$-automorphisms of $\mathfrak{A}$, equipped with the strong topology.

For $\alpha \in \mathscr{A}$, by a representation of $\{\mathfrak{A}, \alpha\}$ we mean a covariant representation (π, U): π is a representation of $\mathfrak{A}$ on a Hilbert space $\mathscr{H}$ and U a strongly continuous unitary representation of G on $\mathscr{H}$ s.t. $U(g)\pi(\cdot)U(g)^{-1} = \pi \circ \alpha_g$, $g \in G$.

If ω is an α-invariant state on $\mathfrak{A}$, (π_ω, U_ω) and ξ_ω denote respectively the G.N.S. covariant representation and the associated cyclic vector s.t. $\omega = (\xi_\omega, \pi_\omega(\cdot)\xi_\omega)$ and $U_\omega(g)\pi_\omega(A)\xi_\omega = \pi_\omega(\alpha_g(A))\xi_\omega$; $g \in G$, $A \in \mathfrak{A}$.

Let $\hat{G}$ denote the dual group of G and $K \subset \hat{G}$ a closed set including the identity of $\hat{G}$.

Let $\mathfrak{J}(\alpha, K)$ denote the smallest left ideal in $\mathfrak{A}$ including the set:

$$\mathfrak{B}(\alpha, K) = \{\alpha_f(A)/A \in \mathfrak{A}\,; \quad f \in L^1(G), \quad \hat{f}|K = 0\},$$

where $\alpha_f(A) = \int f(g)\alpha_g(A)d\mu(g)$ and μ is a Haar measure on G.

The following conditions on $\alpha \in \mathscr{A}$ are equivalent:

(i) there exists an α-invariant state ω on $\mathfrak{A}$ with spectrum $U_\omega \subset K$;

(ii) $\mathfrak{J}(\alpha, K) \neq \mathfrak{A}$.

In Ref. [2] this is proved for $G=\mathbb{R}^4$ and $K=$the future light cone, but that argument has a straightforward generalization to our present case.

Let $\mathscr{A}(K)$ be the set of $\alpha\in\mathscr{A}$ fulfilling (i)/(ii).

1. Proposition. *The set $\mathscr{A}(K)$ is closed in $\mathscr{A}$ for the topology of strong convergence uniformly on compact sets of G. The set $\mathscr{A}(K)$ is sequentially closed in $\mathscr{A}$ for the topology of simple strong convergence.*

Proof. If $\alpha\in\mathscr{A}\backslash\mathscr{A}(K)$ we have $\mathfrak{J}(\alpha, K)=\mathfrak{A}$ i.e.

$$I=\sum_{i=1}^{N} A_i\alpha_{f_i}(B_i) \tag{1}$$

with $f_i\in L^1(G)$, $\hat{f}_i=0$ on K and $A_i, B_i\in\mathfrak{A}, i=1,\dots,N$. Given $\varepsilon>0$ let C be a compact set in G and $\varphi_1,\dots,\varphi_N$ continuous functions with support in C s.t.

$$\|\varphi_i-f_i\|_1<\varepsilon\,,\qquad i=1,\dots,N\,.$$

Let $\mathscr{N}(\alpha)$ be the compact-strong neighbourhood of α in $\mathscr{A}$ defined by

$$\alpha'\in\mathscr{N}(\alpha)\quad\text{if}\quad \sup\{\|\alpha'_g(B_i)-\alpha_g(B_i)\|/g\in C, i=1,\dots,N\}<\varepsilon\,.$$

If $\alpha'\in\mathscr{N}(\alpha)$, $\sum_{i=1}^{N} A_i\alpha'_{f_i}(B_i)\equiv B'\in\mathfrak{J}(\alpha', K)$ and

$$\begin{aligned}\|I-B'\|&\leqq\sum_{i=1}^{N}\|A_i\|\cdot\|(\alpha_{f_i}-\alpha'_{f_i})(B_i)\|\\ &\leqq\varepsilon\cdot\sum_{i=1}^{N}\|A_i\|(2\|B_i\|+\|\varphi_i\|_1)\,;\end{aligned}$$

for small ε, B' is regular in $\mathfrak{A}$ and $\mathfrak{J}(\alpha', K)=\mathfrak{A}$; so the last equation holds for all α' in a compact-strong neighbourhood of α in $\mathscr{A}$.

Let $\{\alpha^{(n)}\}\subset\mathscr{A}$ be a sequence and $\alpha\in\mathscr{A}$ s.t. for any fixed $g\in G$ and $A\in\mathfrak{A}$

$$\|\alpha_g^{(n)}(A)-\alpha_g(A)\|\to 0\quad\text{as}\quad n\to\infty\,.$$

For each $A\in\mathfrak{A}$ and $f\in L^1(G)$, we have that

$$\|(\alpha_f^{(n)}-\alpha_f)(A)\|\leqq\int|f(g)|\cdot\|\alpha_g^{(n)}(A)-\alpha_g(A)\|d\mu(g)$$

and $|f(g)|\cdot\|\alpha_g^{(n)}(A)-\alpha_g(A)\|\leqq 2\|A\|\cdot|f(g)|$; by Lebesgue theorem

$$(\alpha_f^{(n)}-\alpha_f)(A)\to 0\quad\text{as}\quad n\to\infty\,. \tag{2}$$

Assume $\alpha^{(n)}\in\mathscr{A}(K)$ but (1) holds. Setting $B^{(n)}=\sum_{i=1}^{N} A_i\alpha_{f_i}^{(n)}(B_i)$, by (1) and (2)

$$B^{(n)}\to I\quad\text{as}\quad n\to\infty\,; \tag{3}$$

since $B^{(n)}\in\mathfrak{J}(\alpha^{(n)}, K)\neq\mathfrak{A}$ for all n, (3) cannot hold, and also $\mathfrak{J}(\alpha, K)\neq\mathfrak{A}$||.

2. Theorem. *Let $\{\alpha^{(n)}\}\subset\mathscr{A}$ be a sequence s.t. for each $g\in G$; $A\in\mathfrak{A}$; $\alpha_g^{(n)}(A)$ is convergent in $\mathfrak{A}$; then the limit $\alpha_g(A)$ defines on element $\alpha\in\mathscr{A}$. If $\alpha^{(n)}\in\mathscr{A}(K)$, also $\alpha\in\mathscr{A}(K)$.*

Proof. The limit α_g of $\alpha_g^{(n)}$ defines clearly a homomorphism of G into the group of *-automorphisms of $\mathfrak{A}$; by Proposition 1 we need only to prove that if $A\in\mathfrak{A}, g\in G\to\alpha_g(A)$ is continuous. Since $g\in G\to\alpha_g^{(n)}(A)$ is continuous it is also locally strongly measurable and so is $g\in G\to\alpha_0(A)$ by [3, Theorem 3.5.4]. Then continuity follows from local strong measurability by a known generalization of [3, Theorem 10.2.3].

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